Yokohama Publishers

# GENERALIZED VECTOR EQUILIBRIUM-LIKE PROBLEMS WITH APPLICATIONS TO VECTOR OPTIMIZATION PROBLEMS 

LU-CHUAN CENG AND JEN-CHIH YAO*


#### Abstract

In this paper, we studied several kinds of generalized vector equilibriumlike problems. Relationships between the solutions of generalized Minty vector equilibrium-like problem and an efficient solution of a vector optimization problem were established. A generalized Stampacchia vector equilibrium-like problem was considered and its relation with generalized weak Minty vector equilibriumlike problem was derived. Some existence results for solutions of generalized vector equilibrium-like problems were established and applications to vector optimization problems were given.


## 1. Introduction and preliminaries

Since the vector variational inequality in finite-dimensional Euclidean spaces was introduced by F. Giannessi in 1980 [19], various extensions, generalizations and applications have been considered and studied. See, e.g., [1, 3-9, 11, 17, 18, 22, 28] and the references therein. Further in 1998, Giannessi [20] used Minty type vector variational inequality (MVVI) to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem (VOP) for differentiable and convex functions. The research work of F. Giannessi in [8] influenced many researchers to do investigation in this direction. See, e.g, $[4,12,13$, $16,23,26,27]$ and the references therein.

Motivated and inspired by generalized vector equilibrium problems considered by Zeng and Yao $[8,28]$, the work of this manuscript is to further extend the results in [4] to the setting of the generalized vector equilibrium-like problems (GVELPs). The main purpose of this article is to give the solvability of the following GVELPs and their applications to the VOPs. The argument and some notations of this paper follow those in [4].

[^0]Now, several forms of generalized Minty and generalized Stampacchia vector equilibrium-like problems are stated as follows:

Problem (I). Generalized Minty vector equilibrium-like problem (GMVELP): Find $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(w, \bar{x}, x) \not Z_{C(\bar{x}) \backslash\{0\}} 0, \quad \text { for some } w \in A(\bar{x}, T(x)),
$$

which is equivalent to find $\bar{x} \in K$ such that

$$
\Phi(A(\bar{x}, T(x)), \bar{x}, x) \nsubseteq-C(\bar{x}) \backslash\{0\}, \quad \text { for all } x \in K
$$

Problem (II). Generalized weak Minty vector equilibrium-like problem (GWMVELP): Find $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(w, \bar{x}, x) \mathbb{Z}_{\operatorname{int} C(\bar{x})} 0, \quad \text { for some } w \in A(\bar{x}, T(x))
$$

which is equivalent to find $\bar{x} \in K$ such that

$$
\Phi(A(\bar{x}, T(x)), \bar{x}, x) \nsubseteq-\operatorname{int} C(\bar{x}), \quad \text { for all } x \in K
$$

Problem (III). Generalized Stampacchia vector equilibrium-like problem (GSVELP): Find $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(\bar{w}, \bar{x}, x) Z_{C(\bar{x}) \backslash\{0\}} 0, \quad \text { for some } \bar{w} \in A(\bar{x}, T(\bar{x})),
$$

which is equivalent to find $\bar{x} \in K$ such that

$$
\Phi(A(\bar{x}, T(\bar{x})), \bar{x}, x) \nsubseteq-C(\bar{x}) \backslash\{0\}, \quad \text { for all } x \in K
$$

Problem (IV). Generalized weak Stampacchia vector equilibrium-like problem (GWSVELP): Find $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(\bar{w}, \bar{x}, x) \leq_{\operatorname{int} C(\bar{x})} 0, \quad \text { for some } \bar{w} \in A(\bar{x}, T(\bar{x}))
$$

which is equivalent to find $\bar{x} \in K$ such that

$$
\Phi(A(\bar{x}, T(\bar{x})), \bar{x}, x) \nsubseteq-\operatorname{int} C(\bar{x}), \quad \text { for all } x \in K
$$

It can be easily seen that every solution of GMVELP (respectively, GSVELP) is a solution of GWMVELP (respectively, GWSVELP). In particular, if $A(x, u)=u$ for each $(x, u) \in K \times \mathcal{L}(X, Y)$ and $\Phi(w, x, y)=\langle w, \eta(y, x)\rangle$ for each $(w, x, y) \in$ $\mathcal{L}(X, Y) \times K \times K$, where $\eta: K \times K \rightarrow X$ is a function, then Problems (I)-(IV) reduce to corresponding forms of generalized Minty and generalized Stampacchia vector variational-like inequality problems studied in [4].

Next, we recall some concepts and notations. A nonempty subset $C$ of a vector space $Y$ is a convex cone if $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C+C \subseteq C$. A convex cone $C$ is pointed if $C \cap(-C)=\{0\}$. A cone $C$ is proper if it is properly contained in $Y$. Note that $C$ is a proper cone if and only if $0 \notin \operatorname{int} C$, where int $C$ denotes the interior of $C$. A pointed convex cone $C$ induces a partial order $\leq_{C}$ on $Y$ defined by $x \leq_{C} y$ whenever $y-x \in C$. In this case, $\left(Y, \leq_{C}\right)$ is an ordered vector space with an order relation $\leq_{C}$. The weak order $\not_{\operatorname{int} C}$ on an ordered vector space $\left(Y, \leq_{C}\right)$ with $\operatorname{int} C \neq \emptyset$ is defined by $x \not \leq \operatorname{int} C y$ whenever $y-x \notin \operatorname{int} C$.

Let $X$ and $Y$ be two topological spaces. We say that a multifunction $\varphi: X \rightarrow 2^{Y}$ is closed, or has closed graph if its graph given by

$$
\mathcal{G}(\varphi)=\{(x, y) \in X \times Y: y \in \varphi(x)\}
$$

is a closed subset of $X \times Y$.
For any two Hausdorff topological vector spaces (t.v.s.) $X$ and $Y$, let $\mathcal{L}(X, Y)$ denote the family of all continuous linear operators from $X$ into $Y$. When $Y$ is the set $\mathbf{R}$ of real numbers, $\mathcal{L}(X, Y)$ is the usual dual space $X^{*}$ of $X$. For any $x \in X$ and any $u \in \mathcal{L}(X, Y)$, we shall write the value $u(x)$ as $\langle u, x\rangle$. We suppose throughout this paper that $K$ is a nonempty closed convex subset of $X, T: K \rightarrow 2^{\mathcal{L}(X, Y)}$ is a set-valued mapping, $\Phi: \mathcal{L}(X, Y) \times K \times K \rightarrow Y$ and $A: K \times \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Y)$ are two functions, and $\{C(x): x \in K\}$ is a family of closed, convex and pointed cones of $Y$ (i.e., $C: K \rightarrow 2^{Y}$ is a cone mapping) such that $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$.

Let $\mho$ be the family of all bounded subsets of $X$ whose union is total in $X$, i.e., the linear hull of $\bigcup\{S: S \in \mathcal{J}\}$ is dense in $X$. Let $\mathcal{B}$ be a neighborhood base of 0 in $Y$. When $S$ runs through $\mathcal{V}$ and $V$ through $\mathcal{B}$, the family

$$
M(S, V)=\{u \in \mathcal{L}(X, Y):\langle u, x\rangle \in V, \forall x \in S\}
$$

is a neighborhood base of 0 in $\mathcal{L}(X, Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the sets $S \in \mathcal{V}$, or briefly the $\tau$ topology; see [24]. Throughout this paper, we suppose that the space $\mathcal{L}(X, Y)$ is equipped with the $\tau$-topology.

Lemma 1.1 (see [10]). Let $\left(Y, \leq_{C}\right)$ be an ordered topological vector space with a closed, convex and pointed cone $C$ with $\operatorname{int} C \neq \emptyset$. Then for each $x, y \in Y$, one has
(1) $y-x \in \operatorname{int} C$ and $y \notin \operatorname{int} C \Rightarrow x \notin \operatorname{int} C$.
(2) $y-x \in C$ and $y \notin \operatorname{int} C \Rightarrow x \notin \operatorname{int} C$.
(3) $y-x \in-\operatorname{int} C$ and $y \notin-\operatorname{int} C \Rightarrow x \notin-\operatorname{int} C$.
(4) $y-x \in-C$ and $y \notin-\operatorname{int} C \Rightarrow x \notin-\operatorname{int} C$.

We denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of $X$. Let $F: Y \rightarrow$ $2^{X}$ be a set-valued mapping. Then $F$ is said to be transfer closed-valued iff for each $(y, x) \in Y \times X$ with $x \notin F(y)$, there exists $y^{\prime} \in Y$ such that $x \notin \operatorname{cl} F\left(y^{\prime}\right)$. If $B \subseteq Y$ and $A \subseteq X$, then we call $F: B \rightarrow 2^{A}$ transfer closed-valued iff the multi-valued mapping $y \mapsto F(y) \cap A$ is transfer closed-valued. When $X=Y$ and $A=B$, we call $F$ transfer closed-valued on $A$. Let $K$ be a convex subset of a vector space $X$. Then a mapping $F: K \rightarrow 2^{X}$ is called a KKM mapping iff for each nonempty finite subset $A$ of $K, \operatorname{conv} A \subset F(A)$, where conv $A$ denotes the convex hull of $A$, and $F(A)=\bigcup\{F(x): x \in A\}$.

Theorem 1.2 (see [17]). Let $K$ be a nonempty and convex subset of a Hausdorff t.v.s. $X$. Suppose that $\Gamma, \hat{\Gamma}: K \rightarrow 2^{K}$ are two set-valued mappings such that the following conditions are satisfied:
(A1) $\hat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in K ;$
(A2) $\hat{\Gamma}$ is a KKM map;
(A3) for each $A \in \mathcal{F}(K), \Gamma$ is transfer closed-valued on $\operatorname{conv} A$;
(A4) for each $A \in \mathcal{F}(K), \operatorname{cl}_{K}\left(\bigcap_{x \in \operatorname{conv} A} \Gamma(x)\right) \cap \operatorname{conv} A=\left(\bigcap_{x \in \operatorname{conv} A} \Gamma(x)\right) \cap$ conv $A$;
(A5) there is a nonempty compact convex set $B \subseteq K$, such that $\operatorname{cl}_{K}\left(\bigcap_{x \in B} \Gamma(x)\right)$ is compact.
Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

Remark 1.3. Suppose that in addition to conditions (A1), (A2) and (A5), two conditions (A3) and (A4) in Theorem 1.2 are replaced by the condition that for each $x \in K, \Gamma(x)$ is closed. In this case, it is easy to see that the conclusion is still valid, i.e., $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$; see [17] for more details.

## 2. Existence Results for GVELPs

Let $X$ and $Y$ be two Hausdorff topological vector spaces (t.v.s.), and let $\mathcal{L}(X, Y)$ be a t.v.s. equipped with the topology $\tau$ of uniform convergence. Suppose that $K$ is a nonempty closed convex subset of $X, T: K \rightarrow 2^{\mathcal{L}(X, Y)}$ is a set-valued mapping, $\Phi: \mathcal{L}(X, Y) \times K \times K \rightarrow Y$ and $A: K \times \mathcal{L}(X, Y) \rightarrow \mathcal{L}(X, Y)$ are two functions, and $\{C(x): x \in K\}$ is a family of closed, convex and pointed cones of $Y$ (i.e., $C: K \rightarrow 2^{Y}$ is a cone mapping) such that $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. We introduce the following monotonicity concepts.

Definition 2.1. For any $y \in K$, a function $\Phi: \mathcal{L}(X, Y) \times K \times K \rightarrow Y$ is said to be
(a) $C(y)$-pseudomonotone w.r.t. $T$ and $A$ on $K$ if for all $x \in K$

$$
\Phi(A(y, T(x)), x, y) \subseteq C(y) \backslash\{0\} \quad \Rightarrow \quad \Phi(A(y, T(y)), y, x) \subseteq-C(y) \backslash\{0\}
$$

(b) $C(y)$-quasimonotone w.r.t. $T$ and $A$ on $K$ if for all $x \in K$

$$
\Phi(A(y, T(x)), x, y) \subseteq \operatorname{int} C(y) \quad \Rightarrow \quad \Phi(A(y, T(y)), y, x) \subseteq-C(y) \backslash\{0\}
$$

(c) $C(y)$-strictly quasimonotone w.r.t. $T$ and $A$ on $K$ if for all $x \in K$

$$
\Phi(A(y, T(x)), x, y) \subseteq \operatorname{int} C(y) \quad \Rightarrow \quad \Phi(A(y, T(y)), y, x) \subseteq-\operatorname{int} C(y)
$$

(d) $C(y)$-properly quasimonotone w.r.t. $T$ and $A$ on $K$ if for all $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ $\subseteq K$ and for all $y \in \operatorname{conv}\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, there exists $i \in\{1,2, \ldots, n\}$ such that

$$
\Phi\left(A(y, T(y)), y, x_{i}\right) \subseteq C(y) \backslash\{0\}
$$

(e) $C(y)$-weakly properly quasimonotone w.r.t. $T$ and $A$ on $K$ if for all $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\} \subseteq K$ and for all $y \in \operatorname{conv}\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, there exists $i \in\{1,2, \ldots, n\}$ such that

$$
\Phi\left(A(y, T(y)), y, x_{i}\right) \nsubseteq-\operatorname{int} C(y)
$$

that is, $\Phi\left(w, y, x_{i}\right) \not_{\operatorname{int} C(y)} 0$, for some $w \in A(y, T(y))$.
Definition 2.2. $\Phi: \mathcal{L}(X, Y) \times K \times K \rightarrow Y$ is called an equilibrium-like function if $\Phi(u, x, y)+\Phi(u, y, x)=0$ for each $(u, x, y) \in \mathcal{L}(X, Y) \times K \times K$.

We obtain the following existence result for solutions of GSVELPs under $C(y)$-proper quasimonotonicity and for solutions of GMVELPs under $C(y)$ pseudomonotonicity.

Theorem 2.3. For any $y \in K$, let the function $\Phi: \mathcal{L}(X, Y) \times K \times K \rightarrow Y$ be $C(y)$-properly quasimonotone w.r.t. $T$ and $A$. Assume that
(i) the set-valued map $\Gamma: K \rightarrow 2^{K}$ defined by

$$
\Gamma(x)=\left\{y \in K: \Phi(w, y, x) \not \mathbb{Z}_{C(y) \backslash\{0\}} 0 \text { for some } w \in A(y, T(y))\right\}
$$

is closed valued;
(ii) there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $y \in K \backslash M$, there exists $x \in B$ such that $y \notin \Gamma(x)$.
Then Problem (III) holds (i.e., the GSVELP has a solution). Furthermore, if $\Phi$ is $C(y)$-pseudomonotone w.r.t. $T$ and $A$ and is an equilibrium-like function, then Problem (I) also holds (i.e., the GMVELP also has a solution).

Proof. We claim that $\Gamma$ is a KKM mapping on $K$. Indeed, assume $\Gamma$ is not a KKM map, then there exists $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\} \subset K, t_{i} \geq 0, i=1,2, \ldots, n$ with $\sum_{i=1}^{n} t_{i}=1$ such that $y=\sum_{i=1}^{n} t_{i} x_{i} \notin \bigcup_{i=1}^{n} \Gamma\left(x_{i}\right)$. Thus for any $w \in A(y, T(y))$

$$
\Phi\left(w, y, x_{i}\right) \leq_{C(y) \backslash\{0\}} 0, \quad i=1,2, \ldots, n
$$

that is, for any $i=1,2, \ldots, n$

$$
\Phi\left(A(y, T(y)), y, x_{i}\right) \subseteq-C(y) \backslash\{0\}
$$

which contradicts the $C(y)$-proper quasimonotonicity of $\Phi$ w.r.t. $A$ and $T$ on $K$. Hence, $\Gamma$ is a KKM mapping on $K$.

