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FIXED POINT THEOREMS FOR NEW MAPPINGS IN COMPLETE METRIC SPACES

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Abstract. In this paper, we introduce a broad class of mappings in a metric space which contains contractive mappings, Kannan mappings and contractively nonspreading mappings. Then we prove two fixed point theorems for the class of such mappings. One is a fixed point theorem which a fixed point is not necessarily unique. The other is a fixed point theorem which is a generalization of Bogin's fixed point theorem [2]. Using these results, we prove well-known and new fixed point theorems in a metric space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Kocourek, Takahashi and Yao [13] introduced a broad class of mappings $T : C \to C$ called *generalized hybrid* such that for some $\alpha, \beta \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping is also called (α, β)-*generalized hybrid*. We observe that the class of the mappings above covers several well-known mappings. An (α, β)-generalized hybrid mapping is nonexpansive [17] for $\alpha = 1$ and $\beta = 0$, nonspreading [15] for $\alpha = 2$ and $\beta = 1$, and hybrid [18] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Motivated by such mappings, Kawasaki and Takahashi [11] introduced the following nonlinear mapping in a Hilbert space. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \le 0$$

for all $x, y \in C$. Furthermore, Kawasaki and Takahashi [12] defined the following class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0$$

for all $x, y \in C$; see also Takahashi, Wong and Yao [19]. Kawasaki and Takahashi [12] proved fixed point theorems for such mappings in a Hilbert space. On the other

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hand, we know important mappings in a metric space. Let X be a metric space with metric d. A mapping $T : X \to X$ is said to be *contractive* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Such a mapping is also called r-*contractive*. A mapping $T : X \to X$ is said to be *Kannan* [10] if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. A mapping $T : X \to X$ is said to be *contractively nonspreading* [3, 8, 20] if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$.

In this paper, motivated by these mappings, we introduce a broad class of nonlinear mappings in a metric space which contains contractive mappings, Kannan mappings and contractively nonspreading mappings. Then we prove two fixed point theorems for the class of such mappings. One is a fixed point theorem which a fixed point is not necessarily unique. The other is a fixed point theorem which is a generalization of Bogin's fixed point theorem [2]. Using these results, we prove well-known and new fixed point theorems in a metric space.

2. Fixed point theorems in metric spaces

In this section, we first prove a fixed point theorem in a metric space which a fixed point is not necessarily unique.

Theorem 2.1. *Let (X, d) be a complete metric space and let T be a mapping of X into itself. Suppose that there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that*

$$\alpha d(Tx, Ty) + \beta d(x,Ty) + \gamma d(y, Tx) + \delta d(x, y) + \varepsilon d(x, Tx) + \zeta d(y, Ty) \leq 0 \tag{2.1}$$

for all $x, y \in X$, where $\gamma \leq \beta \leq 0$, $\alpha + \beta + \gamma + \delta + \varepsilon + \zeta > 0$ and $\gamma + \delta + \varepsilon \leq 0$. Then

(i) *T has a fixed point in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to a fixed point of T.*

In addition, if $\alpha + \beta + \gamma + \delta > 0$, then a fixed point of T in X is unique.

Proof. Replacing x by $T^n x$ and y by $T^{n+1}x$ in (2.1), we have that

$$\alpha d(T^{n+1}x, T^{n+2}x) + \beta d(T^n x, T^{n+2}x) + \gamma d(T^{n+1}x, T^{n+1}x) + \delta d(T^n x, T^{n+1}x) + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(T^{n+1}x, T^{n+2}x) \leq 0 \tag{2.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. From $d(T^n x, T^{n+2}x) \leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x)$ and $\beta \leq 0$, we have that

$$\beta d(T^n x, T^{n+2}x) \geq \beta d(T^n x, T^{n+1}x) + \beta d(T^{n+1}x, T^{n+2}x). \tag{2.3}$$

From (2.2) and (2.3) we have that

$$(\alpha + \beta + \zeta)d(T^{n+1}x, T^{n+2}x) + (\beta + \delta + \varepsilon)d(T^n x, T^{n+1}x) \leq 0 \tag{2.4}$$

and hence from $\gamma \leq \beta$ and (2.4) that

$$(\alpha + \beta + \zeta)d(T^{n+1}x, T^{n+2}x) + (\gamma + \delta + \varepsilon)d(T^n x, T^{n+1}x) \leq 0. \tag{2.5}$$

