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COMMENTS ON THE KKM THEORY OF METRIC TYPE SPACES

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Abstract. Recent results in the KKM theory of abstract convex spaces and the related multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are applied to deduce generalizations of results on KKM maps in metric type spaces of Khamsi and Hussain [4].

1. Introduction

In 1996, Khamsi [3] established the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex metric spaces and applied it to obtain a Schauder type fixed point theorem. This line of study had been immediately followed by a number of authors; see [13] and the references therein. Later, it is known that most of the results in the KKM theory of hyperconvex metric spaces are simple consequences of much more general results on G-convex spaces due to ourselves; see the references of [14].

Moreover, Amini, Fakhar and Zafarani [1] first attempted to generalize certain results on the KKM theory of hyperconvex metric spaces to the corresponding ones of mere metric spaces. In fact, they introduced the class of KKM-type maps on metric spaces and established some fixed point theorems for this class. They also obtained a generalized Fan matching theorem, a generalized Fan-Browder type fixed point theorem, and a new version of Fan's best approximation theorem.

In 2006, we generalized G-convex spaces to abstract convex spaces and introduced the multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ related to generalizations of KKM maps; see [5-14]. As applications of our new KKM theory of abstract convex spaces, in our previous work [13], we deduced generalizations of recent results on KKM maps on metric spaces in Amini et al. [1] and known generalized KKM theorems on hyperconvex metric spaces.

Recently, Khamsi and Hussain [4] introduced a metric type structure in cone metric spaces and showed that classical proofs related to KKM maps proved in [17] do carry almost identically in these metric type spaces. This approach suggests that any extension of known fixed point results to cone metric spaces is redundant.

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We found that there are some similar results in [4] and [13]. Our principal aim in this article is to obtain generalized results unifying the corresponding ones in [4] and [13].

Section 2 is a preliminary for our abstract convex spaces and the map classes $\mathfrak{KC}$ and $\mathfrak{KO}$. In Section 3, we give a more general concept than metric type spaces introduced by Khamsi and Hussain [4]. Section 4 is the main section of this article devoting to fixed points and matching theorems which are unified generalizations of results in [4] and [13].

2. Abstract convex spaces and the map classes $\mathfrak{KC}$ and $\mathfrak{KO}$

For sets X and Y, a multimap (or simply a map) $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y.

For the concepts on our abstract convex spaces, KKM spaces and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$, we follow [10-12, 14] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* (simply, ACS) whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Remark. Recently the present author was informed that the triple $(E, D; \Gamma)$ was called Γ-convex spaces by Zafarani [18] in 2004.

Definition 2.2. Let $(E, D; \Gamma)$ be an ACS and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.3. The *partial KKM principle* for an ACS $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An ACS is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [15, 16]:

Theorem 2.4. *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*
(1) *$\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$]; and*
(2) *there exists a nonempty compact subset K of E such that either*
(i) *$\bigcap_{z\in M} \overline{G(z)} \subset K$ for some $M \in \langle D\rangle$; or*
(ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z\in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

The following coincidence theorem is given in [13].

Theorem 2.5. *Let $(E, D; \Gamma)$ be an ACS, Y a topological space, $S : D \multimap Y$, $T : E \multimap Y$ maps, and $F \in \mathfrak{KO}(E, Y)$ [resp. $F \in \mathfrak{KC}(E, Y)$]. Suppose that*
(1) *S is open-valued [resp. closed-valued];*
(2) *for each $y \in F(E)$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$F(E) \subset S(N)$ for some $N \in \langle D\rangle$.*
Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

From Theorem 2.5, we obtained the following Ky Fan type matching theorem in [13]:

Theorem 2.6. *Let $(E, D; \Gamma)$ be an ACS, Y a topological space, $S : D \multimap Y$, and $F \in \mathfrak{KO}(E, Y)$ [resp. $F \in \mathfrak{KC}(E, Y)$] satisfying*
(1) *S is open-valued [resp. closed-valued];*
(3) *$F(E) \subset S(N)$ for some $N \in \langle D\rangle$.*
Then there exists an $M \in \langle D\rangle$ such that $F(\Gamma_M) \cap \bigcap\{S(x) \mid x \in M\} \neq \emptyset$.

