Linear and Nonlinear Analysis
Volume 2, Number 1, 2016, 17–28

Yokohama Publishers
ISSN 2188-8167
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RETRACTION FROM A UNIT BALL ONTO ITS SPHERICAL CUP

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Abstract. The Lipschitz constant of an optimal retraction from a unit ball in a Hilbert space onto its spherical cup is reinvestigated to obtain a precise formula for a finite-dimensional Hilbert space, and an improved upper bound for an infinite-dimensional Hilbert space.

1. Introduction and preliminaries

It is generally known that, in an infinite-dimensional normed space, there is a retraction from a unit ball onto its boundary. However, the existence of the lipschitzian version of such a retraction is far from trivial, but was finally accomplished by Nowak [6], and Benyamini and Sternfeld [2]. Since then, the quest for the least possible Lipschitz constant of such a retraction became interesting. To be pricise, for a given normed space X, let B_X and S_X denote the unit ball centered at the origin and its sphere (boundary), respectively. The so-called optimal retraction constant for X, denoted by $k_0(X)$, is defined to be

$$k_0(X) := \inf\{k : \text{there exists a } k\text{-lipschitzian retraction from } \mathrm{B}_X \text{onto } \mathrm{S}_X\}.$$

Although it has been more than thirty years after the birth of this problem, the exact value of $k_0(X)$ is still unknown for a Banach space. Only approximations for some Banach spaces are found; for example, $k_0(\ell_1) \in [4, 8]$, $k_0(C[0, 1]) \in [3, 14.93]$, $k_0(BC(\mathbb{R})) \in [3, 6.83]$, and when H is a Hilbert space, $k_0(H) \in (4.5, 28.99]$ (see [1], [5], [7] and [3]). Until recently, Chaoha, Goebel, and Termwuttipong [4] studied this problem in a Hilbert space by considering only a certain part of the sphere, namely the spherical cup, as the image of the retraction. This leads to a new constant $\kappa(t)$ defined as follows :

Let $(H, \langle\cdot,\cdot\rangle)$ be a (real) Hilbert space, $\mathrm{B} = \mathrm{B}_H$, $\mathrm{S} = \mathrm{S}_H$, $\mathrm{e} \in \mathrm{S}$ and $E = \operatorname{span}\{\mathrm{e}\}^\perp$ the orthogonal complement of e. For each $t \in [-1, 1]$,

- the parallel hyperplane is $E_t := E + t\mathrm{e}$;
- the parallel ball section is $\mathrm{B}_t := \mathrm{B} \cap E_t$;
- the lense cut by E_t is $D_t := \{x \in \mathrm{B} : \langle x, \mathrm{e}\rangle \geq t\}$;
- the spherical cup cut by E_t is $\mathrm{S}_t := D_t \cap \mathrm{S}$.

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Lipschitz retraction, optimal retractions on sphere.

Then, $S_{-1} = S$, and for $t \in (-1, 1]$, S_t is always a retract of B. Define $\kappa : [-1, 1] \to \mathbb{R}$ by

$$\kappa(t) = \inf\{k : \text{there exists a } k\text{-lipschitzian retraction from } B \text{ onto } S_t\}.$$

Following from [4], we have

- $\kappa(1) = 0$;
- if $\dim H < \infty$, $\kappa(-1) = \infty$, and for $-1 < t < 1$, $\kappa(t) \geq \frac{\arccos t}{\sqrt{1-t^2}}$;
- if $\dim H = \infty$, $\kappa(-1) = k_0(H)$, and for $-1 < t < 1$,

$$\kappa(t) \leq \min\left\{\frac{2}{1+t}, (1 + k_0(H))k_0(H)\right\}.$$

Moreover, the following question is still open (see [4]) :

What is the precise formula for $\kappa(t)$ in both cases $\dim H < \infty$ and $\dim H = \infty$?

Notice that, for an infinite-dimensional Hilbert space H, the last inequality above amounts to saying that $(1 + k_0(H))k_0(H)$ is an upper bound of $\kappa(t)$ as $t \to -1^+$. Together with Observation 3.11 [4], we obtain the inequality

$$1 \leq \frac{\kappa(t)}{k_0(H)} \leq 1 + k_0(H),$$

as $t \to -1^+$, which is equivalent to

$$\frac{1}{2}(\sqrt{1 + 4\kappa(t)} - 1) \leq k_0(H) \leq \kappa(t).$$

This certainly gives an approximation of $k_0(H)$ in terms of $\kappa(t)$ when t is closed enough to -1. Therefore, a natural way to improve such an approximation is to consider the upper bound of $\frac{\kappa(t)}{k_0(H)}$ which is currently known to be $1 + k_0(H) \in (5.5, 29.99]$.

