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TWO MODIFIED PROXIMAL POINT ALGORITHMS FOR CONVEX FUNCTIONS IN HADAMARD SPACES

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Abstract. We study the existence and approximation of minimizers of proper lower semicontinuous convex functions in Hadamard spaces through the asymptotic behavior of sequences generated by two modified proximal point algorithms.

1. Introduction

The proximal point algorithm, first introduced by Martinet [22], is an iterative method for approximating zero points of maximal monotone operators in Hilbert spaces. The following is a specialization of the celebrated theorem by Rockafellar [24, Theorem 1] to the convex minimization problem.

Theorem 1.1 ([24, Theorem 1]). *Let X be a real Hilbert space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, $\{\lambda_n\}$ a sequence of positive real numbers such that $\inf_n \lambda_n > 0$, and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right\} \quad (n = 1, 2, \dots). \tag{1.1}$$

Then the set $\operatorname{argmin}_X f$ of all minimizers of f is nonempty if and only if $\{x_n\}$ is bounded. Further, if $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_X f$.

In 1978, Brézis and Lions [7, Théorème 9] obtained the following weak convergence theorem under a weaker condition on the sequence $\{\lambda_n\}$.

Theorem 1.2 ([7, Théorème 9]). *Let X and f be the same as in Theorem 1.1, $\{\lambda_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\{x_n\}$ a sequence defined by $x_1 \in X$ and (1.1). If $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_X f$.*

Later, Güler [11, Corollary 5.1] found a counterexample showing that the sequence $\{x_n\}$ in Theorem 1.1 does not converge strongly in general.

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As is well known, the resolvent J_f of f defined by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} \|y - x\|^2 \right\}$$

for all $x \in X$ is a cornerstone in the convergence analysis of the proximal point algorithm. In fact, the scheme (1.1) can be written as $x_{n+1} = J_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$ and each $J_{\lambda_n f}$ is a well defined single valued firmly nonexpansive mapping of X into itself whose fixed point set is identical with $\operatorname{argmin}_X f$. See [4, 27] for more details on convex analysis in Hilbert spaces.

In 2000, Kamimura and Takahashi [16] investigated the asymptotic behavior of $\{x_n\}$ and $\{y_n\}$ defined by $x_1, y_1 \in X$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n f} x_n \quad (n = 1, 2, \dots), \tag{1.2}$$

and

$$y_{n+1} = \alpha_n y_1 + (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots), \tag{1.3}$$

respectively, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers. They [16, Theorems 6 and 7] showed under some additional assumptions that $\{x_n\}$ converges weakly to an element of $\operatorname{argmin}_X f$ and that $\{y_n\}$ converges strongly to Py_1, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$. See [1, 3, 15, 20] for generalizations of these results to monotone operators in Banach spaces. See also Eckstein and Bertsekas [10] for related results on the iterative scheme (1.2) in Hilbert spaces.

On the other hand, in 1995, Jost [12] proposed a nonlinear generalization of the concept of resolvent in Hadamard spaces. According to [6, Section 2.2], [12, Lemma 2], and [23, Section 1.3], if f is a proper lower semicontinuous convex function of a Hadamard space X into $]-\infty, \infty]$, then the resolvent J_f of f given by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} d(y, x)^2 \right\} \tag{1.4}$$

for all $x \in X$ is a well defined single valued mapping of X into itself. We also know that J_f is nonexpansive and that the equality

$$\mathcal{F}(J_f) = \operatorname*{argmin}_X f \tag{1.5}$$

holds. See [6, 13, 14] for more details on this concept.

Recently, Bačák [5, Theorem 1.4] obtained the following remarkable generalization of Theorem 1.2 in Hadamard spaces.

Theorem 1.3 ([5, Theorem 1.4])**.** *Let X be a Hadamard space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, $\{\lambda_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\{x_n\}$ a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = J_{\lambda_n f} x_n \quad (n = 1, 2, \dots).$$

If $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$.

The aim of this paper is to study the existence and approximation of a minimizer of a proper lower semicontinuous convex function f of a Hadamard X into $]-\infty, \infty]$

through the asymptotic behavior of two iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_1, y_1 \in X$,

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n f} x_n \quad (n = 1, 2, \dots), \tag{1.6}$$

and

$$y_{n+1} = \alpha_n v \oplus (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots), \tag{1.7}$$

respectively, where v is a given point of X, $\{\alpha_n\}$ is a sequence of $[0,1]$, and $\{\lambda_n\}$ is a sequence of positive real numbers. Note that if X is a real Hilbert space and $y_1 = v$, then (1.6) and (1.7) are reduced to (1.2) and (1.3), respectively. In our main results, Theorems 4.2 and 5.1, we show the equivalence of the existence of a minimizer of f and the boundedness of $\{J_{\lambda_n f} x_n\}$ and $\{J_{\lambda_n f} y_n\}$ as well as the convergence of these sequences to minimizers of f, respectively.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{N}$ the set of all positive integers, $\mathbb{R}^2$ the two dimensional Euclidean space with norm $|\cdot|_{\mathbb{R}^2}$, X a metric space with metric d, and $\mathcal{F}(T)$ the set of all fixed points of a mapping T.

We need the following lemma.

Lemma 2.1 ([2, Lemma 2.3])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{t_n\}$ a sequence of real numbers such that $\limsup_n t_n \le 0$. If*

$$s_{n+1} \le (1 - \alpha_n) s_n + \alpha_n t_n \tag{2.1}$$

for all $n \in \mathbb{N}$, then $\lim_n s_n = 0$.

The following variant of Maingé's lemma [21, Lemma 3.1] was first found by Saejung and Yotkaew [25, Lemma 2.6]. Recently, Kimura and Saejung [17, Lemma 2.8] filled in a slight gap of the original proof given in [25]. Note that it was assumed in [17, 25] that $\alpha_n < 1$ for all $n \in \mathbb{N}$. However, the proof given in [17, Lemma 2.8] is valid to the case below without any change.

Lemma 2.2 ([17, Lemma 2.8] and [25, Lemma 2.6])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of $]0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{t_n\}$ a sequence of real numbers such that $\limsup_i t_{n_i} \le 0$ whenever $\{n_i\}$ is an increasing sequence of $\mathbb{N}$ satisfying*

$$\limsup_{i \to \infty} \bigl(s_{n_i} - s_{n_i+1}\bigr) \le 0.$$

If (2.1) holds for all $n \in \mathbb{N}$, then $\lim_n s_n = 0$.