By condition (ii), $\Gamma(x)$ is a closed subset of a compact set and hence compact. Then by Theorem 1.2

$$
\bigcap_{x \in K} \Gamma(x) \neq \emptyset
$$

that is, there exists $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(\bar{w}, \bar{x}, x) \not Z_{C(\bar{x}) \backslash\{0\}} 0 \text { for some } \bar{w} \in A(\bar{x}, T(\bar{x})) .
$$

Hence the GSVELP has a solution.
Further, suppose that $\bar{x}$ is not a solution of the GMVELP. Then there exists $x \in K$ such that

$$
\Phi(w, \bar{x}, x) \leq_{C(\bar{x}) \backslash\{0\}} 0, \quad \text { for all } w \in A(\bar{x}, T(x))
$$

Since $\Phi$ is an equilibrium-like function, we have

$$
0 \leq_{C(\bar{x}) \backslash\{0\}} \Phi(w, x, \bar{x}), \quad \text { for all } w \in A(\bar{x}, T(x))
$$

that is, $\Phi(A(\bar{x}, T(x)), x, \bar{x}) \subseteq C(\bar{x}) \backslash\{0\}$. By the $C(x)$-pseudomonotonicity of $\Phi$ w.r.t. $T$ and $A$, we have

$$
\Phi(A(\bar{x}, T(\bar{x})), \bar{x}, x) \subseteq-C(\bar{x}) \backslash\{0\}
$$

and thus, $\bar{x} \in K$ is not a solution of the GSVELP, a contradiction.
Remark 2.4. In Theorem 2.3, if we put $A(x, u)=u$ for each $(x, u) \in K \times \mathcal{L}(X, Y)$ and $\Phi(w, x, y)=\langle w, \eta(y, x)\rangle$ for each $(w, x, y) \in \mathcal{L}(X, Y) \times K \times K$, where $\eta$ : $K \times K \rightarrow X$ is a function, then Theorem 2.3 reduces to Theorem 8 of [4]. In this case, if $\eta$ is affine in the first variable and $\eta(x, x)=0$ for all $x \in K$, then trivially $\Phi$ is $C$-properly quasimonotone w.r.t. $T$ and $A$. Therefore, condition 4 in Theorem 2.3 in [29] is superfluous, since it can be easily deduced from condition 3 in this theorem. Therefore, Theorem 2.3 also improves and generalizes Theorems 2.1 and 2.3 in [29] in the settings of t.v.s. and GVELPs.

Simple examples can be easily constructed to show that the $C(x)$-properly quasimonotonicity of $\Phi$ w.r.t. $T$ and $A$ does not imply that the affineness of $\Phi$ in the third variable. When $X$ and $Y$ are normed spaces, we establish the following existence result for a solution of the GWMVELP.
Theorem 2.5. For all $y \in K$, let $T: K \rightarrow 2^{\mathcal{L}(X, Y)}$ be compact-valued, and $\Phi$ be $C(y)$-properly quasimonotone w.r.t. $T$ and $A$, and $C(y)$-strictly quasimonotone w.r.t. $T$ and $A$. Assume that
(i) the set-valued mapping $W: K \rightarrow 2^{Y}$ defined by $W(x)=Y \backslash(\operatorname{int} C(x))$ is closed;
(ii) for each $z \in K, \Phi(\cdot, z, \cdot): \mathcal{L}(X, Y) \times K \rightarrow Y$ is continuous, and $A$ is continuous;
(iii) there exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \backslash M$, there exists $y \in B$ such that $y \notin \Gamma(x):=\left\{y \in K: \Phi(w, y, x) \not \mathbb{Z}_{\operatorname{int} C(y)} 0\right.$ for some $\left.w \in A(y, T(y))\right\}$.
Then Problem (II) holds (i.e., the GWMVELP has a solution).
Proof. Repeating the same argument as the first part of the proof of Theorem 2.3, we know that $\Gamma$ is a KKM mapping.

We claim that the set-valued mapping $\hat{\Gamma}$ defined by

$$
\hat{\Gamma}(x):=\left\{y \in K: 0 \not \leq_{\operatorname{int} C(y)} \Phi(w, x, y) \text { for some } w \in A(y, T(x))\right\}, \quad \forall x \in K
$$

is closed valued.
Let $\left\{y_{n}\right\}$ be a sequence in $\hat{\Gamma}(x)$ convergent to $y \in K$. Then

$$
0 \not \mathbb{Z}_{\operatorname{int} C\left(y_{n}\right)} \Phi\left(w_{n}, x, y_{n}\right) \text { for some } w_{n} \in A\left(y_{n}, T(x)\right)
$$

and therefore, there exists $u_{n} \in T(x)$ such that $w_{n}=A\left(y_{n}, u_{n}\right)$ and

$$
z_{n}=\Phi\left(A\left(y_{n}, u_{n}\right), x, y_{n}\right) \notin \operatorname{int} C\left(y_{n}\right)
$$

Then $z_{n} \in W\left(y_{n}\right)$, and hence, $\left(y_{n}, z_{n}\right) \in \operatorname{Graph}(W)$. Since $T(x)$ is compact, $\left\{u_{n}\right\}$ has a convergent subsequence in $T(x)$. Let $\left\{u_{n_{k}}\right\}$ be a subsequence of $\left\{u_{n}\right\}$ that converges to $u_{0} \in T(x)$. By the continuity of $A,\left\{A\left(y_{n_{k}}, u_{n_{k}}\right)\right\}$ converges to $A\left(y, u_{0}\right)$. Also, since $\Phi(\cdot, x, \cdot): \mathcal{L}(X, Y) \times K \rightarrow Y$ is continuous, it follows that

$$
z_{0}=\lim _{k \rightarrow \infty} z_{n_{k}}=\lim _{k \rightarrow \infty} \Phi\left(A\left(y_{n_{k}}, u_{n_{k}}\right), x, y_{n_{k}}\right)=\Phi\left(A\left(y, u_{0}\right), x, y\right)
$$

Since $\operatorname{Graph}(W)$ is closed, we have $\left(y, z_{0}\right) \in \operatorname{Graph}(W)$, and hence,

$$
0 \not \mathbb{Z}_{\operatorname{int} C(y)} \Phi(w, x, y) \quad \text { with } w=A\left(y, u_{0}\right) \in A(y, T(x))
$$

Thus, $y \in \hat{\Gamma}(x)$.
Since $\Phi$ is $C(x)$-strictly quasimonotone w.r.t. $T$ and $A$, we have $\Gamma(x) \subseteq \hat{\Gamma}(x)$ for all $x \in K$. Therefore, $\hat{\Gamma}$ is also a KKM mapping. By Theorem 1.2,

$$
\bigcap_{x \in K} \hat{\Gamma}(x) \neq \emptyset
$$

Therefore, there exists $\bar{x} \in K$ such that for each $x \in K$

$$
0 \not \mathbb{Z}_{\mathrm{int} C(\bar{x})} \Phi(w, x, \bar{x}) \quad \text { for some } w \in A(\bar{x}, T(x))
$$

Hence the GWMVELP has a solution.

Remark 2.6. When $K$ is compact, then the condition (iii) of Theorem 2.5 is trivially satisfied.

## 3. Applications of GMVELPs to VOPs

Throughout this section, unless otherwise specified, we assume that $K$ is a nonempty subset of $\mathbf{R}^{\mathbf{n}}$ and $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ is a given map. The interior of $K$ is denoted by $\operatorname{int} K$.

Let $f=\left(f_{1}, \ldots, f_{\ell}\right): \mathbf{R}^{\mathbf{n}} \rightarrow \mathbf{R}^{\ell}$ be a vector-valued function. We consider the following vector optimization problem (VOP):

Minimize $f(x)=\left(f_{1}(x), \ldots, f_{\ell}(x)\right)$ subject to $x \in K$.
A point $\bar{x} \in K$ is said to be an efficient (or Pareto) solution of the VOP if

$$
f(y){\not \mathbb{Z}_{+}^{\ell} \backslash\{0\}}^{f(\bar{x}), \quad \forall y \in K, ~}
$$

where $\mathbf{R}_{+}^{\ell}$ is the nonnegative orthant of $\mathbf{R}^{\ell}$ and 0 is the zero vector of $\mathbf{R}^{\ell}$. It is equivalent to find $\bar{x} \in K$ such that

$$
f(y)-f(\bar{x})=\left(f_{1}(y)-f_{1}(\bar{x}), \ldots, f_{\ell}(y)-f_{\ell}(\bar{x})\right) \notin-\mathbf{R}_{+}^{\ell} \backslash\{0\}, \quad \text { for all } y \in K
$$

Definition 3.1 (see [15]). Let $g: K \rightarrow \mathbf{R}$ be locally Lipschitz at a given point $x \in K$. The Clarke's generalized directional derivative of $g$ at $x \in K$ in the direction of a vector $v \in K$, denoted by $g^{\circ}(x ; v)$, is defined by

$$
g^{\circ}(x ; v)=\limsup _{y \rightarrow x, t \downarrow 0} \frac{g(y+t v)-g(y)}{t}
$$