From $\alpha+\beta+\gamma+\delta+\varepsilon+\zeta > 0$, we have that $\alpha+\beta+\zeta > -(\gamma+\delta+\varepsilon)$. Furthermore, from $\gamma + \delta + \varepsilon \leq 0$, we obtain that

$$\alpha + \beta + \zeta > -(\gamma + \delta + \varepsilon) \geq 0. \tag{2.6}$$

Then we have from (2.5) and (2.6) that

$$d(T^{n+1}x, T^{n+2}x) \leq \frac{-(\gamma + \delta + \varepsilon)}{\alpha + \beta + \zeta} d(T^n x, T^{n+1}x); \tag{2.7}$$

$$0 \leq \frac{-(\gamma + \delta + \varepsilon)}{\alpha + \beta + \zeta} < 1. \tag{2.8}$$

Putting $\lambda = \frac{-(\gamma+\delta+\varepsilon)}{\alpha+\beta+\zeta}$ in (2.8), we have from (2.7) that for any $m, n \in \mathbb{N}$ with $m \geq n$,

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^m x) \\ &\leq \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &\leq \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \lambda^{n+2} d(x, Tx) + \cdots \\ &= d(x, Tx)\lambda^n (1 + \lambda + \lambda^2 + \cdots) \\ &= d(x, Tx)\frac{\lambda^n}{1 - \lambda}. \end{aligned}$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\{T^n x\}$ converges. Let $T^n x \to u$. We also have from (2.1) that

$$\begin{aligned} &\alpha d(T^{n+1}x, Tu) + \beta d(T^n x, Tu) + \gamma d(T^{n+1}x, u) \\ &\quad + \delta d(T^n x, u) + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(u, Tu) \leq 0. \end{aligned} \tag{2.9}$$

Since $T^n x \to u$, we have from (2.9) that

$$\begin{aligned} &\alpha d(u, Tu) + \beta d(u, Tu) + \gamma d(u, u) \\ &\quad + \delta d(u, u) + \varepsilon d(u, u) + \zeta d(u, Tu) \leq 0 \end{aligned} \tag{2.10}$$

and hence from (2.10) that

$$(\alpha + \beta + \zeta)d(u, Tu) \leq 0.$$

From $\alpha + \beta + \zeta > 0$, we have that $d(u, Tu) \leq 0$ and hence $Tu = u$.

In addition, suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\begin{aligned} &\alpha d(Tp_1, Tp_2) + \beta d(p_1, Tp_2) + \gamma d(Tp_1, p_2) + \delta d(p_1, p_2) \\ &\quad + \varepsilon d(p_1, Tp_1) + \zeta d(p_2, Tp_2) \leq 0 \end{aligned}$$

and hence $(\alpha+\beta+\gamma+\delta)d(p_1, p_2) \leq 0$. We have from $\alpha+\beta+\gamma+\delta > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. □

Using Theorem 2.1, we have the following fixed point theorem for contractively generalized hybrid mappings in a complete metric space.

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be an (a,b,r)-contractively generalized hybrid mapping, i.e., there exist $a, b \in \mathbb{R}$ and $r \in [0, 1)$ such that*

$$ad(Tx, Ty) + (1 - a)d(x, Ty) \leq r\{bd(Tx, y) + (1 - b)d(x, y)\}$$

for all $x, y \in X$. Suppose that $1 \leq a \leq 1 + rb$. Then the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u.*

Proof. Since $T : X \to X$ is an (a,b,r)-contractively generalized hybrid mapping, we have that

$$ad(Tx, Ty) + (1 - a)d(x, Ty) - rbd(Tx, y) - r(1 - b)d(x, y) \leq 0$$

for all $x, y \in X$. Since $1 \leq a \leq 1 + rb$ and $0 \leq r < 1$, we have that

$$\beta = 1 - a \leq 0;$$

$$\gamma = -rb \leq 1 - a = \beta;$$

$$\alpha + \beta + \gamma + \delta + \varepsilon + \zeta = a + (1 - a) - rb - r(1 - b) + 0 + 0 = 1 - r > 0;$$

$$\gamma + \delta + \varepsilon = -rb - r(1 - b) + 0 = -r \leq 0;$$

$$\alpha + \beta + \gamma + \delta = 1 - r > 0$$

in Theorem 2.1. Therefore, we have the desired result from Theorem 2.1. □

Theorem 2.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contractive mapping, i.e., there exists a real number r with $0 \leq r < 1$ such that*