A metric space X is said to be uniquely geodesic if for each $x, y \in X$, there exists a unique mapping c of $[0, l]$ into X such that $c(0) = x$, $c(l) = y$, and

$$d\bigl(c(t), c(t')\bigr) = \bigl|t - t'\bigr|$$

for all $t, t' \in [0, l]$, where $l = d(x, y)$. The image of c is denoted by $[x, y]$ and is called the geodesic segment between x and y. For each $\alpha \in [0,1]$, the point $c\bigl((1-\alpha)l\bigr)$ is denoted by $\alpha x \oplus (1-\alpha) y$. A uniquely geodesic metric space is simply called a uniquely geodesic space. A subset C of a uniquely geodesic space X is said to be convex if $[x, y]$ is contained by C for all $x, y \in C$.

If X is a uniquely geodesic space and x_1, x_2, x_3 are points in X, then there exist $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{R}^2$ such that $d(x_i, x_j) = |\bar{x}_i - \bar{x}_j|_{\mathbb{R}^2}$ for all $i, j \in \{1, 2, 3\}$. The sets Δ and $\bar{\Delta}$ defined by

$$\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1] \quad \text{and} \quad \bar{\Delta} = [\bar{x}_1, \bar{x}_2] \cup [\bar{x}_2, \bar{x}_3] \cup [\bar{x}_3, \bar{x}_1]$$

are called a geodesic triangle with vertices x_1, x_2, x_3 and a comparison triangle for Δ, respectively. A point $\bar{p} \in \bar{\Delta}$ is said to be a comparison point for $p \in \Delta$ if

$$p \in [x_i, x_j], \quad \bar{p} \in [\bar{x}_i, \bar{x}_j], \quad \text{and} \quad d(x_i, p) = |\bar{x}_i - \bar{p}|_{\mathbb{R}^2}$$

for some distinct $i, j \in \{1, 2, 3\}$.

A metric space X is said to be a CAT(0) space if it is uniquely geodesic and

$$d(p, q) \leq |\bar{p} - \bar{q}|_{\mathbb{R}^2}$$

whenever Δ is a geodesic triangle with vertices $x_1, x_2, x_3 \in X$, $\bar{\Delta}$ is a comparison triangle for Δ, and $\bar{p}, \bar{q} \in \bar{\Delta}$ are comparison points for $p, q \in \Delta$, respectively. Every complete CAT(0) space is particularly called a Hadamard space. If X is a CAT(0) space, then

$$d\big(\alpha x \oplus (1 - \alpha) y, z\big) \leq \alpha d(x, z) + (1 - \alpha) d(y, z) \tag{2.2}$$

and

$$d\big(\alpha x \oplus (1 - \alpha) y, z\big)^2 \leq \alpha d(x, z)^2 + (1 - \alpha) d(y, z)^2 - \alpha(1 - \alpha) d(x, y)^2 \tag{2.3}$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$. The inequality (2.2) implies that every CAT(0) space is a convex metric space in the sense of Takahashi [26]. It is known that every nonempty closed convex subset of a real Hilbert space and every open unit ball of a real Hilbert space with the hyperbolic metric are Hadamard spaces; see Bačák [6] and Bridson and Haefliger [8] for more details on CAT(0) spaces and CAT(κ) spaces with $\kappa \in \mathbb{R}$, respectively.

If C is a nonempty closed convex subset of a Hadamard space X, then for each $x \in X$, there exists a unique $\hat{x} \in C$ such that

$$d(\hat{x}, x) = \inf_{y \in C} d(y, x).$$

The metric projection P_C of X onto C is defined by $P_C(x) = \hat{x}$ for all $x \in X$.

Let X be a CAT(0) space and $\{x_n\}$ a sequence of X. Then the asymptotic center $\mathcal{A}\big(\{x_n\}\big)$ of the sequence $\{x_n\}$ is defined by

$$\mathcal{A}\big(\{x_n\}\big) = \left\{ z \in X : \limsup_{n \to \infty} d(z, x_n) = \inf_{y \in X} \limsup_{n \to \infty} d(y, x_n) \right\}.$$

The sequence $\{x_n\}$ is said to be Δ-convergent if there exists $p \in X$ such that

$$\mathcal{A}\big(\{x_{n_i}\}\big) = \{p\}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, $\{x_n\}$ is said to be Δ-convergent to p. If $\{x_n\}$ is Δ-convergent to $p \in X$, then it is bounded and its every subsequence is Δ-convergent to p. We denote by $\omega_\Delta\big(\{x_n\}\big)$ the set of all $q \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to q. If $\{x_n\}$ is a sequence of a real Hilbert space H and $p \in H$, then $\{x_n\}$ is Δ-convergent to p if and only if it is weakly convergent to p. See [6, 9, 19] for more details on Δ-convergence. Since

$$\limsup_{n\to\infty} d(y, x_n)^2 = \left(\limsup_{n\to\infty} d(y, x_n)\right)^2$$

for each $y \in X$, the concept of weak convergence discussed in [6, Chapter 3] coincides with that of Δ-convergence.

The following lemmas are of fundamental importance.

Lemma 2.3 ([9, Proposition 7]; see also [6, Section 3.1])**.** *The set $\mathcal{A}(\{x_n\})$ is a singleton for each bounded sequence $\{x_n\}$ of a Hadamard space.*

Lemma 2.4 ([19, Section 3]; see also [6, Proposition 3.1.2])**.** *Every bounded sequence of a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.5 ([18, Proposition 3.1])**.** *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence of X such that*

$$\inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) < \frac{\pi}{2}.$$

If the sequence $\{d(z, x_n)\}$ is convergent for each element z in $\omega_\Delta(\{x_n\})$, then $\{x_n\}$ is Δ-convergent to an element of X.

Using Lemma 2.5, we can show the following lemma.

Lemma 2.6. *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence of X. If the sequence $\{d(z, x_n)\}$ is convergent for each element z of $\omega_\Delta(\{x_n\})$, then $\{x_n\}$ is Δ-convergent to an element of X.*

Proof. Since $\{x_n\}$ is bounded, there exists $\kappa \in \,]0, \infty[$ such that

$$\inf_{y\in X} \limsup_{n\to\infty} \sqrt{\kappa} d(y, x_n) < \frac{\pi}{2}.$$

Since (X, d) is a Hadamard space, it is also a complete CAT(κ) space and hence $(X, \sqrt{\kappa} d)$ is a complete CAT(1) space. It is easy to verify that

$$\omega_\Delta(\{x_n\}) = \omega'_\Delta(\{x_n\}),$$

where the right hand side is the set of all $q \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to q in $(X, \sqrt{\kappa} d)$. Thus Lemma 2.5 implies that there exists $p \in X$ such that $\{x_n\}$ is Δ-convergent to some p in $(X, \sqrt{\kappa} d)$, that is,

$$\begin{aligned}\{p\} &= \left\{z \in X : \limsup_{i\to\infty} \sqrt{\kappa} d(z, x_{n_i}) = \inf_{y\in X} \limsup_{i\to\infty} \sqrt{\kappa} d(y, x_{n_i})\right\} \\ &= \left\{z \in X : \limsup_{i\to\infty} d(z, x_{n_i}) = \inf_{y\in X} \limsup_{i\to\infty} d(y, x_{n_i})\right\}\end{aligned}$$

for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Therefore, the sequence $\{x_n\}$ is Δ-convergent to p also in the space (X, d). □