Definition 3.2 (see [15]). Let $g: K \rightarrow \mathbf{R}$ be locally Lipschitz at a given point $x \in K$. The Clarke's generalized subdifferential of $g$ at $x \in K$, denoted by $\partial^{c} g(x)$, is defined by

$$
\partial^{c} g(x)=\left\{\xi \in \mathbf{R}^{\mathbf{n}}: g^{\circ}(x ; v) \geq\langle\xi, v\rangle, \forall v \in \mathbf{R}^{\mathbf{n}}\right\}
$$

where $\langle\cdot, \cdot\rangle$ denotes the scalar product in $\mathbf{R}^{\mathbf{n}}$.
We note that $\partial^{c} g(x)$ is a nonempty, convex and compact subset of $\mathbf{R}^{\mathbf{n}}$ if $g$ is locally Lipschitz on $K$. Next, let $f=\left(f_{1}, \ldots, f_{\ell}\right): K \rightarrow \mathbf{R}^{\ell}$ be a vector-valued function. Let each component $f_{i}$ of $f$ be locally Lipschitz on $K$ for $i \in\{1,2, \ldots, \ell\}$. Then the Clarke's generalized subdifferential of $f$ at $x \in K$ is the set

$$
\partial^{c} f(x)=\partial^{c} f_{1}(x) \times \partial^{c} f_{2}(x) \times \cdots \times \partial^{c} f_{\ell}(x)
$$

Let $\phi: \mathbf{R}^{\mathbf{n}} \times K \times K \rightarrow \mathbf{R}$ be an equilibrium-like function, that is, $\phi(u, x, y)+$ $\phi(u, y, x)=0$ for all $(u, x, y) \in \mathbf{R}^{\mathbf{n}} \times K \times K$.
Definition 3.3. Let $x$ and $y$ be points in $K \subseteq \mathbf{R}^{\mathbf{n}}$ and suppose that $g: K \rightarrow \mathbf{R}$ is locally Lipschitz on an open set containing the line segment $[x, y]$. Then $g$ is said to satisfy the Lebourg mean value condition with respect to $\phi$ if there exists a point $z \in(x, y)$ such that

$$
g(x)-g(y) \in \phi\left(\partial^{c} g(z), y, x\right)
$$

where $(x, y)$ denotes the line segment joining $x$ and $y$ excluding the end points $x$ and $y$.

A mapping $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ is said to be skew if for all $x, y \in K$,

$$
\eta(y, x)+\eta(x, y)=0 .
$$

Definition 3.4. Let $x$ be an arbitrary point of $K$. The set $K$ is said to be invex at $x$ with respect to $\eta$ if for all $y \in K$,

$$
x+t \eta(y, x) \in K, \quad \text { for all } t \in[0,1] .
$$

$K$ is said to be invex with respect to $\eta$ if $K$ is invex at every point $x \in K$ with respect to $\eta$.

Condition C. Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ be an invex set with respect to $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$. Then $\eta$ is said to satisfy Condition C with respect to $\phi$ if for all $x, y \in K$ and $t \in[0,1]$,
(a) $\eta(x, x+t \eta(y, x))=-t \eta(y, x)$ and $\phi(u, x+t \eta(y, x), x)=-t \phi(u, x, y), \forall u \in$ $\mathbf{R}^{\mathrm{n}}$;
(b) $\eta(y, x+t \eta(y, x))=(1-t) \eta(y, x)$ and $\phi(u, x+t \eta(y, x), y)=(1-t) \phi(u, x, y)$, $\forall u \in \mathbf{R}^{\mathbf{n}}$.

Obviously, if we put $\eta(y, x)=y-x$ and $\phi(u, x, y)=\langle u, \eta(y, x)\rangle$ for all $(u, x, y) \in$ $\mathbf{R}^{\mathbf{n}} \times K \times K$, then $\eta$ satisfies Condition $\mathbf{C}$ with respect to $\phi$, where $\langle\cdot, \cdot\rangle$ denotes the scalar product in $\mathbf{R}^{\mathbf{n}}$. In addition, the examples of the map $\eta$ that satisfies Condition C with respect to $\phi$ can be constructed according to [24,25]. Furthermore, we also consider the following Condition $\mathrm{C}^{\dagger}$.

Condition $\mathbf{C}^{\dagger}$. Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ be an invex set with respect to $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$. We say that the mapping $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ satisfies the Condition $\mathrm{C}^{*}$ with respect to $\phi$ if for all $x, y \in K$ and $t \in[0,1]$,
(a) $\eta(x, x+t \eta(y, x))=-\alpha(t) \eta(y, x)$ and $\phi(u, x+t \eta(y, x), x)=-\alpha(t) \phi(u, x, y)$, $\forall u \in \mathbf{R}^{\mathbf{n}}$;
(b) $\eta(y, x+\operatorname{t\eta }(y, x))=\beta(t) \eta(y, x)$ and $\phi(u, x+\operatorname{t\eta }(y, x), y)=\beta(t) \phi(u, x, y), \forall u \in$ $\mathbf{R}^{\mathrm{n}}$,
where $\alpha(t)>0, \beta(t)>0$ for all $t \in(0,1)$.
Remark 3.5. We note that if $\eta$ satisfies the Condition C with respect to $\phi$, then it satisfies the Condition $\mathrm{C}^{\dagger}$ with respect to $\phi$. However, the converse is not true in general. Simple examples can be constructed eaasily by using Examples in [18].

Definition 3.6. Let $\phi: \mathbf{R}^{\mathbf{n}} \times K \times K \rightarrow \mathbf{R}$ be an equilibrium-like function and $g: K \rightarrow \mathbf{R}$ is locally Lipschitz on $K$. Then $g$ is said to be
(a) invex with respect to $\phi$ on $K$ if $\phi(\xi, x, y) \leq g(y)-g(x)$ for all $x, y \in K$ and $\xi \in \partial^{c} g(x)$;
(b) pseudoinvex with respect to $\phi$ on $K$ if $\phi(\xi, x, y) \geq 0 \Rightarrow g(y) \geq g(x)$ for all $x, y \in K$ and $\xi \in \partial^{c} g(x)$;
(c) strictly pseudoinvex with respect to $\phi$ on $K$ if $\Phi(\xi, x, y) \geq 0 \Rightarrow g(y)>g(x)$ for all $x, y \in K$ with $x \neq y$ and $\xi \in \partial^{c} g(x)$;
(d) quasiinvex with respect to $\phi$ on $K$ if $g(y) \leq g(x) \Rightarrow \phi(\xi, x, y) \leq 0$ for all $x, y \in K$ and $\xi \in \partial^{c} g(x)$.

Definition 3.7 (see [4]). Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ be an invex set with respect to $\eta$. A function $g: K \rightarrow \mathbf{R}$ is said to be
(a) preinvex with respect to $\eta$ if
$g(x+\operatorname{t\eta }(y, x)) \leq t g(y)+(1-t) g(x), \quad$ for all $x, y \in K$ and $t \in[0,1] ;$
(b) prequasiinvex with respect to $\eta$ on $K$ if for all $x, y \in K, 0 \leq t \leq 1$,

$$
g(x+\operatorname{t\eta }(y, x)) \leq \max \{g(x), g(y)\}
$$

(c) semi-strictly prequasiinvex with respect to $\eta$ on $K$ if for all $x, y \in K, 0<$ $t<1$ with $g(x) \neq g(y)$,

$$
g(x+\operatorname{t\eta }(y, x))<\max \{g(x), g(y)\} .
$$

Definition 3.8. Let $\phi: \mathbf{R}^{\mathbf{n}} \times K \times K \rightarrow \mathbf{R}$ be an equilibrium-like function and $g: K \subseteq \mathbf{R}^{\mathbf{n}} \rightarrow \mathbf{R}$ be locally Lipschitz on $K$. Then $\partial^{c} g$ is said to be quasimonotone with respect to $\phi$ if for all $x, y \in K, \xi \in \partial^{c} g(x)$ and $\zeta \in \partial^{c} g(y)$, one has $\phi(\xi, x, y)>0$ $\Rightarrow \phi(\zeta, y, x) \leq 0$.