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.*

Proof. Putting $\alpha = 1$, $\beta = \gamma = 0$, $\delta = -r$ and $\varepsilon = \zeta = 0$ in Theorem 2.1, we have that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Furthermore, we have that $\gamma = \beta \leq 0$,

$$\alpha + \beta + \gamma + \delta + \varepsilon + \zeta = \alpha + \beta + \gamma + \delta = 1 - r > 0$$

and $\gamma + \delta + \varepsilon = -r \leq 0$. From Theorem 2.1, we have the desired result. □

Theorem 2.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be contractively nonspreading, i.e., there exists a real number γ with $0 \leq r < \frac{1}{2}$ such that*

$$d(Tx, Ty) \leq r\{d(Tx, y) + d(Ty, x)\}$$

for all $x, y \in X$. Then the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.*

Proof. Putting $\alpha = 1$, $\beta = -r$, $\gamma = -r$ and $\delta = \varepsilon = \zeta = 0$ in Theorem 2.1, we have that

$$d(Tx, Ty) \leq r\{d(Tx, y) + d(Ty, x)\}$$

for all $x, y \in X$. Furthermore, we have that $\gamma = \beta = -r \leq 0$,

$$\alpha + \beta + \gamma + \delta + \varepsilon + \zeta = \alpha + \beta + \gamma + \delta = 1 - 2r > 0$$

and $\gamma + \delta + \varepsilon = -r \leq 0$. From Theorem 2.1, we have the desired result. □

Theorem 2.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be contractively hybrid, i.e., there exists a real number r with $0 \leq r < \frac{1}{3}$ and*

$$d(Tx, Ty) \leq r\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$. Then the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.*

Proof. Putting $\alpha = 1$, $\beta = \gamma = \delta = -r$ and $\varepsilon = \zeta = 0$ in Theorem 2.1, we have that

$$d(Tx, Ty) \leq r\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$. Furthermore, we have that $\gamma = \beta = -r \leq 0$,

$$\alpha + \beta + \gamma + \delta + \varepsilon + \zeta = \alpha + \beta + \gamma + \delta = 1 - 3r > 0$$

and $\gamma + \delta + \varepsilon = -2r \leq 0$. From Theorem 2.1, we have the desired result. □

Next, we prove a fixed point theorem in a metric space which is a generalization of Bogin's fixed point theorem [2].

Theorem 2.6. *Let (X, d) be a complete metric space and let T be a mapping of X into itself. Suppose that there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that*

$$\begin{aligned} \alpha d(Tx, Ty) + \beta d(x, Ty) + \gamma d(y, Tx) \\ + \delta d(x, y) + \varepsilon d(x, Tx) + \zeta d(y, Ty) \leq 0 \end{aligned} \tag{2.11}$$

for all $x, y \in X$, where

$$\gamma \leq \beta < 0,\ \delta \leq 0,\ \alpha + \beta + \gamma + \delta > 0,\ \alpha + \beta + \gamma + \delta + \varepsilon + \zeta \geq 0 \text{ and } \varepsilon = r\zeta$$

for some $r \in \mathbb{R}$ with $1 \leq r$. Then the following hold:

(i) *T has a unique fixed point u in X;*
(ii) *for every $z \in X$, the sequence $\{T^n z\}$ converges to a fixed point u of T.*

Proof. Replacing x by $T^n x$ and y by $T^{n+1} x$ in (2.11), we have that

$$\begin{aligned} \alpha d(T^{n+1}x, T^{n+2}x) + \beta d(T^n x, T^{n+2}x) + \gamma d(T^{n+1}x, T^{n+1}x) \\ + \delta d(T^n x, T^{n+1}x) + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(T^{n+1}x, T^{n+2}x) \leq 0 \end{aligned} \tag{2.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. From $d(T^n x, T^{n+2}x) \leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x)$ and $\beta < 0$, we have that

$$\beta d(T^n x, T^{n+2}x) \geq \beta d(T^n x, T^{n+1}x) + \beta d(T^{n+1}x, T^{n+2}x). \tag{2.13}$$