Let X be a CAT(0) space. A function $f\colon X \to \,]-\infty, \infty]$ is said to be proper if $f(x)$ is finite for some $x \in X$. It is also said to be convex if

$$f\big(\alpha x \oplus (1-\alpha) y\big) \le \alpha f(x) + (1-\alpha) f(y)$$

whenever $x, y \in X$ and $\alpha \in]0,1[$. We denote by $\Gamma_0(X)$ the set of all proper lower semicontinuous convex functions of X into $]-\infty, \infty]$. It is known [6, Lemma 3.2.3] that if X is a Hadamard space and f is an element of $\Gamma_0(X)$, then f is Δ-lower semicontinuous, that is,

$$f(p) \le \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\}$ is a sequence of X which is Δ-convergent to $p \in X$. If C is a nonempty closed convex subset of X, then the indicator function i_C of C, which is defined by $i_C(x) = 0$ if $x \in C$ and ∞ if $x \in X \setminus C$, is an element of $\Gamma_0(X)$. See Bačák [6] for more details on convex analysis in Hadamard spaces.

3. Fundamental properties of resolvents of convex functions

In this section, we study some fundamental properties of resolvents of proper lower semicontinuous convex functions in Hadamard spaces.

Let X be a Hadamard space and f an element of $\Gamma_0(X)$. According to [6, Section 2.2], [12, Lemma 2], and [23, Section 1.3], for each $x \in X$, there exists a unique $\hat{x} \in X$ such that

$$f(\hat{x}) + \frac{1}{2}d(\hat{x}, x)^2 = \inf_{y\in X}\left\{f(y) + \frac{1}{2}d(y, x)^2\right\}.$$

The resolvent J_f of f is defined by $J_f x = \hat{x}$ for all $x \in X$. In other words, J_f is given by (1.4). It is also known that J_f is nonexpansive and (1.5) holds. See [6, 13, 14] for more details on this concept. If f is particularly the indicator function i_C of a nonempty closed convex subset C of X, then J_f coincides with the metric projection P_C of X onto C.

We first show the following lemma. The inequality (3.2) is a counterpart of [3, Lemma 3.1] in the Hadamard space setting.

Lemma 3.1. *Let X be a Hadamard space and f an element of $\Gamma_0(X)$. If $\lambda, \mu > 0$ and $x, y \in X$, then the inequalities*

$$d(J_{\lambda f}x, J_{\mu f}y)^2 + d(J_{\lambda f}x, x)^2 + 2\lambda\big(f(J_{\lambda f}x) - f(J_{\mu f}y)\big) \le d(J_{\mu f}y, x)^2 \tag{3.1}$$

and

$$\begin{aligned}&(\lambda+\mu)d(J_{\lambda f}x, J_{\mu f}y)^2 + \mu d(J_{\lambda f}x, x)^2 + \lambda d(J_{\mu f}y, y)^2\\ &\le \lambda d(J_{\lambda f}x, y)^2 + \mu d(J_{\mu f}y, x)^2\end{aligned} \tag{3.2}$$

hold.

Proof. Let $\lambda, \mu > 0$ and $x, y \in X$ be given. In order to show (3.1), we set

$$z_t = tJ_{\mu f}y \oplus (1-t)J_{\lambda f}x$$

for all $t \in]0,1[$. By the definition of $J_{\lambda f}$, the convexity of λf, and (2.3), we have

$$\begin{aligned}
&\lambda f(J_{\lambda f}x) + \frac{1}{2}d(J_{\lambda f}x, x)^2 \\
&\quad \le \lambda f(z_t) + \frac{1}{2}d(z_t, x)^2 \\
&\quad \le t\lambda f(J_{\mu f}y) + (1-t)\lambda f(J_{\lambda f}x) \\
&\qquad + \frac{1}{2}\big(td(J_{\mu f}y, x)^2 + (1-t)d(J_{\lambda f}x, x)^2 - t(1-t)d(J_{\mu f}y, J_{\lambda f}x)^2\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&2\lambda\big(f(J_{\lambda f}x) - f(J_{\mu f}y)\big) \\
&\quad \le d(J_{\mu f}y, x)^2 - (1-t)d(J_{\lambda f}x, J_{\mu f}y)^2 - d(J_{\lambda f}x, x)^2.
\end{aligned}$$

Letting $t \downarrow 0$, we obtain (3.1). It then follows from (3.1) that

$$\begin{aligned}
&\mu d(J_{\lambda f}x, J_{\mu f}y)^2 + \mu d(J_{\lambda f}x, x)^2 + 2\lambda\mu\big(f(J_{\lambda f}x) - f(J_{\mu f}y)\big) \\
&\quad \le \mu d(J_{\mu f}y, x)^2
\end{aligned}$$

and

$$\begin{aligned}
&\lambda d(J_{\mu f}y, J_{\lambda f}x)^2 + \lambda d(J_{\mu f}y, y)^2 + 2\lambda\mu\big(f(J_{\mu f}y) - f(J_{\lambda f}x)\big) \\
&\quad \le \lambda d(J_{\lambda f}x, y)^2.
\end{aligned}$$

Adding these inequalities, we obtain (3.2). □

The following corollary follows from Lemma 3.1. Note that (3.4) is a well known fact that each $J_{\lambda f}$ is nonexpansive; see [6, Theorem 2.2.22] and [12, Lemma 4].

Corollary 3.2. *Let X be a Hadamard space and f an element of $\Gamma_0(X)$. Then*

$$2d(J_{\lambda f}x, J_{\lambda f}y)^2 + d(J_{\lambda f}x, x)^2 + d(J_{\lambda f}y, y)^2 \le d(J_{\lambda f}x, y)^2 + d(J_{\lambda f}y, x)^2 \tag{3.3}$$

and

$$d(J_{\lambda f}x, J_{\lambda f}y) \le d(x, y) \tag{3.4}$$

for all $\lambda > 0$ and $x, y \in X$.