Let $K$ be a nonempty subset of $\mathbf{R}^{\mathbf{n}}$ and $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ be a given map. Let $f=\left(f_{1}, f_{2}, \ldots, f_{\ell}\right): \mathbf{R}^{\mathbf{n}} \rightarrow \mathbf{R}^{\ell}$ be a vector-valued function such that each $f_{i}$ is locally Lipschitz on $K$, that is, $f$ has the Clarke's generalized subdifferential on $K$. Let $\Phi: \mathcal{L}\left(\mathbf{R}^{\mathrm{n}}, \mathbf{R}^{\ell}\right) \times K \times K \rightarrow \mathbf{R}^{\ell}$ be an equilibrium-like function, that is, $\Phi(u, x, y)+\Phi(u, y, x)=0$ for all $(u, x, y) \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K$. We consider the following generalized Minty vector equilibrium-like problems (GMVELPs):
(P1) GMVELP. Find $\bar{x} \in K$ such that for each $x \in K$, there exists $\zeta=$ $\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\ell}\right) \in \partial^{c} f(x)$ satisfying

$$
\Phi(\zeta, \bar{x}, x) \leq_{\mathbf{R}_{+}^{e} \backslash\{0\}} 0 .
$$

(P2) GGMVELP. Find $\bar{x} \in K$ such that for each $x \in K$

$$
\Phi(\zeta, \bar{x}, x) \mathbb{Z}_{\mathbf{R}_{+}^{\ell} \backslash\{0\}} 0, \quad \forall \zeta \in \partial^{c} f(x),
$$

where $\zeta=\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\ell}\right) \in \partial^{c} f(x)$.
(P3) GWMVELP. Find $\bar{x} \in K$ such that for each $x \in K$, there exists $\zeta=$ $\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\ell}\right) \in \partial^{c} f(x)$ satisfying

$$
\Phi(\zeta, \bar{x}, x) \not \mathbb{Z}_{\mathrm{int} \mathbf{R}_{+}^{\ell}} 0
$$

In particular, if $\Phi(u, x, y)=\langle u, \eta(y, x)\rangle_{\ell}$ for all $(u, x, y) \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K$, then the above GMVELP, GGMVELP and GWMVELP reduce to the GMVVLIP, GGMVVLIP and GWMVVLIP considered and studied in [4]. The GGMVVLIP and GWMVVLIP are considered and studied in [19] with further applications to the VOP. The relationship between a solution of the GMVVLIP and an efficient solution of the VOP is established in [1] under the condition that each $f_{i}$ is preinvex. The existence of solutions of the GGMVVLIP is studied in [2]. When $\eta(y, x)=$ $y-x$, then the GGMVVLIP reduces to the generalized Minty vector variational inequality problem considered and studied in [4]. Of course, the GGMVVLIP is more general than the GMVVLIP as every solution of the GGMVVLIP is a solution of the GMVVLIP.

Theorem 3.9. Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ be a nonempty invex set with respect to $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ such that $\eta$ is skew and satisfies Condition $C$ with respect to each $\phi_{i}, i=1,2, \ldots, \ell$, where $\Phi(u, x, y)=\left(\phi_{1}\left(u_{1}, x, y\right), \phi_{2}\left(u_{2}, x, y\right), \ldots, \phi_{\ell}\left(u_{\ell}, x, y\right)\right)$ for all $(u, x, y) \in$ $\mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K$ with $u=\left(u_{1}, u_{2}, \ldots, u_{\ell}\right)$. For each $i \in\{1,2, \ldots, \ell\}$, let $f_{i}: K \rightarrow \mathbf{R}$ be pseudoinvex and quasiinvex with respect to $\phi_{i}$, locally Lipschitz on $K$. Suppose that for $i=1,2, \ldots, \ell$
(i) each $f_{i}$ is both prequasiinvex and semi-strictly prequasiinvex with respect to $\eta$ on $K$;
(ii) each $f_{i}$ satisfies the Lebourg mean value condition with respect to $\phi_{i}$;
(iii) each $\partial^{c} f_{i}$ is quasimonotone with respect to $\phi_{i}$.

Then we have
(a) If $\bar{x} \in K$ is a solution of the GGMVELP, then it is an efficient solution of the VOP.
(b) If $\bar{x} \in K$ is an efficient solution of the VOP, then it is a solution of the GGMVELP.

Proof. (a) Let $\bar{x}$ be a solution of the GGMVELP but not an efficient solution of the VOP. Then there exists $x_{0} \in K$ such that $f\left(x_{0}\right) \leq_{\mathbf{R}_{+}^{e} \backslash\{0\}} f(\bar{x})$; that is,

$$
\begin{equation*}
f(\bar{x})-f\left(x_{0}\right)=\left(f_{1}(\bar{x})-f_{1}\left(x_{0}\right), \ldots, f_{\ell}(\bar{x})-f_{\ell}\left(x_{0}\right)\right) \in \mathbf{R}_{+}^{\ell} \backslash\{0\} . \tag{3.1}
\end{equation*}
$$

Since $K$ is invex with respect to $\eta$, we have $x(t):=\bar{x}+t \eta\left(x_{0}, \bar{x}\right) \in K$ for all $t \in[0,1]$. Note that each $f_{i}$ is both prequasiinvex and semi-strictly prequasiinvex with respect to the same $\eta$ on $K$ for $i=1,2, \ldots, n$. Then by using prequasiinvexity, semi-strict prequasiinvexity and (3.1), we get

$$
f(\bar{x})-f(x(t)) \in \mathbf{R}_{+}^{\ell} \backslash\{0\}, \quad \text { for all } t \in(0,1),
$$

that is,

$$
\begin{equation*}
f(x(0))-f(x(t)) \in \mathbf{R}_{+}^{\ell} \backslash\{0\}, \quad \text { for all } t \in(0,1) \tag{3.2}
\end{equation*}
$$

Since each $f_{i}$ satisfies the Lebourg mean value condition with respect to $\phi_{i}$ for $i=1,2, \ldots, \ell$, there exist $t_{i} \in(0,1)$ and $\xi_{i} \in \partial^{c} f_{i}\left(x\left(t_{i}\right)\right)$ for all $i \in\{1,2, \ldots, \ell\}$ such that

$$
f_{i}(x(0))-f_{i}(x(t))=-t \phi_{i}\left(\xi_{i}, \bar{x}, x_{0}\right), \quad \text { for all } i \in\{1,2, \ldots, \ell\}
$$

By using (3.2), we obtain

$$
\begin{equation*}
\phi_{i}\left(\xi_{i}, \bar{x}, x_{0}\right) \leq 0, \quad \text { for all } i \in\{1,2, \ldots, \ell\}, \tag{3.3}
\end{equation*}
$$

and one of which becomes a strict inequality. Note that each $\phi_{i}$ is an equilibrium-like function for $i=1,2, \ldots, \ell$. So, from Condition C (a), we have

$$
\phi_{i}\left(\xi_{i}, \bar{x}, x\left(t_{i}\right)\right)=t_{i} \phi_{i}\left(\xi_{i}, \bar{x}, x_{0}\right), \quad \text { for all } i \in\{1,2, \ldots, \ell\}
$$

and hence

$$
\phi_{i}\left(\xi_{i}, \bar{x}, x\left(t_{i}\right)\right) \leq 0, \quad \text { for all } i \in\{1,2, \ldots, \ell\}
$$

and one of which becomes a strict inequality. Therefore,

$$
\begin{equation*}
\left(\phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right), \ldots, \phi_{\ell}\left(\xi_{\ell}, \bar{x}, x\left(t_{\ell}\right)\right)\right) \in-\mathbf{R}_{+}^{\ell} \backslash\{0\} \tag{3.4}
\end{equation*}
$$

Suppose that $t_{1}, t_{2}, \ldots, t_{\ell}$ are all equal. Then it follows from (3.4) that $\bar{x} \in K$ is not a solution of the GGMVELP, a contradiction to our supposition. Consider the case where $t_{1}, t_{2}, \ldots, t_{\ell}$ are all not equal.