From (2.12) and (2.13) we have that

$$(\alpha + \beta + \zeta) d(T^{n+1}x, T^{n+2}x) + (\beta + \delta + \varepsilon) d(T^n x, T^{n+1}x) \leq 0. \tag{2.14}$$

We have from $\gamma \le \beta$ and (2.14) that

$$(\alpha+\beta+\zeta)d(T^{n+1}x, T^{n+2}x) + (\gamma+\delta+\varepsilon)d(T^n x, T^{n+1}x) \le 0. \tag{2.15}$$

Suppose that $\zeta \ge 0$. Then we have from (2.15) and $\varepsilon = r\zeta$ that

$$(\alpha+\beta)d(T^{n+1}x, T^{n+2}x) + (\gamma+\delta)d(T^n x, T^{n+1}x) \le 0. \tag{2.16}$$

From $\alpha+\beta+\gamma+\delta > 0$, we have that $\alpha+\beta > -(\gamma+\delta)$. Furthermore, from $\gamma < 0$ and $\delta \le 0$, we have that $\alpha+\beta > -(\gamma+\delta) > 0$. Then we have from (2.16) that

$$d(T^{n+1}x, T^{n+2}x) \le \frac{-(\gamma+\delta)}{\alpha+\beta} d(T^n x, T^{n+1}x); \tag{2.17}$$

$$0 < \frac{-(\gamma+\delta)}{\alpha+\beta} < 1. \tag{2.18}$$

Putting $\lambda = \frac{-(\gamma+\delta)}{\alpha+\beta}$ in (2.18), we have from (2.20) that for any $m, n \in \mathbb{N}$ with $m \ge n$,

$$\begin{aligned} d(T^n x, T^m x) &\le d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^m x) \\ &\le \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &\le \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \lambda^{n+2} d(x, Tx) + \cdots \\ &= d(x, Tx)\lambda^n (1 + \lambda + \lambda^2 + \cdots) \\ &= d(x, Tx)\frac{\lambda^n}{1-\lambda}. \end{aligned}$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\{T^n x\}$ converges. Let $T^n x \to u$. We also have from (2.11) that

$$\begin{aligned} &\alpha d(T^{n+1}x, Tu) + \beta d(T^n x, Tu) + \gamma d(u, T^{n+1}x) \\ &\quad + \delta d(T^n x, u) + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(u, Tu) \le 0. \end{aligned}$$

Since $T^n x \to u$, we have that

$$\begin{aligned} &\alpha d(u, Tu) + \beta d(u, Tu) + \gamma d(u, u) \\ &\quad + \delta d(u, u) + \varepsilon d(u, u) + \zeta d(u, Tu) \le 0 \end{aligned}$$

and hence

$$(\alpha+\beta+\zeta)d(u, Tu) \le 0.$$

From $\alpha+\beta+\zeta > 0$, we have that $d(u, Tu) \le 0$ and hence $Tu = u$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\begin{aligned} &\alpha d(Tp_1, Tp_2) + \beta d(p_1, Tp_2) + \gamma d(Tp_1, p_2) + \delta d(p_1, p_2) \\ &\quad + \varepsilon d(p_1, Tp_1) + \zeta d(p_2, Tp_2) \le 0 \end{aligned}$$

and hence $(\alpha+\beta+\gamma+\delta)d(p_1, p_2) \le 0$. We have from $\alpha+\beta+\gamma+\delta > 0$ that $p_1 = p_2$. Therefore, a fixed point of T is unique.