Proof. Let $\lambda > 0$ and $x, y \in X$ be given. It directly follows from Lemma 3.1 that (3.3) holds. According to Bačák [6, Corollary 1.2.5], the inequality

$$d(p,s)^2 + d(q,r)^2 - d(p,r)^2 - d(q,s)^2 \le 2d(p,q)d(r,s)$$

holds for all $p, q, r, s \in X$. Thus (3.3) implies that

$$\begin{aligned}
2d(J_{\lambda f}x, J_{\lambda f}y)^2 &\le d(J_{\lambda f}x, y)^2 + d(J_{\lambda f}y, x)^2 - d(J_{\lambda f}x, x)^2 - d(J_{\lambda f}y, y)^2 \\
&\le 2d(J_{\lambda f}x, J_{\lambda f}y)d(x, y).
\end{aligned}$$

Hence $J_{\lambda f}$ is nonexpansive. □

Using Lemma 3.1, we next show the following lemma.

Lemma 3.3. *Let X be a Hadamard space, f an element of $\Gamma_0(X)$, $\{\lambda_n\}$ a sequence of positive real numbers, and p an element of X. Then the following hold.*

(i) *If* $\inf_n \lambda_n > 0$ *and* $\mathcal{A}\big(\{z_n\}\big) = \{p\}$ *for some sequence* $\{z_n\}$ *of* X *satisfying* $d(J_{\lambda_n f} z_n, z_n) \to 0$, *then* p *is an element of* $\operatorname{argmin}_X f$*;*

(ii) *if* $\lim_n \lambda_n = \infty$ *and* $\mathcal{A}\big(\{J_{\lambda_n f} z_n\}\big) = \{p\}$ *for some bounded sequence* $\{z_n\}$ *of* X*, then* p *is an element of* $\operatorname{argmin}_X f$*.*

Proof. We first show (i). Suppose that $\inf_n \lambda_n > 0$ and $\mathcal{A}\big(\{z_n\}\big) = \{p\}$ for some sequence $\{z_n\}$ of X satisfying $d(J_{\lambda_n f} z_n, z_n) \to 0$. Then the sequences $\{z_n\}$ and $\{J_{\lambda_n f} z_n\}$ are bounded. It follows from (3.2) that

$$(\lambda_n + 1) d(J_{\lambda_n f} z_n, J_f p)^2 \leq \lambda_n d(J_{\lambda_n f} z_n, p)^2 + d(J_f p, z_n)^2$$

and hence

$$\begin{aligned} & d(J_{\lambda_n f} z_n, J_f p)^2 \\ & \quad \leq d(J_{\lambda_n f} z_n, p)^2 + \frac{1}{\lambda_n}\big(d(J_f p, z_n)^2 - d(J_f p, J_{\lambda_n f} z_n)^2\big) \\ & \quad \leq d(J_{\lambda_n f} z_n, p)^2 + \frac{1}{\lambda_n} d(z_n, J_{\lambda_n f} z_n)\big(d(J_f p, z_n) + d(J_f p, J_{\lambda_n f} z_n)\big) \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{1/\lambda_n\}$ is bounded, $d(J_{\lambda_n f} z_n, z_n) \to 0$, and both $\{z_n\}$ and $\{J_{\lambda_n f} z_n\}$ are bounded, we have

$$\limsup_{n\to\infty} d(J_{\lambda_n f} z_n, J_f p)^2 \leq \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, p)^2. \tag{3.5}$$

On the other hand, it follows from $d(J_{\lambda_n f} z_n, z_n) \to 0$ that

$$\limsup_{n\to\infty} d(z_n, y) = \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, y) \tag{3.6}$$

for all $y \in X$. Thus, by (3.5) and (3.6), we obtain

$$\begin{aligned} \Big(\limsup_{n\to\infty} d(z_n, J_f p)\Big)^2 &= \Big(\limsup_{n\to\infty} d(J_{\lambda_n f} z_n, J_f p)\Big)^2 \\ &= \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, J_f p)^2 \\ &\leq \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, p)^2 \\ &= \Big(\limsup_{n\to\infty} d(J_{\lambda_n f} z_n, p)\Big)^2 = \Big(\limsup_{n\to\infty} d(z_n, p)\Big)^2 \end{aligned}$$

and hence

$$\limsup_{n\to\infty} d(z_n, J_f p) \leq \limsup_{n\to\infty} d(z_n, p) = \inf_{y\in X} \limsup_{n\to\infty} d(z_n, y).$$

Thus we have $J_f p \in \mathcal{A}\big(\{z_n\}\big) = \{p\}$ and hence we obtain $J_f p = p$. Therefore, it follows from (1.5) that $p \in \operatorname{argmin}_X f$.

We next show (ii). Suppose that $\lim_n \lambda_n = \infty$ and $\mathcal{A}\big(\{J_{\lambda_n f} z_n\}\big) = \{p\}$ for some bounded sequence $\{z_n\}$ of X. Using (3.2), we can see that

$$d(J_{\lambda_n f} z_n, J_f p)^2 \leq d(J_{\lambda_n f} z_n, p)^2 + \frac{1}{\lambda_n} d(J_f p, z_n)^2$$

for all $n \in \mathbb{N}$. Since $\lim_n \lambda_n = \infty$ and $\{z_n\}$ is bounded, we have

$$\limsup_{n\to\infty} d(J_{\lambda_n f} z_n, J_f p)^2 \leq \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, p)^2$$

and hence

$$\limsup_{n\to\infty} d(J_{\lambda_n f} z_n, J_f p) \leq \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, p) = \inf_{y\in X} \limsup_{n\to\infty} d(J_{\lambda_n f} z_n, y).$$

This gives us that $p \in \operatorname{argmin}_X f$. □

4. A Δ-convergent modified proximal point algorithm

In this section, we study the asymptotic behavior of the sequence $\{x_n\}$ generated by (1.6).

Before obtaining one of our two main results in this paper, we show the following convex minimization theorem.

Theorem 4.1. *Let X be a Hadamard space, f an element of $\Gamma_0(X)$, $\{z_n\}$ a bounded sequence of X, $\{\beta_n\}$ a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \beta_n = \infty$, and g the real function defined by*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k d(y, z_k)^2$$

for all $y \in X$. Then g is a continuous and convex function such that $\operatorname{argmin}_X g$ *is a singleton.*

Proof. Set $\sigma_n = \sum_{l=1}^{n} \beta_l$ for all $n \in \mathbb{N}$ and let g_n be the function defined by

$$g_n(y) = \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k d(y, z_k)^2$$

for all $n \in \mathbb{N}$ and $y \in X$.