The following type of ACSs is due to ourselves [9-11]:

Definition 2.7. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

3. KKM maps in metric type spaces

We give a more general concept than metric spaces introduced by Khamsi and Hussain [4] as follows:

Definition 3.1. Let M be a set. Let $\delta : M \times M \to [0, \infty)$ be a function which satisfies
(1) $\delta(x, y) = 0$ if and only if $x = y$;

(2) $\delta(x,y) = \delta(y,x)$ for any $x, y \in M$;
(3) $\delta(x,y) \leq k(\delta(x,z) + \delta(z,y))$ for any points $x, y, z \in M$, for some constant $k > 0$.
Then the pair (M, δ) is called a *metric type space*.

In [4], examples of metric type spaces are given and, for metric type spaces, the concepts of convergence, completeness, openness, closedness, closure, topology, compactness, totally boundedness, and others are defined as usual. Moreover, the following are defined in [4]:
Let A be a nonempty bounded subset of a metric type space (M, δ). Then we define as follows:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap \{B \subset M \mid B \text{ is a closed ball in } M \text{ such that } A \subset B\}$.
(ii) $\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.
A is called *subadmissible*, if for each $N \in \langle A \rangle$, $\mathrm{ad}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

For an $x \in M$ and $\varepsilon > 0$, let

$$B(x,\varepsilon) := \{y \in M \mid \delta(x,y) \leq \varepsilon\} \quad \text{and} \quad N(x,\varepsilon) := \{y \in M \mid \delta(x,y) < \varepsilon\}.$$

It is amazing that, in metric type spaces, when we do not know whether open balls are open and closed balls are closed; see [4].
Here, we need the following extra requirement:

(∗) $\overline{N(x,\varepsilon)} \subset B(x,\varepsilon)$ for all $x \in M$ and $\varepsilon > 0$.

This condition holds for any metric spaces.
We introduce new definitions:

Definition 3.2. An ACS $(M, D; \Gamma)$ is called simply a *metric type space* if (M, δ) is a metric type space, $D \subset M$ is a nonempty subset, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map such that $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle D \rangle$. A map $G : D \multimap M$ is a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.
A Γ-convex subset of $(M \supset D; \Gamma)$ is said to be *subadmissible*.

Remark. 1. For a metric space M, $(M \supset D; \Gamma)$ is given in [3], where $\Gamma_A := \mathrm{ad}(A)$. This is a metric type space.
2. Let M be a metric space and D a nonempty set. For each $A := \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$, choose a subset $B := \{x_0, x_1, \ldots, x_n\} \in \langle M \rangle$ and define $\Gamma_A := \mathrm{ad}(B)$. Then $(M, D; \Gamma)$ is not a metric type space. For this space, the so-called generalized $gKKM$ mapping in [2] is not a KKM map.

4. Fixed point and matching theorems

Let X be a nonempty subset of a metric type space (M, δ). A map $F : X \multimap M$ is said to have the *almost fixed point property* (simply, a.f.p.p.) if for any $\varepsilon > 0$, there exists an $x_\varepsilon \in X$ such that $F(x_\varepsilon) \cap B(x_\varepsilon, \varepsilon) \neq \emptyset$.
In the following, we show that some results on metric spaces in [13] can be extended to metric type spaces.
The following is an almost fixed point property of the $\mathfrak{KC}$-maps or $\mathfrak{KO}$-maps:

Theorem 4.1. *Let $(M \supset D; \Gamma)$ be a metric type space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. Suppose that $F \in \mathfrak{KC}(X,X)$ [resp. $F \in \mathfrak{KO}(X,X)$ with the condition $(*)$] such that $F(X)$ is totally bounded. Then F has the a.f.p.p.*

