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A MINIMAX THEOREM IN INFINITE-DIMENSIONAL TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, we obtain a minimax theorem by means of which, in turn, we prove the following result:

Let E be an infinite-dimensional reflexive real Banach space, $T : E \to E$ a non-zero compact linear operator, $\varphi : E \to \mathbf{R}$ a lower semicontinuous, convex and coercive functional, $I \subset \mathbf{R}$ a compact interval, with $0 \in I$, $\psi : I \to \mathbf{R}$ a lower semicontinuous convex function.

Then, for each $r > \varphi(0)$, one has

$$\sup_{x\in X}\inf_{\lambda\in I}(\varphi(T(x)-\lambda x)+\psi(\lambda)) = r+\psi(0)\ ,$$

where

$$X = \{x \in E : \varphi(T(x)) \leq r\}\ .$$

Let E be a topological space and X a non-empty subset of E. A function $f : X \to \mathbf{R}$ is said to be relatively inf-compact (resp. relatively sequentially inf-compact) in E, provided that, for each $r \in \mathbf{R}$, the sub-level set $f^{-1}(]-\infty, r])$ is relatively compact (resp. relatively sequentially compact) in E, that is its closure in E is compact (resp. sequentially compact). A real-valued function f on a convex subset of a vector space is said to be quasi-convex if, for each $r \in \mathbf{R}$, the set $f^{-1}(]-\infty, r])$ is convex.

The aim of this very short note is to highlight the following minimax result:

Theorem 1. *Let E be a real Hausdorff topological vector space and let $X \subseteq E$ be an infinite-dimensional convex set whose interior in its closed affine hull is non-empty. Moreover, let $I \subset \mathbf{R}$ be a compact interval and $f : X \times I \to \mathbf{R}$ a function which is lower semicontinuous in $X \times I$ and quasi-convex in I. Finally, assume that there is a set $D \subset I$, dense in I, such that, for each $\lambda \in D$, the function $f(\cdot, \lambda)$ is relatively inf-compact (resp. relatively sequentially inf-compact) in E .*

Then, one has

$$\sup_{x\in X}\inf_{\lambda\in I} f(x,\lambda) = \inf_{\lambda\in I}\sup_{x\in X} f(x,\lambda)\ .$$

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Theorem 1 can be qualified as unconventional in the sense that, in most of the known minimax theorems, lower semicontinuity and inf-compactness are related to the variable with respect one takes the *inf*, while it is quasi-concavity that one generally assumes with respect to the other variable (see, for instance, [4]). It is natural to ask whether the two assumptions made on the convex set X are necessary. We start just presenting two examples related to such a question.

The first example concerns the infinite-dimensionality of X.

Example 1. Let E be a finite-dimensional normed space and let $f : E \times [0,1] \to \mathbf{R}$ be the function defined by

$$f(x,\lambda) = |\|x\| - \lambda(\|x\|^2 + 1)|$$

for all $(x,\lambda) \in E \times [0,1]$.

Of course, f is convex in $[0,1]$, inf-compact in E and, just because $\dim(E) < \infty$, continuous in $E \times [0,1]$. Further, notice that, for each $x \in E$, taking $\lambda = \frac{\|x\|}{\|x\|^2+1}$, we have $\lambda \in [0,1]$ and $f(x,\lambda) = 0$. This implies that

$$\sup_{x\in E} \inf_{\lambda\in[0,1]} f(x,\lambda) = 0 \ .$$

On the other hand, we clearly have

$$\inf_{\lambda\in[0,1]} \sup_{x\in E} f(x,\lambda) = +\infty \ .$$

So, the conclusion of Theorem 1 can fail if X if finite-dimensional.

The second example deals with the non-emptyness of the interior of X in its closed affine hull.

Example 2. Let E be an infinite-dimensional reflexive real Banach space, let X be the open unit ball in E and let $\varphi \in E^*$, with $\|\varphi\|_{E^*} = 1$. Consider the function $f : X \times [0,1] \to \mathbf{R}$ defined by

$$f(x,\lambda) = \left| \frac{1}{1-\varphi(x)} - \lambda \left(\left(\frac{1}{1-\varphi(x)} \right)^2 + 1 \right) \right|$$

for all $(x,\lambda) \in X \times [0,1]$.