We first show the continuity of g. Let $\{x_m\}$ be a sequence of X converging to $a \in X$. Since $\{z_n\}$ and $\{x_m\}$ are bounded, we have a real number M such that $d(a, z_k) \leq M$ and $d(x_m, z_k) \leq M$ for all $k, m \in \mathbb{N}$. Then the triangle inequality implies that

$$\begin{aligned} g_n(x_m) &\leq \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \big(d(x_m, a)^2 + 2d(x_m, a)d(a, z_k) + d(a, z_k)^2\big) \\ &\leq \big(d(x_m, a) + 2M\big) d(x_m, a) + g_n(a) \end{aligned} \tag{4.1}$$

for all $m, n \in \mathbb{N}$. Taking the upper limit in (4.1) with respect to n, we obtain

$$g(x_m) \leq \big(d(x_m, a) + 2M\big) d(x_m, a) + g(a).$$

Similarly, we can see that

$$g(a) \leq \big(d(a, x_m) + 2M\big) d(a, x_m) + g(x_m).$$

Thus we obtain

$$|g(x_m) - g(a)| \leq \big(d(x_m, a) + 2M\big) d(x_m, a)$$

for all $m \in \mathbb{N}$. This implies that $g(x_m) \to g(a)$ and hence g is continuous.

We next show the convexity of g. If $y_1, y_2 \in X$ and $\alpha \in \,]0,1[$, then we have from (2.3) that

$$\begin{aligned} &g_n\big(\alpha y_1 \oplus (1-\alpha)y_2\big) \\ &= \frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k d\big(\alpha y_1 \oplus (1-\alpha)y_2, z_k\big)^2 \\ &\le \frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k\big(\alpha d(y_1,z_k)^2 + (1-\alpha)d(y_2,z_k)^2 - \alpha(1-\alpha)d(y_1,y_2)^2\big) \\ &= \alpha g_n(y_1) + (1-\alpha)g_n(y_2) - \alpha(1-\alpha)d(y_1,y_2)^2 \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$g\big(\alpha y_1 \oplus (1-\alpha)y_2\big) \le \alpha g(y_1) + (1-\alpha)g(y_2) - \alpha(1-\alpha)d(y_1,y_2)^2 \tag{4.2}$$

and hence g is convex.

We next show that $\operatorname{argmin}_X g$ is nonempty. It is obvious that

$$l = \inf g(X) \in [0,\infty[\,.$$

Then there exists a sequence $\{y_n\}$ of X such that $g(y_n) \to l$ and $g(y_n) \ge g(y_{n+1})$ for all $n \in \mathbb{N}$. If $m \ge n$, then it follows from (4.2) that

$$l \le g\Big(\frac{1}{2}y_n \oplus \frac{1}{2}y_m\Big) \le \frac{1}{2}g(y_n) + \frac{1}{2}g(y_m) - \frac{1}{4}d(y_n,y_m)^2$$

and hence

$$d(y_n,y_m) \le 2\sqrt{g(y_n) - l}.$$

This implies that $\{y_n\}$ is a Cauchy sequence and hence it converges to some $p \in X$. Since g is continuous and $g(y_n) \to l$, we obtain $g(p) = \lim_n g(y_n) = l$ and hence p belongs to $\operatorname{argmin}_X g$.

We finally show that $\operatorname{argmin}_X g$ is a singleton. If p and p' belong to $\operatorname{argmin}_X g$, then it follows from (4.2) that

$$l \le g\Big(\frac{1}{2}p \oplus \frac{1}{2}p'\Big) \le \frac{1}{2}g(p) + \frac{1}{2}g(p') - \frac{1}{4}d(p,p')^2 = l - \frac{1}{4}d(p,p')^2$$

and hence $p = p'$. Thus $\operatorname{argmin}_X g$ is a singleton. □

Now, we are ready to prove one of our two main results in this paper.

Theorem 4.2. *Let X be a Hadamard space, f an element of $\Gamma_0(X)$, and $\{x_n\}$ a sequence of X defined by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)J_{\lambda_n f}x_n \quad (n = 1,2,\dots),$$

where $\{\alpha_n\}$ is a sequence of $[0,1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n = \infty$. Then the following hold.

(i) *The sequence $\{J_{\lambda_n f}x_n\}$ is bounded if and only if $\operatorname{argmin}_X f$ is nonempty;*
(ii) *if $\sup_n \alpha_n < 1$ and $\operatorname{argmin}_X f$ is nonempty, then $\{x_n\}$ and $\{J_{\lambda_n f}x_n\}$ are Δ-convergent to an element x_∞ of $\operatorname{argmin}_X f$.*

Proof. Set $z_n = J_{\lambda_n f} x_n$ for all $n \in \mathbb{N}$. Using Theorem 4.1, we show the only if part of (i). Suppose that $\{z_n\}$ is bounded. Set

$$\beta_n = (1-\alpha_n)\lambda_n \quad \text{and} \quad \sigma_n = \sum_{l=1}^{n} \beta_l \tag{4.3}$$

for all $n \in \mathbb{N}$. According to Theorem 4.1, the real function g on X defined by

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k d(y, z_k)^2$$

for all $y \in X$ has a unique minimizer $p \in X$. By the definition of $\{x_n\}$ and (2.3), we have

$$\begin{aligned} d(J_f p, x_{k+1})^2 &= d\big(J_f p, \alpha_k x_k \oplus (1-\alpha_k) z_k\big)^2 \\ &\le \alpha_k d(J_f p, x_k)^2 + (1-\alpha_k) d(J_f p, z_k)^2. \end{aligned} \tag{4.4}$$

It follows from (3.2) that

$$(\lambda_k + 1) d(z_k, J_f p)^2 \le \lambda_k d(z_k, p)^2 + d(J_f p, x_k)^2. \tag{4.5}$$

Thus, by (4.3), (4.4), and (4.5), we obtain

$$\begin{aligned} &\beta_k d(z_k, J_f p)^2 \\ &\le \beta_k d(z_k, p)^2 + (1-\alpha_k)\big(d(J_f p, x_k)^2 - d(J_f p, z_k)^2\big) \\ &= \beta_k d(z_k, p)^2 + d(J_f p, x_k)^2 - \big(\alpha_k d(J_f p, x_k)^2 + (1-\alpha_k) d(J_f p, z_k)^2\big) \\ &\le \beta_k d(z_k, p)^2 + d(J_f p, x_k)^2 - d(J_f p, x_{k+1})^2 \end{aligned}$$

and hence

$$\frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k d(z_k, J_f p)^2 \le \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k d(z_k, p)^2 + \frac{1}{\sigma_n} d(J_f p, x_1)^2$$

for all $n \in \mathbb{N}$. Thus, it follows from $\lim_n \sigma_n = \infty$ that $g(J_f p) \le g(p)$. Since p is the unique minimizer of g, we have $J_f p = p$ and hence it follows from (1.5) that p is an element of $\operatorname{argmin}_X f$. Therefore, $\operatorname{argmin}_X f$ is nonempty.