Suppose that $\zeta < 0$. Then from $\varepsilon = r\zeta$ and $1 \le r$ we have $\zeta \ge r\zeta = \varepsilon$. From (2.15), we know that

$$(\alpha+\beta+\zeta)d(T^{n+1}x, T^{n+2}x) + (\gamma+\delta+\varepsilon)d(T^n x, T^{n+1}x) \le 0.$$

Since $\alpha+\beta+\gamma+\delta+\varepsilon+\zeta \geq 0$ and $\gamma+\delta+\varepsilon<0$, we obtain that

$$\alpha+\beta+\zeta \geq -(\gamma+\delta+\varepsilon)>0. \tag{2.19}$$

Then we have from (2.19) that

$$d(T^{n+1}x, T^{n+2}x) \leq \frac{-(\gamma+\delta+\varepsilon)}{\alpha+\beta+\zeta} d(T^n x, T^{n+1}x); \tag{2.20}$$

$$0<\frac{-(\gamma+\delta+\varepsilon)}{\alpha+\beta+\zeta} \leq 1. \tag{2.21}$$

Putting $\lambda=\frac{-(\gamma+\delta+\varepsilon)}{\alpha+\beta+\zeta}$ in (2.21), we have from (2.20) that

$$d(T^{n+1}x, T^{n+2}x) \leq \lambda d(T^n x, T^{n+1}x) \leq d(T^n x, T^{n+1}x) \tag{2.22}$$

and hence $\{d(T^n x, T^{n+1}x)\}$ is a decreasing sequence. We have from (2.11) that

$$\begin{aligned} &\alpha d(T^{n+1}x, T^{n+3}x)+\beta d(T^n x, T^{n+3}x)+\gamma d(T^{n+1}x, T^{n+2}x) \\ &\quad +\delta d(T^n x, T^{n+2}x)+\varepsilon d(T^n x, T^{n+1}x)+\zeta d(T^{n+2}x, T^{n+3}x) \leq 0. \end{aligned}$$

Since $\delta \leq 0$ and $\beta<0$, we have that

$$\begin{aligned} &\alpha d(T^{n+1}x, T^{n+3}x)+\beta d(T^n x, T^{n+1}x)+\beta d(T^{n+1}x, T^{n+3}x) \\ &\quad +\gamma d(T^{n+1}x, T^{n+2}x)+\delta d(T^n x, T^{n+1}x)+\delta d(T^{n+1}x, T^{n+2}x) \\ &\quad +\varepsilon d(T^n x, T^{n+1}x)+\zeta d(T^{n+2}x, T^{n+3}x) \leq 0. \end{aligned}$$

Using (2.22), we have that

$$(\alpha+\beta) d(T^{n+1}x, T^{n+3}x)+(\beta+\gamma+2\delta+\varepsilon+\zeta) d(T^n x, T^{n+1}x) \leq 0$$

and hence from $\gamma \leq \beta$ and $\varepsilon \leq \zeta$

$$(\alpha+\beta) d(T^{n+1}x, T^{n+3}x)+(2\gamma+2\delta+2\varepsilon) d(T^n x, T^{n+1}x) \leq 0.$$

Since $\alpha+\beta>\alpha+\beta+\zeta \geq -(\gamma+\delta+\varepsilon)>0$, we have that

$$d(T^{n+1}x, T^{n+3}x) \leq \frac{-2(\gamma+\delta+\varepsilon)}{\alpha+\beta} d(T^n x, T^{n+1}x);$$

$$0<\frac{-2(\gamma+\delta+\varepsilon)}{\alpha+\beta}<2.$$

We also have from (2.11) that

$$\begin{aligned} &\alpha d(T^{n+2}x, T^{n+3}x)+\beta d(T^{n+1}x, T^{n+3}x)+\gamma d(T^{n+2}x, T^{n+2}x) \\ &\quad +\delta d(T^{n+1}x, T^{n+2}x)+\varepsilon d(T^{n+1}x, T^{n+2}x)+\zeta d(T^{n+2}x, T^{n+3}x) \leq 0 \end{aligned}$$

and hence

$$\begin{aligned} &\alpha d(T^{n+2}x, T^{n+3}x)+\beta d(T^{n+1}x, T^{n+3}x) \\ &\quad +\delta d(T^{n+1}x, T^{n+2}x)+\varepsilon d(T^{n+1}x, T^{n+2}x)+\zeta d(T^{n+2}x, T^{n+3}x) \leq 0. \end{aligned}$$

Putting $m=\frac{-2(\gamma+\delta+\varepsilon)}{\alpha+\beta}$, we have that $0<m<2$ and

$$\begin{aligned} &\alpha d(T^{n+2}x, T^{n+3}x)+\beta m d(T^n x, T^{n+1}x) \\ &\qquad +\delta d(T^n x, T^{n+1}x)+(\varepsilon+\zeta) d(T^n x, T^{n+1}x) \leq 0. \end{aligned}$$