Proof. Let $Y := \overline{F(X)}$. Then, for each $\varepsilon > 0$, there exists a finite subset A of $X \cap D$ such that $Y \subset \bigcup_{x\in A} \operatorname{Int} N(x,\varepsilon) \subset \bigcup_{x \in A} \overline{\operatorname{Int} N(x,\varepsilon)}$. Let us define a map $G : X \multimap Y$ by

$$G(x) := Y \setminus \operatorname{Int} N(x,\varepsilon) \quad [\text{resp. } G(x) := Y \setminus \overline{\operatorname{Int} N(x,\varepsilon)}] \quad \text{for } x \in X.$$

Then each $G(x)$ is closed [resp. open] and $\bigcap_{x\in A} G(x) = \emptyset$. Hence G is not a KKM map with respect to $F \in \mathfrak{KC}(X,Y)$ [resp. $F \in \mathfrak{KO}(X,Y)$]; that is, there exists a set $B = \{x_0, x_1, \ldots, x_m\} \in \langle X \cap D\rangle$ such that $F(\Gamma_B) \not\subset G(B)$. Therefore, there exists an $x_\varepsilon \in F(\Gamma_B)$ such that $x_\varepsilon \notin G(B)$; that is, $x_\varepsilon \in \operatorname{Int} N(x_i,\varepsilon) \subset B(x_i,\varepsilon)$ [resp. $x_\varepsilon \in \overline{\operatorname{Int} N(x_i,\varepsilon)} \subset \overline{N(x_i,\varepsilon)} \subset B(x_i,\varepsilon)$ by $(*)$] for all $i \in B$. Hence $x_i \in N(x_\varepsilon,\varepsilon)$ [resp. $x_i \in B(x_\varepsilon,\varepsilon)$] for all $i \in B$. Therefore $x_i \in B(x_\varepsilon,\varepsilon)$ for all $i \in B$ and hence $\Gamma_B = \mathrm{BI}(B) \subset B(x_\varepsilon,\varepsilon)$. Since $x_\varepsilon \in F(\Gamma_B)$, $x_\varepsilon \in F(x'_\varepsilon)$ for some $x'_\varepsilon \in \Gamma_B = \mathrm{BI}(B) \subset B(x_\varepsilon,\varepsilon)$ or equivalently $x_\varepsilon \in \Gamma_B \subset B(x'_\varepsilon,\varepsilon)$. Therefore, $x_\varepsilon \in F(x'_\varepsilon) \cap B(x'_\varepsilon,\varepsilon) \neq \emptyset$. □

Recalling that any metric type space is Hausdorff, from Theorem 4.1, it is routine to deduce the following fixed point theorem:

Theorem 4.2. *Let $(M \supset D; \Gamma)$ be a metric type space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. Then any compact closed map $F \in \mathfrak{KC}(X,X)$ [resp. $\mathfrak{KO}(X,X)$ with the condition $(*)$] has a fixed point.*

Example 4.3. We give some known particular or similar cases of Theorem 4.1. Each case also has a corresponding form of Theorem 4.2.

1. [1, Theorem 2.1] Let (M,d) be a metric space and X a nonempty subadmissible subset of M. Suppose that $F \in \mathfrak{KC}(X,X)$ such that $\overline{F(X)}$ is totally bounded.
2. [15, Theorem 13] Let (M,δ) be a cone metric space and X a nonempty subadmissible subset of M. Suppose $F \in \mathfrak{KC}(X,X)$ such that $\overline{F(X)}$ is totally bounded.
3. [4, Theorem 4.1] Let (M,δ) be a metric type space and X a nonempty subadmissible subset of M. Let $F \in \mathfrak{KC}(X,X)$ such that $\overline{F(X)}$ is totally bounded.
4. [13, Theorem 4.1] Let $(M \supset D; \Gamma)$ be a metric space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. Suppose that $F \in \mathfrak{KC}(X,X)$ [resp. $F \in \mathfrak{KO}(X,X)$] such that $F(X)$ is totally bounded.