Case 1 (i). If $t_{1}>t_{2}$, and in (3.4) the inequality is strict for $k=1$, then

$$
\phi_{1}\left(\xi_{1}, x\left(t_{1}\right), x\left(t_{2}\right)\right)=\frac{t_{2}-t_{1}}{t_{1}} \phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right)>0
$$

Indeed, by Condition C, we have

$$
\begin{aligned}
\phi_{1}\left(\xi_{1}, x\left(t_{1}\right), x\left(t_{2}\right)\right)= & \phi_{1}\left(\xi_{1}, \bar{x}+t_{1} \eta\left(x_{0}, \bar{x}\right), \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right)\right) \\
= & \phi_{1}\left(\xi_{1}, \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right)+\left(t_{1}-t_{2}\right) \eta\left(x_{0}, \bar{x}\right), \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right)\right) \\
= & \phi_{1}\left(\xi_{1}, \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right)+\frac{t_{1}-t_{2}}{1-t_{2}} \eta\left(x_{0}, \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right)\right), \bar{x}\right. \\
& \left.+t_{2} \eta\left(x_{0}, \bar{x}\right)\right) \\
= & \frac{t_{2}-t_{1}}{1-t_{2}} \phi_{1}\left(\xi_{1}, \bar{x}+t_{2} \eta\left(x_{0}, \bar{x}\right), x_{0}\right)=\left(t_{2}-t_{1}\right) \phi_{1}\left(\xi_{1}, \bar{x}, x_{0}\right)
\end{aligned}
$$

and

$$
\phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right)=-\phi_{1}\left(\xi_{1}, \bar{x}+t_{1} \eta\left(x_{0}, \bar{x}\right), \bar{x}\right)=t_{1} \phi_{1}\left(\xi_{1}, \bar{x}, x_{0}\right)
$$

Combining the above relationships, we obtain the assertion.
Note that $\partial^{c} f_{1}$ is quasimonotone with respect to $\phi_{1}$. Thus by virtue of pseudoinvexity of $f_{1}$ with respect to $\phi_{1}$, we have for all $\zeta_{1} \in \partial^{c} f_{1}\left(x\left(t_{2}\right)\right)$,

$$
\phi_{1}\left(\zeta_{1}, x\left(t_{2}\right), x\left(t_{1}\right)\right) \leq 0
$$

From Condition C, we deduce

$$
\begin{equation*}
\phi_{1}\left(\zeta_{1}, \bar{x}, x\left(t_{2}\right)\right)=\frac{t_{2}}{t_{1}-t_{2}} \phi_{1}\left(\zeta_{1}, x\left(t_{2}\right), x\left(t_{1}\right)\right) \leq 0 \tag{3.5}
\end{equation*}
$$

Therefore, from (3.4) and (3.5), for all $\zeta_{1} \in \partial^{c} f_{1}\left(x\left(t_{2}\right)\right)$ and $\xi_{2} \in \partial^{c} f_{2}\left(x\left(t_{2}\right)\right)$, we obtain

$$
\begin{equation*}
\phi_{1}\left(\zeta_{1}, \bar{x}, x\left(t_{2}\right)\right) \leq 0 \quad \text { and } \quad \phi_{2}\left(\xi_{2}, \bar{x}, x\left(t_{2}\right)\right) \leq 0 \tag{3.6}
\end{equation*}
$$

Case 1 (ii). If $t_{1}<t_{2}$ and in (3.4) the inequality is strict for $k=1$. From Condition C, we have

$$
\phi_{2}\left(\xi_{2}, x\left(t_{2}\right), x\left(t_{1}\right)\right)=\frac{t_{1}-t_{2}}{t_{2}} \phi_{2}\left(\xi_{2}, \bar{x}, x\left(t_{2}\right)\right) \geq 0
$$

The pseudoinvexity of $f_{2}$ with respect to $\phi_{2}$ implies that $f_{2}\left(x\left(t_{1}\right)\right) \geq f_{2}\left(x\left(t_{2}\right)\right)$. Therefore, by the quasiinvexity of $f_{2}$ w.r.t. $\phi_{2}$, we know that for any $\xi_{1}^{\prime} \in \partial^{c} f_{2}\left(x\left(t_{1}\right)\right)$,

$$
\phi_{2}\left(\xi_{1}^{\prime}, x\left(t_{1}\right), x\left(t_{2}\right)\right) \leq 0
$$

Thus from (3.4) and the assumption that the strict inequality holds in (3.4) for $k=1$, we have for all $\xi_{1} \in \partial^{c} f_{1}\left(x\left(t_{1}\right)\right)$ satisfies $\phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right)<0$. Therefore, for all $\xi_{1} \in \partial^{c} f_{1}\left(x\left(t_{1}\right)\right)$ and $\xi_{1}^{\prime} \in \partial^{c} f_{2}\left(x\left(t_{1}\right)\right)$, we have

$$
\phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right)<0 \quad \text { and } \quad \phi_{2}\left(\xi_{1}^{\prime}, \bar{x}, x\left(t_{1}\right)\right) \leq 0
$$

The above inequalities contradict the fact that $\bar{x}$ is a solution of the GGMVELP.

Case 2 (i). If $t_{1}<t_{2}$ and in (3.4) the inequality is strict for $k=2$, then from Condition C, we have

$$
\phi_{2}\left(\xi_{2}, x\left(t_{2}\right), x\left(t_{1}\right)\right)=\frac{t_{2}-t_{1}}{t_{2}} \phi_{2}\left(\xi_{2}, \bar{x}, x\left(t_{2}\right)\right)>0
$$

Note that $\partial^{c} f_{2}$ is quasimonotone with respect to $\phi_{2}$. Thus by virtue of the pseudoinvexity of $f_{2}$ with respect to $\phi_{2}$, for all $\xi_{1}^{\prime} \in \partial^{c} f_{2}\left(x\left(t_{1}\right)\right)$, we have

$$
\phi_{2}\left(\xi_{1}^{\prime}, x\left(t_{1}\right), x\left(t_{2}\right)\right) \leq 0
$$

From Condition C, we deduce that

$$
\begin{equation*}
\phi_{2}\left(\xi_{1}^{\prime}, x\left(t_{1}\right), x\left(t_{2}\right)\right)=\frac{t_{2}-t_{1}}{t_{1}} \phi_{2}\left(\xi_{1}^{\prime}, \bar{x}, x\left(t_{1}\right)\right) \tag{3.7}
\end{equation*}
$$

Therefore, from (3.4) and (3.7), for all $\xi_{1} \in \partial^{c} f_{1}\left(x\left(t_{1}\right)\right)$ and $\xi_{1}^{\prime} \in \partial^{c} f_{2}\left(x\left(t_{1}\right)\right)$ we obtain

$$
\begin{equation*}
\phi_{1}\left(\xi_{1}, \bar{x}, x\left(t_{1}\right)\right) \leq 0 \quad \text { and } \quad \phi_{2}\left(\xi_{1}^{\prime}, \bar{x}, x\left(t_{1}\right)\right) \leq 0 \tag{3.8}
\end{equation*}
$$

Case 2 (ii). If $t_{1}>t_{2}$ and in (3.4) the inequality is strict for $k=2$, by the similar method to that in Case 1 (i), we reach a contradiction.

Hence for $t_{1} \neq t_{2}$, let $t_{0}=\min \left\{t_{1}, t_{2}\right\}$. Then from (3.6) and (3.8), for $\gamma_{i} \in$ $\partial^{c} f_{i}\left(x\left(t_{0}\right)\right), i=1,2$ we have

$$
\phi_{i}\left(\gamma_{i}, \bar{x}, x\left(t_{0}\right)\right) \leq 0, \quad \text { for } i=1,2
$$

By continuing this process, we can find $t^{*} \in(0,1)$ such that for $\tau_{i} \in \partial^{c} f_{i}\left(x\left(t^{*}\right)\right), i=$ $1,2, \ldots, \ell$

$$
\phi_{i}\left(\tau_{i}, \bar{x}, x\left(t^{*}\right)\right) \leq 0
$$

This contradicts the fact that $\bar{x} \in K$ is a solution of the GGMVELP.
(b) Let $\bar{x}$ be an efficient solution of the VOP but not a solution of the GMVELP. Then there exists $x_{0} \in K$ such that

$$
\Phi\left(\zeta, \bar{x}, x_{0}\right)=\left(\phi_{1}\left(\zeta_{1}, \bar{x}, x_{0}\right), \ldots, \phi_{\ell}\left(\zeta_{\ell}, \bar{x}, x_{0}\right)\right) \in-\mathbf{R}_{+}^{\ell} \backslash\{0\}
$$

for all $\zeta_{i} \in \partial^{c} f_{i}\left(x_{0}\right), i=1,2, \ldots, \ell$. Since each $\phi_{i}$ is an equilibrium-like function for $i=1,2, \ldots, \ell$, we have

$$
\Phi\left(\zeta, x_{0}, \bar{x}\right)=\left(\phi_{1}\left(\zeta_{1}, x_{0}, \bar{x}\right), \ldots, \phi_{\ell}\left(\zeta_{\ell}, x_{0}, \bar{x},\right)\right) \in \mathbf{R}_{+}^{\ell} \backslash\{0\}
$$

for all $\zeta_{i} \in \partial^{c} f_{i}\left(x_{0}\right), i=1,2, \ldots, \ell$. From the pseudoinvexity of each $f_{i}$ with respect to $\phi_{i}$, it follows that

$$
f(\bar{x})-f\left(x_{0}\right) \in \mathbf{R}_{+}^{\ell} \backslash\{0\} ;
$$

that is, $f\left(x_{0}\right) \leq_{\mathbf{R}_{+}^{\ell} \backslash\{0\}} f(\bar{x})$, contradicting the fact that $\bar{x}$ is an efficient solution of the VOP.