On the other hand, since

$$\alpha \geq -(\beta + \gamma + \delta + \varepsilon + \zeta) \geq -(2\beta + \delta + \varepsilon + \zeta) > -(\beta m + \delta + \varepsilon + \zeta) > 0,$$

we have that

$$d(T^{n+2}x, T^{n+3}x) \leq \frac{-(\beta m + \delta + \varepsilon + \zeta)}{\alpha} d(T^n x, T^{n+1}x);$$

$$0 < \frac{-(\beta m + \delta + \varepsilon + \zeta)}{\alpha} < 1.$$

Putting $k = \frac{-(\beta m+\delta+\varepsilon+\zeta)}{\alpha}$, we have that $0 < k < 1$ and

$$d(T^{n+2}x, T^{n+3}x) \leq k d(T^n x, T^{n+1}x).$$

Therefore, for any even integer n, we have that

$$d(T^n x, T^{n+1}x) \leq k^{\frac{n}{2}} d(x, Tx) \leq k^{\frac{n-1}{2}} d(x, Tx).$$

For any odd integer n, we have that

$$d(T^n x, T^{n+1}x) \leq k^{\frac{n-1}{2}} d(Tx, T^2 x) \leq k^{\frac{n-1}{2}} d(x, Tx).$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since X is complete, $\{T^n x\}$ converges. Let $T^n x \to u$. We have from (2.11) that

$$\alpha d(T^{n+1}x, Tu) + \beta d(T^n x, Tu) + \gamma d(T^{n+1}x, u) + \delta d(T^n x, u) + \varepsilon d(T^n x, T^{n+1}x) + \zeta d(u, Tu) \leq 0.$$

Since $T^n x \to u$, we have that

$$\alpha d(u, Tu) + \beta d(u, Tu) + \gamma d(u, u) + \delta d(u, u) + \varepsilon d(u, u) + \zeta d(u, Tu) \leq 0$$

and hence

$$(\alpha + \beta + \zeta) d(u, Tu) \leq 0.$$

From $\alpha + \beta + \zeta > 0$, we have that $d(u, Tu) \leq 0$ and hence $Tu = u$. Let p_1 and p_2 be fixed points of T. Then we have that

$$\alpha d(Tp_1, Tp_2) + \beta d(p_1, Tp_2) + \gamma d(Tp_1, p_2) + \delta d(p_1, p_2) + \varepsilon d(p_1, Tp_1) + \zeta d(p_2, Tp_2) \leq 0$$

and hence $(\alpha+\beta+\gamma+\delta) d(p_1, p_2) \leq 0$. We have from $\alpha+\beta+\gamma+\delta > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. □

Using Theorem 2.6, we obtain the following fixed point theorem which was proved by Bogin [2].

Theorem 2.7 ([2]). *Let (X, d) be a complete metric space and let T be a mapping of X into itself. Suppose that there exist $a, b, c \in \mathbb{R}$ such that*

$$d(Tx, Ty) \leq a d(x, y) + b\big(d(x, Tx) + d(y, Ty)\big) + c\big(d(x, Ty) + d(y, Tx)\big) \tag{2.23}$$

for all $x, y \in X$, where

$$a \geq 0,\ b > 0,\ c > 0 \text{ and } a + 2b + 2c = 1.$$

Then T has a unique fixed point u in X.

Proof. We have from (2.23) that

$$d(Tx,Ty) - c\big(d(x,Ty) + d(y,Tx)\big) - ad(x,y) - b\big(d(x,Tx) + d(y,Ty)\big) \le 0$$

for all $x, y \in X$. Putting $\alpha = 1$, $\beta = \gamma = -c$, $\delta = -a$ and $\varepsilon = \zeta = -b$ in Theorem 2.6, we have that $\beta = \gamma = -c < 0$, and $\delta = -a \le 0$. Furthermore, we have that

$$\alpha + \beta + \gamma + \delta = 1 - 2c - a = 2b > 0, \ \alpha + \beta + \gamma + \delta + \varepsilon + \zeta = 1 - 2c - a - 2b = 0$$

and $\varepsilon = \zeta$. Therefore, from Theorem 2.6, we have the desired result. □

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