In [8], we had the following:

Lemma 4.4. *Let $(E, D; \Gamma)$ be an ACS, Z, W two topological spaces, $F \in \mathfrak{KC}(E,Z)$ and $f : Z \to W$ a continuous function. Then $fF \in \mathfrak{KC}(E,W)$. This also holds for $\mathfrak{KO}$ instead of $\mathfrak{KC}$.*

As a consequence of Theorem 4.2 and Lemma 4.4, we obtain a Schauder type fixed point theorem for metric type spaces:

Theorem 4.5. *Let $(M \supset D; \Gamma)$ be a metric type space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. If the identity map $1_X \in \mathfrak{KC}(X, X)$ [resp. $1_X \in \mathfrak{KO}(X, X)$], then any compact continuous function $f : X \to X$ has a fixed point.*

Note that Theorems 4.1, 4.2, and 4.5 properly generalize the corresponding ones in [4]. In fact, their $\mathfrak{KC}$ cases reduce to corresponding ones in [4] whenever $M = D$, and our proofs of Theorem 4.1 is shorter than that of [4]. Moreover, the authors of [1] claimed that acyclic maps defined on G-convex spaces have the KKM property; but it was already shown by Park and Kim in 1997.

From Theorem 4.2 and Lemma 4.4, we have the following coincidence theorem:

Theorem 4.6. *Let $(M \supset D; \Gamma)$ be a metric type space, X a Γ-convex subset of M such that $X \cap D$ is dense in X, and Z a compact topological space. Let $T : Z \multimap X$ be a map having a continuous selection and $F \in \mathfrak{KC}(X, Z)$ [resp. $F \in \mathfrak{KO}(X, Z)$] a closed map. Then there exist $x_0 \in X$ and $z_0 \in Z$ such that $z_0 \in F(x_0)$ and $x_0 \in T(z_0)$.*

Proof. Let f be a continuous selection of T. Then $fF \in \mathfrak{KC}(X, X)$ by Lemma 4.4 and it is compact and u.s.c. Hence, by Theorem 4.2, it has a fixed point $x_0 \in X$ such that $x_0 \in fF(x_0)$. Therefore, $x_0 = f(z_0) \in T(z_0)$ for some $z_0 \in F(x_0)$. □

From Theorem 2.6, we deduce the following Ky Fan type matching theorem:

Theorem 4.7. *Let $(M \supset D; \Gamma)$ be a metric type space, X a Γ-convex subset of M, and Z a topological space. Let $S : X \cap D \multimap Z$ be an open-valued [resp. closed-valued] map and $F \in \mathfrak{KC}(X, Z)$ [resp. $F \in \mathfrak{KO}(X, Z)$] such that $F(X) \subset S(N)$ for some $N \in \langle X \cap D \rangle$. Then there exists an $A \in \langle X \cap D \rangle$ such that $F(\Gamma_A) \cap \bigcap\{T(x) \mid x \in A\} \neq \emptyset$.*

Note that Theorems 4.6 and 4.7 are far-reaching extensions of [1, Theorems 2.5 and 2.7], resp.

From Theorem 2.5 with $E = Y$ and $F = 1_E$, we have the following Fan-Browder type fixed point theorem:

Theorem 4.8. *Let $(E, D; \Gamma)$ be an ACS such that $1_E \in \mathfrak{KO}(E, E)$ [resp. $1_E \in \mathfrak{KC}(E, E)$], and $S : D \multimap E$, $T : E \multimap E$ maps. Suppose that*

(1) *S is open-valued [resp. closed-valued];*
(2) *for each $y \in E$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$E = S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $\bar{x} \in T(\bar{x})$.

Theorem 4.8 generalizes [1, Corollary 2.8].

In the last part of [1], the authors introduced $\mathcal{NR}$-metric spaces, which are G-convex spaces and hence KKM-spaces. For those spaces, some known results on hyperconvex metric spaces are extended in [14].

Corrections of our previous works. In this occasion, we apologize for the following incorrect statements: Theorem 4.2 [7]; Corollary 3.4 [12]; Statements (V), (VI), Theorem 4, (XVI), and (XVII) [14]. Those statements in [14] can be easily corrected.

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