Consider E equipped with the weak topology. Clearly, the affine hull of X is the whole E and, since $\dim(E) = \infty$, the interior of X in the weak topology is empty. Since, by reflexivity, X is relatively weakly compact, the function f is relatively weakly inf-compact in E. Since $\varphi \in E^*$, f is weakly continuous in $X \times [0,1]$, besides being convex in $[0,1]$. As in Example 1, it is seen that

$$\sup_{x\in X} \inf_{\lambda\in[0,1]} f(x,\lambda) = 0$$

and

$$\inf_{\lambda\in[0,1]} \sup_{x\in X} f(x,\lambda) = +\infty \ .$$

So, the conclusion of Theorem 1 can fail if the interior of X in its closed affine hull is empty.

Our proof of Theorem 1 is fully based on the joint use of three previous results of ours. We now recall them.

Theorem A ([2], Proposition 3)**.** *Let E be a real Hausdorff topological vector space, let $X \subseteq E$ be an infinite-dimensional convex set whose interior in its closed affine hull is non-empty and let $K \subseteq E$ be a relatively compact (resp. relatively sequentially compact) set.*

Then, the set $X \setminus K$ is connected.

Remark 2. Notice that, in [2], such a result was proved for the relatively compact case only. The same proof shows the validity of the result also in the relatively sequentially compact case, in view of the fact that any Hausdorff topological vector space possessing a sequentially compact neighbourhood of 0 is finite-dimensional.

Theorem B ([3], Proposition 5.3)**.** *Let X, Y be two topological spaces, with X connected, and let $F : X \to 2^Y$ be a lower semicontinuous multifunction with non-empty values. Assume the set*

$$\{x \in X : F(x) \text{ is connected}\}$$

is dense in X.

Then, the set

$$\{(x, y) \in X \times Y : y \in F(x)\}$$

is connected.

For a generic set $S \subseteq X \times I$, for each $(x, \lambda) \in X \times I$, we set

$$S_x = \{\mu \in I : (x, \mu) \in S\}$$

and

$$S^\lambda = \{u \in X : (u, \lambda) \in S\} .$$

Theorem C ([1], Theorem 2.3)**.** *Let X be a topological space, $I \subseteq \mathbf{R}$ a compact interval and $S, T \subseteq X \times I$. Assume that S is connected and $S^\lambda \neq \emptyset$ for all $\lambda \in I$, while T is closed and T_x is non-empty and connected for all $x \in X$.*

Then, one has $S \cap T \neq \emptyset$.

Proof of Theorem 1. Arguing by contradiction, assume that

$$\sup_{x \in X} \inf_{\lambda \in I} f(x, \lambda) < \inf_{\lambda \in I} \sup_{x \in X} f(x, \lambda) .$$

Fix ρ satisfying

$$\sup_{x \in X} \inf_{\lambda \in I} f(x, \lambda) < \rho < \inf_{\lambda \in I} \sup_{x \in X} f(x, \lambda) \tag{1}$$

and put

$$S = \{(x, \lambda) \in X \times I : f(x, \lambda) > \rho\}$$

and

$$T = \{(x, \lambda) \in X \times I : f(x, \lambda) \leq \rho\} .$$

Since f is lower semicontinuous, the set T is closed. Moreover, for each $x \in X$, the set T_x is non-empty by (1) and connected by the quasi-convexity of $f(x, \cdot)$. By (1) again, $S^\lambda \neq \emptyset$ for all $\lambda \in I$. Fix $\lambda \in D$. Since

$$S^\lambda = X \setminus \{x \in X : f(x, \lambda) \leq \rho\}$$

and $\{x \in X : f(x,\lambda) \leq \rho\}$ is relatively compact (resp. relatively sequentially compact) in E, in view of Theorem A, the set S^λ turns out to be connected. On the other hand, since S_x is open for all $x \in X$, the multifunction $\lambda \to S^\lambda$ is lower semicontinuous in I. At this point, we can apply Theorem B to realize that the set

$$\{(\lambda, x) \in I \times X : (x,\lambda) \in S\}$$

is connected. But such a set is clearly homeomorphic to S, and so S is connected. As a consequence, each assumption of Theorem C is satisfied, and hence we would have $S \cap T \neq \emptyset$ which is clearly false. Such a contradiction completes the proof. □

We conclude with the following application of Theorem 1. We first introduce a notation. Namely, if Y is a topological space and τ is the topology of Y, we denote by τ_s the topology on Y whose members are the sequentially open subsets of Y. Let us recall that a set $A \subseteq Y$ is said to be sequentially open if, for every sequence $\{y_n\}$ in Y converging to a point of A, there is $\nu \in \mathbf{N}$ such that $y_n \in A$ for all $n \geq \nu$. A functional φ on a real normed space is said to be coercive if $\lim_{\|x\|\to+\infty} \varphi(x) = +\infty$.