We next show the if part of (i). If $\operatorname{argmin}_X f$ is nonempty, then there exists an element u of $\operatorname{argmin}_X f$. By (1.5), (2.2), and (3.4), we have

$$\begin{aligned} d(u, z_{n+1}) = d(u, J_{\lambda_{n+1} f} x_{n+1}) &\le d(u, x_{n+1}) \\ &= d\big(u, \alpha_n x_n \oplus (1-\alpha_n) z_n\big) \\ &\le \alpha_n d(u, x_n) + (1-\alpha_n) d(u, z_n) \\ &\le d(u, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ and $\{z_n\}$ are bounded.

We finally show (ii). Suppose that $\sup_n \alpha_n < 1$ and $\operatorname{argmin}_X f$ is nonempty. Let u be an element of $\operatorname{argmin}_X f$. Then, it follows from (2.3) and (3.1) that

$$\begin{aligned}
&d(u, x_{n+1})^2 \\
&\leq \alpha_n d(u, x_n)^2 + (1-\alpha_n) d(u, z_n)^2 \\
&\leq \alpha_n d(u, x_n)^2 + (1-\alpha_n)\Big(d(u, x_n)^2 - d(z_n, x_n)^2 - 2\lambda_n\big(f(z_n) - f(u)\big)\Big) \\
&= d(u, x_n)^2 - (1-\alpha_n)\Big(d(z_n, x_n)^2 + 2\lambda_n\big(f(z_n) - \inf f(X)\big)\Big) \\
&\leq d(u, x_n)^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus $\{d(u, x_n)\}$ is convergent for each $u \in \operatorname{argmin}_X f$. Since

$$(1-\alpha_n)\Big(d(z_n, x_n)^2 + 2\lambda_n\big(f(z_n) - \inf f(X)\big)\Big) \leq d(u, x_n)^2 - d(u, x_{n+1})^2,$$

we have

$$\sum_{n=1}^{\infty}(1-\alpha_n)\Big(d(z_n, x_n)^2 + 2\lambda_n\big(f(z_n) - \inf f(X)\big)\Big) \leq d(u, x_1)^2.$$

This implies that

$$\sum_{n=1}^{\infty}(1-\alpha_n) d(z_n, x_n)^2 < \infty \tag{4.6}$$

and

$$\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n\big(f(z_n) - \inf f(X)\big) < \infty. \tag{4.7}$$

By $\sup_n \alpha_n < 1$ and (4.6), we have

$$\sum_{n=1}^{\infty} d(z_n, x_n)^2 < \infty. \tag{4.8}$$

By $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n = \infty$ and (4.7), we also have

$$\liminf_{n\to\infty}\big(f(z_n) - \inf f(X)\big) = 0. \tag{4.9}$$

On the other hand, the definition of $J_{\lambda_n f}$ implies that

$$f(z_n) \leq f(z_n) + \frac{1}{2\lambda_n} d(z_n, x_n)^2 \leq f(x_n)$$

for all $n \in \mathbb{N}$. Thus, by the definition of $\{x_n\}$ and the convexity of f, we obtain

$$-\infty < \inf f(X) \leq f(x_{n+1}) \leq \alpha_n f(x_n) + (1-\alpha_n) f(z_n) \leq f(x_n)$$

for all $n \in \mathbb{N}$. Accordingly, the sequence $\{f(x_n)\}$ is convergent to some real number β and hence $\{f(z_n)\}$ is bounded. Let $\{f(z_{n_i})\}$ be any subsequence of $\{f(z_n)\}$. Since $\sup_n \alpha_n < 1$, we have a subsequence $\{\alpha_{n_{i_j}}\}$ of $\{\alpha_{n_i}\}$ which tends to some $\gamma \in [0, 1[$. Then, by letting $j \to \infty$ in

$$\frac{1}{1-\alpha_{n_{i_j}}}\Big(f(x_{n_{i_j}+1}) - \alpha_{n_{i_j}} f(x_{n_{i_j}})\Big) \leq f(z_{n_{i_j}}) \leq f(x_{n_{i_j}}),$$

we know that $\{f(z_{n_{i_j}})\}$ tends to β. Thus we conclude that

$$\lim_{n\to\infty} f(z_n) = \beta = \lim_{n\to\infty} f(x_n). \tag{4.10}$$

Consequently, using (4.9) and (4.10), we obtain

$$\lim_{n\to\infty} f(x_n) = \inf f(X). \tag{4.11}$$

Let z be an element of $\omega_\Delta\big(\{x_n\}\big)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is Δ-convergent to z. Since f is Δ-lower semicontinuous, it follows from (4.11) that

$$f(z) \le \liminf_{i\to\infty} f(x_{n_i}) = \lim_{n\to\infty} f(x_n) = \inf f(X)$$

and hence $z \in \operatorname{argmin}_X f$. Thus we know that $\omega_\Delta\big(\{x_n\}\big)$ is contained by $\operatorname{argmin}_X f$. Combining this property with the fact that $\{d(z, x_n)\}$ is convergent for each z in $\operatorname{argmin}_X f$, we know that $\{d(z, x_n)\}$ is convergent for each z in $\omega_\Delta\big(\{x_n\}\big)$. Accordingly, Lemma 2.6 ensures that $\{x_n\}$ is Δ-convergent to an element x_∞ of X. Since it follows from (4.8) that $d(z_n, x_n) \to 0$, the sequence $\{z_n\}$ is also Δ-convergent to x_∞. Finally, since

$$\{x_\infty\} = \omega_\Delta\big(\{x_n\}\big) \subset \operatorname*{argmin}_X f,$$

we conclude that x_∞ is an element of $\operatorname{argmin}_X f$. $\square$

As direct consequences of Theorem 4.2, we obtain the following two corollaries.

Corollary 4.3. *Let X and f be the same as in Theorem 4.2 and $\{x_n\}$ a sequence of X defined by $x_1 \in X$ and*

$$x_{n+1} = J_{\lambda_n f} x_n \quad (n = 1, 2, \dots),$$

where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $\sum_{n=1}^\infty \lambda_n = \infty$. Then the following hold.

(i) *The sequence $\{x_n\}$ is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if* $\operatorname{argmin}_X f$ *is nonempty, then $\{x_n\}$ is Δ-convergent to an element of* $\operatorname{argmin}_X f$.

Remark 4.4. Note that (ii) is the result due to Bačák [5, Theorem 1.4].

Corollary 4.5. *Let X be a Hilbert space, f an element of $\Gamma_0(X)$, and $\{x_n\}$ a sequence of X defined by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n f} x_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ is a sequence of $[0, 1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying $\sum_{n=1}^\infty (1 - \alpha_n)\lambda_n = \infty$.