Remark 3.10. Theorem 3.9 improves and generalizes Theorem 6 in [4] at a great extent. In Theorem 3.9, we establish the conclusion that every efficient solution of the VOP is a solution of the GMVELP for pseudoinvex functions with some extra conditions while it is proven in [4] for pseudoinvex functions with the continuity condition of $\eta$ in the second variable. Also in the proof of part (b) of Theorem 3.9, we assume that $K$ is an invex set with some other condition while it is only assumed to be invex in [4, Theorem 6].

It is worth to mention that we use the simple mean value condition for invex functions to establish the above Theorem 3.9; however, in the proof of Theorem 6 of [4], the authors used simple mean value theorem for Clarke's generalized subdifferentials. Therefore, the arguments of Theorem 3.9 and Theorem 6 in [4] are different. In addition, Theorem 3.9 also improves and extends [1, Theorem 3.1] in the settings of GGMVELPs and GMVELPs.

Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ and $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$. Let $f=\left(f_{1}, f_{2}, \ldots, f_{\ell}\right): \mathbf{R}^{\mathbf{n}} \rightarrow \mathbf{R}^{\ell}$ be a vector-valued function such that each $f_{i}$ is locally Lipschitz on $K$, that is, $f$ has the Clarke's generalized subdifferential on $K$. Let $\Phi: \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K \rightarrow \mathbf{R}^{\ell}$ be an equilibrium-like function, i.e., $\Phi(u, x, y)+\Phi(u, y, x)=0$ for all $(u, x, y) \in$ $\mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K$. Now we consider the perturbed form of generalized weak Stampacchia vector equilibrium-like problem (PGWSVELP): find $\bar{x} \in K$ for which there exists $t_{0} \in(0,1)$ such that

$$
\Phi\left(\partial^{c} f(\bar{x}+t \eta(x, \bar{x})), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}, \quad \text { for all } x \in K \text { and } t \in\left(0, t_{0}\right]
$$

It is equivalent to find $\bar{x} \in K$ for which there exists $t_{0} \in(0,1)$ such that for all $x \in K$ and $t \in\left(0, t_{0}\right]$, there exists $\xi_{i} \in \partial^{c} f_{i}(\bar{x}+t \eta(x, \bar{x})), i=1,2, \ldots, \ell$, satisfying

$$
\Phi(\xi, \bar{x}, x)=\left(\phi_{1}\left(\xi_{1}, \bar{x}, x\right), \ldots, \phi_{\ell}\left(\xi_{\ell}, \bar{x}, x\right)\right) \notin-\operatorname{int} \mathbf{R}_{+}^{\ell}
$$

where $\phi_{i}: \mathbf{R}^{\mathbf{n}} \times K \times K \rightarrow \mathbf{R}$ for each $i=1,2, \ldots, \ell$. Inspired by Gang and Liu [18], we introduce the following Condition $\mathrm{C}^{*}$.

Condition $\mathbf{C}^{*}$. Let $\Phi: \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K \rightarrow \mathbf{R}^{\ell}$. Let $K \subseteq \mathbf{R}^{\mathbf{n}}$ be an invex set with respect to $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$. We say that $\eta$ satisfies the Condition $\mathrm{C}^{*}$ with respect to $\Phi$ if for all $x, y \in K$ and $t \in[0,1]$,
(a) $\Phi(u, x+t \eta(y, x), x)=-\alpha(t) \Phi(u, x, y), \forall u \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right)$;
(b) $\Phi(u, x+\operatorname{t\eta }(y, x), y)=\beta(t) \Phi(u, x, y), \forall u \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right)$, where $\alpha(t)>0, \beta(t)>$ 0 for all $t \in(0,1)$.
The following result provides the relationship between solutions of the PGWSVELP and ones of the GWMVELP.

Theorem 3.11. Let $K$ be an invex set with respect to $\eta: K \times K \rightarrow \mathbf{R}^{\mathbf{n}}$ such that $\eta$ is skew and satisfies Condition $C^{*}$ with respect to $\Phi$. Let $\partial^{c} f$ is strictly $\mathbf{R}_{+}^{\ell}$-quasimonotone with respect to $\Phi$, that is,

$$
\Phi\left(\partial^{c} f(x), x, y\right) \subseteq \operatorname{int} \mathbf{R}_{+}^{\ell} \Rightarrow \Phi\left(\partial^{c} f(y), y, x\right) \subseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}
$$

for all $x, y \in K$. Then $\bar{x} \in K$ is a solution of the PGWSVELP if and only if it is a solution of the GWMVELP.

Proof. Let $\bar{x}$ be a solution of the PGWSVELP. Then there exists $t_{0} \in(0,1)$ such that

$$
\begin{equation*}
\Phi\left(\partial^{c} f(\bar{x}+t \eta(x, \bar{x})), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell} \tag{3.9}
\end{equation*}
$$

for all $x \in K$ and $t \in\left(0, t_{0}\right]$. By the Condition $\mathrm{C}^{*}$, we have

$$
\begin{equation*}
\Phi(u, \bar{x}+t \eta(x, \bar{x}), x)=\beta(t) \Phi(u, \bar{x}, x), \quad \forall u \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \tag{3.10}
\end{equation*}
$$

where $\beta(t)>0$ for all $t \in(0,1)$. It follows from (3.9) that

$$
\Phi\left(\partial^{c} f(\bar{x}+\operatorname{t\eta }(x, \bar{x})), \bar{x}+\operatorname{t\eta }(x, \bar{x}), x\right)=\beta(t) \Phi\left(\partial^{c} f(\bar{x}+\operatorname{t\eta }(x, \bar{x})), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}
$$

for all $x \in K$ and $t \in\left(0, t_{0}\right]$. By the strict $\mathbf{R}_{+}^{\ell}$-quasimonotonicity of $\partial^{c} f$ with respect to $\Phi$, we have

$$
\Phi\left(\partial^{c} f(x), x, \bar{x}+t \eta(x, \bar{x})\right) \nsubseteq \operatorname{int} \mathbf{R}_{+}^{\ell}
$$

Note that $\Phi: \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right) \times K \times K \rightarrow \mathbf{R}^{\ell}$ is an equilibrium-like function. Thus, by (3.10) we obtain

$$
\begin{aligned}
\beta(t) \Phi\left(\partial^{c} f(x), \bar{x}, x\right) & =\Phi\left(\partial^{c} f(x), \bar{x}+\operatorname{t\eta }(x, \bar{x}), x\right) \\
& =-\Phi\left(\partial^{c} f(x), x, \bar{x}+\operatorname{t\eta }(x, \bar{x})\right) \\
& \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}
\end{aligned}
$$

which immediately yields

$$
\Phi\left(\partial^{c} f(x), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}
$$

that is, for each $x \in K$, there exists $\zeta=\left(\zeta_{1}, \zeta_{2}, \ldots, \zeta_{\ell}\right) \in \partial^{c} f(x)$ satisfying $\Phi(\zeta, \bar{x}, x) \mathbb{Z}_{\mathrm{int} \mathbf{R}_{+}^{\ell}} 0$. Hence $\bar{x} \in K$ is a solution of the GWMVELP.