Theorem 3. *Let E be an infinite-dimensional reflexive real Banach space, $T : E \to E$ a non-zero compact linear operator, $\varphi : E \to \mathbf{R}$ a continuous, convex and coercive functional, $I \subset \mathbf{R}$ a compact interval, with $0 \in I$, $\psi : I \to \mathbf{R}$ a continuous convex function.*

Then, for each $r > \varphi(0)$, one has

$$\sup_{x\in X} \inf_{\lambda\in I} (\varphi(T(x) - \lambda x) + \psi(\lambda)) = r + \psi(0) \ ,$$

where

$$X = \{x \in E : \varphi(T(x)) \leq r\} \ .$$

Proof. Since $T(E)$ is a non-zero linear subspace, the set $\varphi(T(E))$ is unbounded above as φ is coercive. As a consequence, since $T(E)$ is connected, we have

$$\varphi(T(E)) = \left(\inf_{T(E)} \varphi, +\infty\right[.$$

From this, we clearly infer that

$$\sup_{x\in X} \varphi(T(x)) = r \ . \tag{2}$$

Next, consider the function $f : X \times I \to \mathbf{R}$ defined by

$$f(x, \lambda) = \varphi(T(x) - \lambda x) + \psi(\lambda)$$

for all $(x, \lambda) \in X \times I$. Now, denote by τ the weak topology of E. Notice that T, being linear and compact, turns out to be sequentially continuous from E with the topology τ to E with the strong topology. It is easy to check that this is equivalent to the continuity of T from E with the topology τ_s to E with the strong topology. Of course, (E, τ_s) is a Hausdorff topological vector space. Now, we are going to apply Theorem 1 to the function f considering E with the topology τ_s. First, notice that the set X is convex and its interior in τ_s is non-empty. Actually, X contains the non-empty set $T^{-1}(\varphi^{-1}(]-\infty, r[))$ which is open in τ_s, by the remarks above. Next, observe that, for each $\lambda \in \mathbf{R}$, the function $x \to T(x) - \lambda x$, being continuous and linear, is continuous from the weak to the weak topology, and so, *a* *fortiori*, from the τ_s to the weak topology. Of course, this implies that the function $(x, \lambda) \to T(x) - \lambda x$ is continuous from the product of τ_s and the topology of $\mathbf{R}$ to the weak topology. But then, since φ is weakly lower semicontinuous, the function f is lower semicontinuous in $X \times I$ with respect to the considered topology. Of course, f is convex in I. Finally, by a classical result, the spectrum of T is countable, and so the set, say D, of all $\lambda \in I$ such that $x \to T(x) - \lambda x$ is a homeomorphism between E (with the strong topology) and itself is dense in I. Fix $\lambda \in D$. Of course, since φ is coercive, for each $\rho \in \mathbf{R}$, the set

$$\{x \in E : \varphi(T(x) - \lambda x) \leq \rho\}$$

is bounded. Hence, due to the reflexivity of E, the sub-level sets of $f(\cdot, \lambda)$ are weakly compact and so, by the Eberlein-Smulyan theorem, sequentially weakly compact which is equivalent to sequentially τ_s-compact. Therefore, each assumption of Theorem 1 is satisfied and hence we have

$$\sup_{x \in X} \inf_{\lambda \in I} (\varphi(T(x) - \lambda x) + \psi(\lambda)) = \inf_{\lambda \in I} \sup_{x \in X} (\varphi(T(x) - \lambda x) + \psi(\lambda)) \ . \tag{3}$$

Now, observe that if $\lambda \in I \setminus \{0\}$, we have

$$\sup_{x \in X} \varphi(T(x) - \lambda x) = +\infty \ . \tag{4}$$

Indeed, since the τ_s-interior of X is non-empty and E is reflexive and infinite-dimensional, X turns out to be unbounded. But $T(X)$ is bounded (since φ is coercive) and so, since $\lambda \neq 0$,

$$\sup_{x \in X} \|T(x) - \lambda x\| = +\infty$$

which yields (4) by the coercivity of φ again. At this point, th conclusion follows directly from (2), (3) and (4). □

Remark 4. Notice that both infinite-dimensionality of E and compactness of T cannot be dropped in Theorem 3. The this connection, it is enough to take $T(x) = x$, $I = [0, 1]$, $\varphi(x) = \|x\|$, and $\psi = 0$.

Remark 5. At present, we do not know any example showing that the reflexivity of E cannot be dropped. However, we conjecture that such an example can be constructed in infinite-dimensional Banach spaces with the Schur property.

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