(i) *The sequence $\{J_{\lambda_n f} x_n\}$ is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if $\sup_n \alpha_n < 1$ and* $\operatorname{argmin}_X f$ *is nonempty, then $\{x_n\}$ and $\{J_{\lambda_n f} x_n\}$ are weakly convergent to an element x_∞ of* $\operatorname{argmin}_X f$.

Remark 4.6. Note that the special case of (ii) where $\lim_n \lambda_n = \infty$ is also a corollary of the weak convergence theorem due to Kamimura and Takahashi [16, Theorem 3].

5. A convergent modified proximal point algorithm

In this section, we study the asymptotic behavior of the sequence $\{y_n\}$ generated by (1.7).

The following is the other of our two main results in this paper.

Theorem 5.1. *Let X be a Hadamard space, f an element of $\Gamma_0(X)$, v an element of X, and $\{y_n\}$ a sequence of X defined by $y_1 \in X$ and*

$$y_{n+1} = \alpha_n v \oplus (1-\alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots), \tag{5.1}$$

where $\{\alpha_n\}$ is a sequence of $[0,1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\lim_n \lambda_n = \infty$. Then the following hold.

(i) *The sequence $\{J_{\lambda_n f} y_n\}$ is bounded if and only if $\operatorname{argmin}_X f$ is nonempty;*
(ii) *if $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\operatorname{argmin}_X f$ is nonempty, then $\{y_n\}$ and $\{J_{\lambda_n f} y_n\}$ are convergent to Pv, where P denotes the metric projection of X onto $\operatorname{argmin}_X f$.*

Proof. Set $z_n = J_{\lambda_n f} y_n$ for all $n \in \mathbb{N}$. We first show the only if part of (i). Suppose that $\{z_n\}$ is bounded. By Lemma 2.3, we know that $\mathcal{A}(\{z_n\}) = \{p\}$ for some $p \in X$. It follows from (2.2) that

$$\begin{aligned} d(p, y_{n+1}) &= d(p, \alpha_n v \oplus (1-\alpha_n) z_n) \\ &\le \alpha_n d(p, v) + (1-\alpha_n) d(p, z_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Thus the boundedness of $\{z_n\}$ implies that of $\{y_n\}$. Noting that $\lim_n \lambda_n = \infty$ and $\mathcal{A}(\{J_{\lambda_n f} y_n\}) = \{p\}$, we have from (ii) of Lemma 3.3 that p is an element of $\operatorname{argmin}_X f$. Thus $\operatorname{argmin}_X f$ is nonempty.

We next show the if part of (i). If $\operatorname{argmin}_X f$ is nonempty, then it follows from (1.5), (2.2) and (3.4) that

$$\begin{aligned} d(Pv, y_{n+1}) &= d(Pv, \alpha_n v \oplus (1-\alpha_n) z_n) \\ &\le \alpha_n d(Pv, v) + (1-\alpha_n) d(Pv, z_n) \\ &\le \max\{d(Pv, v), d(Pv, y_n)\} \end{aligned} \tag{5.2}$$

and hence

$$d(Pv, y_n) \le \max\{d(Pv, v), d(Pv, y_1)\} \tag{5.3}$$

for all $n \in \mathbb{N}$. This implies that $\{y_n\}$ is bounded. Since

$$d(Pv, z_n) = d(Pv, J_{\lambda_n f} y_n) \le d(Pv, y_n), \tag{5.4}$$

the sequence $\{z_n\}$ is also bounded.

We next show (ii). Suppose that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\operatorname{argmin}_X f$ is nonempty. Then we know that (5.2), (5.3), and (5.4) hold. Hence $\{y_n\}$ and $\{z_n\}$ are bounded. Using (1.5), (2.3), (3.4), and (5.4), we can see that

$$\begin{aligned} & d(Pv, y_{n+1})^2 \\ & = d(Pv, \alpha_n v \oplus (1-\alpha_n) z_n)^2 \\ & \le \alpha_n d(Pv, v)^2 + (1-\alpha_n) d(Pv, z_n)^2 - \alpha_n (1-\alpha_n) d(v, z_n)^2 \\ & \le (1-\alpha_n) d(Pv, y_n)^2 + \alpha_n \big(d(Pv, v)^2 - d(v, z_n)^2 + \alpha_n d(v, z_n)^2 \big) \end{aligned} \tag{5.5}$$

for all $n \in \mathbb{N}$. Thus, letting

$$s_n = d(Pv, y_n)^2 \quad \text{and} \quad t_n = d(Pv, v)^2 - d(v, z_n)^2 + \alpha_n d(v, z_n)^2, \tag{5.6}$$

we have

$$s_{n+1} \le (1 - \alpha_n) s_n + \alpha_n t_n \tag{5.7}$$

for all $n \in \mathbb{N}$.

Since $\{z_n\}$ is bounded, by Lemma 2.4, we have a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ which is Δ-convergent to some $q \in X$ and

$$\lim_{i \to \infty} d(v, z_{n_i})^2 = \liminf_{n \to \infty} d(v, z_n)^2. \tag{5.8}$$

Since $\lim_i \lambda_{n_i} = \infty$, $\mathcal{A}\big(\{J_{\lambda_{n_i} f} y_{n_i}\}\big) = \{q\}$, and $\{y_{n_i}\}$ is bounded, it follows from (ii) of Lemma 3.3 that q is an element of $\operatorname{argmin}_X f$. Using the Δ-lower semicontinuity of the function $d(v, \cdot)^2$ and (5.8), we then obtain

$$d(v, q)^2 \le \liminf_{i \to \infty} d(v, z_{n_i})^2 = \lim_{i \to \infty} d(v, z_{n_i})^2 = \liminf_{n \to \infty} d(v, z_n)^2. \tag{5.9}$$

Since $d(Pv, v) \le d(q, v)$, $\alpha_n \to 0$, and $\{z_n\}$ is bounded, it follows from (5.9) that

$$\begin{aligned} \limsup_{n \to \infty} t_n &\le d(Pv, v)^2 - \liminf_{n \to \infty} d(v, z_n)^2 + \limsup_{n \to \infty} \alpha_n d(v, z_n)^2 \\ &\le d(Pv, v)^2 - d(v, q)^2 \le 0. \end{aligned}$$

Consequently, Lemma 2.1 implies that $\lim_n s_n = 0$ and hence $\{y_n\}$ is convergent to Pv. It follows from (5.4) that $\{z_n\}$ is also convergent to Pv. □

As a direct consequence of Theorem 5.1, we obtain the following.