Conversely, let $\bar{x}$ be a solution of the GWMVELP. Then, for each $x \in K$, there exists $\zeta \in \partial^{c} f(x)$ satisfying $\Phi(\zeta, \bar{x}, x) \mathbb{Z}_{\mathrm{int} \mathbf{R}_{+}^{\ell}} 0$; that is, for all $x \in K$,

$$
\Phi\left(\partial^{c} f(x), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell}
$$

which immediately implies that

$$
\begin{equation*}
\Phi\left(\partial^{c} f(\bar{x}+\operatorname{t\eta }(x, \bar{x})), \bar{x}+\operatorname{t\eta }(x, \bar{x}), \bar{x}\right) \nsubseteq \operatorname{int} \mathbf{R}_{+}^{\ell} \tag{3.11}
\end{equation*}
$$

for all $x \in K$ and $t \in\left(0, t_{0}\right]$ because $\Phi$ is an equilibrium-like function.
By Condition* (a), we have

$$
\Phi(u, \bar{x}+\operatorname{t\eta }(x, \bar{x}), \bar{x})=-\alpha(t) \Phi(u, \bar{x}, x), \quad \forall u \in \mathcal{L}\left(\mathbf{R}^{\mathbf{n}}, \mathbf{R}^{\ell}\right)
$$

where $\alpha(t)>0$ for all $t \in(0,1)$. It follows from (3.11) that for all $x \in K$ and $t \in\left(0, t_{0}\right]$

$$
-\alpha(t) \Phi\left(\partial^{c} f(\bar{x}+t \eta(x, \bar{x})), \bar{x}, x\right)=\Phi\left(\partial^{c} f(\bar{x}+t \eta(x, \bar{x})), \bar{x}+t \eta(x, \bar{x}), \bar{x}\right) \nsubseteq \operatorname{int} \mathbf{R}_{+}^{\ell}
$$

Hence, for all $x \in K$ and $t \in\left(0, t_{0}\right.$ ]

$$
\Phi\left(\partial^{c} f(\bar{x}+t \eta(x, \bar{x})), \bar{x}, x\right) \nsubseteq-\operatorname{int} \mathbf{R}_{+}^{\ell} .
$$

Thus, $\bar{x}$ is a solution of the PGWSVELP.
Remark 3.12. Theorem 3.11 develops and improves Theorem 7 in [4] because we generalize and extend the GWMVVLIP and PGWSVVLIP in [4, Theorem 7] to the GWMVELP and PGWSVELP, respectively. Moreover, Theorem 3.11 generalizes and extends Proposition 2 in [20] and Theorem 3.2 in [26] for nondifferentiable and pseudoinvex functions.

## References

[1] S. Al-Homidan and Q. H. Ansari, Generalized Minty vector variational-like inequalities and vector optimization problems, J. Optim. Theory Appl. 144 (2010), 1-11.
[2] Q. H. Ansari, A note on generalized vector variational-like inequalities, Optimization 41 (1997), 197-205.
[3] Q. H. Ansari and G. M. Lee, Nonsmooth vector optimization problems and Minty vector variational inequalities, J. Optim. Theory Appl. 145 (2010), 1-16.
[4] Q. H. Ansari, M. Rezaie and J. Zafarani, Generalized vector variational-like inequalities and vector optimization, J. Glob. Optim. 53 (2012), 271-284.
[5] L. C. Ceng, G. Y. Chen, X. X. Huang and J.C. Yao, Existence theorems for generalized vector variational inequalities with pseudomonotonicity and their applications, Taiwanese J. Math. 12 (2008), 151-172.
[6] L.C. Ceng, P. Cubiotti and J. C. Yao, Existence of vector mixed variational inequalities in Banach spaces, Nonlinear Anal. 70 (2009), 1239-1256.
[7] L. C. Ceng, G. Mastroeni and J. C. Yao, Existence of solutions and variational principles for generalized vector systems, J. Optim. Theory Appl. 137 (2008), 485-495.
[8] L. C. Zeng and J. C. Yao, An existence result for generalized vector equilibrium problems without pseudomonotonicity, Appl. Math. Lett. 19 (2006), 1320-1326.
[9] L. C. Ceng and J. C. Yao, Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces, J. Glob. Optim. 36 (2006), 483-497.
[10] G. Y. Chen, Existence of solutions for a vector variational inequality: An extension of Hartman-Stampacchia theorem, J. Optim. Theory Appl. 74 (1992), 445-456.
[11] J. W. Chen, S. Li, Z. Wan and J. C. Yao, Vector variational-like inequalities with constraints: separation and alternative, J. Optim. Theory Appl. 166 (2015), 460-479.
[12] T. D. Chuong, B. Mordukhovich and J. C. Yao, Hybrid approximate proximal algorithms for efficient solutions in vector optimization, J. Nonlinear Convex Analy. 12 (2011), 257-286.
[13] D. T. Chuong and J. C. Yao, Fréchet subdifferentials of efficient point multifunctions in parametric vector optimization, J. Global Optim. 57 (2013), 1229-1243.
[14] T. D. Chuong and J. C. Yao, Isolated and proper efficiencies in semi-infinite vector optimization problems, J. Optim. Theory Appl. 162 (2014), 447-462.
[15] F. H. Clarke, Optimization and Nonsmooth Analysis, SIAM, Philadelphia, Pennsylvania, 1990.
[16] G. P. Crespi, I. Ginchev and M. Rocca, Some remarks on the Minty vector variational principle, J. Math. Anal. Appl. 345 (2008), 165-175.
[17] M. Fakhar and J. Zafarani, Generalized vector equilibrium problems for pseudomonotone bifunctions, J. Optim. Theory Appl. 126 (2005), 109-124.
[18] X. Gang and S. Liu, On Minty vector variational-like inequality, Comput. Math. Appl. 56 (2008), 311-323.
[19] F. Giannessi, Theorems of alternative, quadratic programmes and complementarity problems, in: Variational Inequalities and Complementarity Problems, R. W. Cottle, F. Giannessi, J. L. Lions (eds.), Wiley, Chichester, 1980, pp. 151-186.
[20] F. Giannessi, On Minty variational principle, in: New Trends in Mathematical Programming, F. Giannessi, S. Komloski, T. Tapcsack (eds.), Kluwer Academic Publisher, Dordrecht, Holland, 1998, pp. 93-99.
[21] G. M. Lee, On relations between vector variational inequality and vector optimization problem, in: Progress in Optimization, II: Contributions from Australasia, X. Q. Yang, A. I. Mees, M. E. Fisher, L. S. Jennings (eds.), Kluwer Academic Publisher, Dordrecht, Holland, 2000, pp. 167-179.
[22] J. W. Peng and J. C. Yao, Levitin-Polyak well-posedness of the system of weak generalized vector equilibrium problems, Fixed Point Theory 15 (2014), 529-544.
[23] M. Rezaie and J. Zafarani, Vector optimization and variational-like inequalities, J. Global Optim. 43 (2009), 47-66.
[24] X. M. Yang, X. Q. Yang and K.L. Teo, Generalizations and applications of prequasi-invex functions, J. Optim. Theory Appl. 110 (2001), 645-668.
[25] X. M. Yang, X. Q. Yang and K. L. Teo, Generalized invexity and generalized invariant monotonicity, J. Optim. Theory Appl. 117 (2003), 607-625.
[26] X. M. Yang, X. Q. Yang and K. L. Teo, Some remarks on the Minty vector variational inequality, J. Optim. Theory Appl. 121 (2004), 193-201.
[27] J. Zeng and S. J. Li, On vector variational-like inequalities and set-valued optimization problems, Optim. Lett. 5 (2011), 55-69.
[28] L. C. Zeng and J. C. Yao, Generalized Minty's lemma for generalized vector equilibrium problems, Appl. Math. Lett. 20 (2007), 32-37.
[29] Y. Zhao and Z. Xia, Existence results for systems of vector variational-like inequalities, Nonlinear Anal. Real World Appl. 8 (2007), 1370-1378.

Manuscript received 31 January 2016 revised 18 February 2016

Lu-Chuan Ceng
Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
E-mail address: zenglc@hotmail.com
Jen-Chif Yao
Center for General Education, China Medical University, Taichung 40402, Taiwan; and Research
Center for Interneural Computing, China Medical University Hospital, Taichung 40447, Taiwan.
The research of this author was supported in part by the National Science Council of Taiwan under grant MOST 102-2115-M-039-003-MY3.

E-mail address: yaojc@mail.cmu.edu.tw


[^0]:    2010 Mathematics Subject Classification. 49J40, 47H04, 47H05, 49J53.
    Key words and phrases. Generalized vector equilibrium-like problems, vector optimization problems, pseudoinvexity, quasimonotonicity, properly quasimonotonicity, Clarke's generalized subdifferential.

    This research was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002), and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).
    *Corresponding author.