Corollary 5.2. *Let X be a Hilbert space, f an element of $\Gamma_0(X)$, v an element of X, and $\{y_n\}$ a sequence of X defined by $y_1 \in X$ and*

$$y_{n+1} = \alpha_n v + (1 - \alpha_n) J_{\lambda_n f} y_n \quad (n = 1, 2, \dots),$$

where $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\lim_n \lambda_n = \infty$. Then the following hold.

(i) *The sequence $\{J_{\lambda_n f} y_n\}$ is bounded if and only if* $\operatorname{argmin}_X f$ *is nonempty;*
(ii) *if $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and* $\operatorname{argmin}_X f$ *is nonempty, then $\{y_n\}$ and $\{J_{\lambda_n f} y_n\}$ are strongly convergent to Pv, where P denotes the metric projection of X onto* $\operatorname{argmin}_X f$.

Remark 5.3. Note that (ii) is also a corollary of the strong convergence theorem due to Kamimura and Takahashi [16, Theorem 1].

Applying Lemma 2.2, we can also show the following convergence theorem under a different type of coefficient conditions on the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$.

Theorem 5.4. *Let X be a Hadamard space, f an element of $\Gamma_0(X)$ such that* $\operatorname{argmin}_X f$ *is nonempty, v an element of X, and $\{y_n\}$ a sequence of X defined by $y_1 \in X$ and (5.1), where $\{\alpha_n\}$ is a sequence of $]0, 1]$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\lim_n \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\inf_n \lambda_n > 0$. Then $\{y_n\}$ and $\{J_{\lambda_n f} y_n\}$ are convergent to Pv, where P denotes the metric projection of X onto* $\operatorname{argmin}_X f$.

Proof. Set $z_n = J_{\lambda_n f} y_n$ for all $n \in \mathbb{N}$. We denote by $\{s_n\}$ and $\{t_n\}$ the sequences defined by (5.6). Note that (5.4), (5.5) and (5.7) hold also in this case. Let $\{n_i\}$ be any increasing sequence of $\mathbb{N}$ such that

$$\limsup_{i\to\infty}\bigl(s_{n_i} - s_{n_i+1}\bigr) \le 0. \tag{5.10}$$

Using (3.3), (5.5), (5.10), and $\alpha_{n_i} \to 0$, we have

$$\begin{aligned}
&\limsup_{i\to\infty} d(z_{n_i}, y_{n_i})^2 \\
&\quad \le \limsup_{i\to\infty}\bigl(d(Pv, y_{n_i})^2 - d(Pv, z_{n_i})^2\bigr) \\
&\quad \le \limsup_{i\to\infty}\Bigl(d(Pv, y_{n_i})^2 - d(Pv, y_{n_i+1})^2 + \alpha_{n_i}\bigl(d(Pv, v)^2 - d(Pv, z_{n_i})^2\bigr)\Bigr) \\
&\quad = \limsup_{i\to\infty}\bigl(s_{n_i} - s_{n_i+1}\bigr) \le 0
\end{aligned}$$

and hence we obtain

$$\lim_{i\to\infty} d(z_{n_i}, y_{n_i}) = 0. \tag{5.11}$$

Since $\{z_{n_i}\}$ is bounded, Lemma 2.4 implies that there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which is Δ-convergent to some $q \in X$ and

$$\lim_{j\to\infty} d(v, z_{n_{i_j}})^2 = \liminf_{i\to\infty} d(v, z_{n_i})^2. \tag{5.12}$$

It follows from (5.11) that

$$\mathcal{A}\bigl(\{y_{n_{i_j}}\}\bigr) = \mathcal{A}\bigl(\{z_{n_{i_j}}\}\bigr) = \{q\}. \tag{5.13}$$

Since $\inf_j \lambda_{n_{i_j}} \ge \inf_n \lambda_n > 0$ and both (5.11) and (5.13) hold, it follows from (i) of Lemma 3.3 that q is an element of $\operatorname{argmin}_X f$. Since $d(v, \cdot)^2$ is Δ-lower semicontinuous and (5.12) holds, we have

$$d(v, q)^2 \le \liminf_{j\to\infty} d(v, z_{n_{i_j}})^2 = \lim_{j\to\infty} d(v, z_{n_{i_j}})^2 = \liminf_{i\to\infty} d(v, z_{n_i})^2.$$

This implies that

$$\begin{aligned}
\limsup_{i\to\infty} t_{n_i} &= \limsup_{i\to\infty}\bigl(d(Pv, v)^2 - d(v, z_{n_i})^2 + \alpha_{n_i} d(v, z_{n_i})^2\bigr) \\
&= d(Pv, v)^2 - \liminf_{i\to\infty} d(v, z_{n_i})^2 \\
&\le d(Pv, v)^2 - d(v, q)^2 \le 0.
\end{aligned}$$

Consequently, Lemma 2.2 implies that $\lim_n s_n = 0$ and hence $\{y_n\}$ is convergent to Pv. It follows from (5.4) that $\{z_n\}$ is also convergent to Pv. $\square$

References

[1] K. Aoyama, Y. Kimura and F. Kohsaka, *Strong convergence theorems for strongly relatively nonexpansive sequences and applications*, J. Nonlinear Anal. Optim. **3** (2012), 67–77.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Proximal point methods for monotone operators in Banach spaces*, Taiwanese J. Math. **15** (2011), 259–281.

[4] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[5] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701.

[6] M. Bačák, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter, Berlin, 2014.

[7] H. Brézis and P.-L. Lions, *Produits infinis de résolvantes*, Israel J. Math. **29** (1978), 329–345.

[8] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, Berlin, 1999.

[9] S. Dhompongsa, W. A. Kirk and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. **65** (2006), 762–772.

[10] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Math. Program. **55** (1992), 293–318.

[11] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[12] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[13] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Birkhäuser Verlag, Basel, 1997.

[14] J. Jost, *Nonlinear Dirichlet forms*, Amer. Math. Soc., Providence, RI, 1998, pp. 1–47.

[15] S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.

[16] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.

[17] Y. Kimura and S. Saejung, *Strong convergence for a common fixed point of two different generalizations of cutter operators*, Linear Nonlinear Anal. **1** (2015), 53–65.

[18] Y. Kimura, S. Saejung and P. Yotkaew, *The Mann algorithm in a complete geodesic space with curvature bounded above*, Fixed Point Theory Appl. **2013** 2013:336, 1–13.

[19] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.

[20] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. (2004), 239–249.

[21] P.-E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[22] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158 (French).

[23] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[24] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[25] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

[26] W. Takahashi, *A convexity in metric space and nonexpansive mappings. I*, Kōdai Math. Sem. Rep. **22** (1970), 142–149.

[27] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

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