In this work, we will answer the open problem mentioned above for a finite-dimensional Hilbert space by giving a concrete construction of an $(\frac{\arccos t}{\sqrt{1-t^2}})$-lipschitzian retraction, and give a sharper upper bound of $\kappa(t)$ for an infinite-dimensional Hilbert space that leads to a better approximation of $k_0(H)$ in term of $\kappa(t)$ when t is closed enough to -1.

Throughout this work, all retractions are assumed to be lipschitzian. Denote by P_A the nearest point projection onto A, which is nonexpansive if A is convex, and by $\nu(x)$ the unit vector $\frac{x}{\|x\|}$ for every $x \in H - \{0\}$ and $\nu(0) = 0$.

2. Roof maps

We first introduce the notion of roof maps which is essential for our construction in the next section. Let H be the Euclidean plane $\mathbb{R}^2$ throughout this section, and $e = (0, 1)$. Denote by $\triangle PQR$ the *isosceles triangle* in $\mathbb{R}^2$ whose base is the segment QR and legs are segments PQ and PR (see Figure 1(A)), and by $\blacktriangle PQR$ the closed region bounded by $\triangle PQR$ (see Figure 1(B)).

Definition. A selfmap $r : \blacktriangle PQR \to PQ \cup PR$ is called a *roof map* if

$$r = (P_{QR}|_{PQ\cup PR})^{-1} \circ P_{QR}$$

(see Figure 1(C)). Notice that $r(\blacktriangle PQR) = PQ \cup PR$; i.e., r is surjective onto legs of $\blacktriangle PQR$.

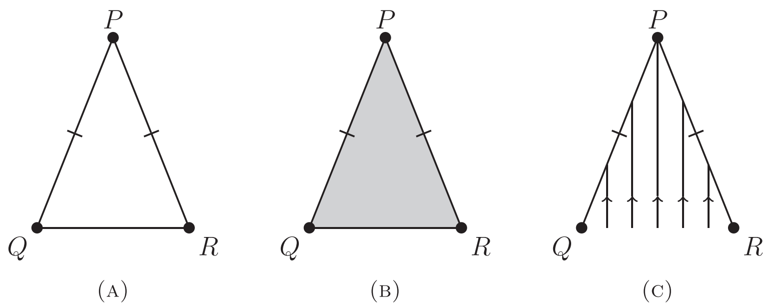

FIGURE 1. $\triangle PQR$, $\blacktriangle PQR$ and the roof map.

Properties 2.1. *Let r be the roof map defined on $\blacktriangle PQR$. Then*

(i) *$r|_{QR}$ is both injective and expansive; i.e., for each $x, y \in QR$, $\|r(x) - r(y)\| \geq \|x - y\|$;*
(ii) *for each $x \in \blacktriangle PQR$, $r(x) = r \circ P_{QR}(x)$.*

For a fixed $-1 < t < 1$, we recall that $\mathrm{B}_t = [-\sqrt{1-t^2}, \sqrt{1-t^2}] \times \{t\}$, $D_t = \{(x,s) : \|(x,s)\| \leq 1, s \geq t\}$ and $\mathrm{S}_t = \{(x,s) : \|(x,s)\| = 1, s \geq t\}$ (see Figure 2(A)). For each $\phi \in \mathbb{R}$ and $P, Q \in \mathrm{S}_t$, let ρ_ϕ denote $(-\sqrt{1-t^2}\cos\phi + t\sin\phi, \sqrt{1-t^2}\sin\phi + t\cos\phi)$ (see Figure 2(B)), and $\sphericalangle(P,Q)$ denote the central angle of the arc $\widehat{PQ}$, i.e., $\sphericalangle(P,Q) = \arccos\langle P, Q\rangle \in [0, \pi]$. Then $|\phi - \theta| = \sphericalangle(\rho_\phi, \rho_\theta)$ for all $\phi, \theta \in [0, 2\arccos t]$, and the homeomorphism $\varphi : [0,1] \to \mathrm{S}_t$ defined by $\varphi(a) = \varphi_a = \rho_{2a\arccos t}$ clearly satisfies:

$$\frac{\sphericalangle(\varphi_a, \varphi_b)}{\sphericalangle(\varphi_c, \varphi_d)} = \frac{\sphericalangle(\rho_{2a\arccos t}, \rho_{2b\arccos t})}{\sphericalangle(\rho_{2c\arccos t}, \rho_{2d\arccos t})} = \frac{2|a-b|\arccos t}{2|c-d|\arccos t} = \frac{|a-b|}{|c-d|},$$

for all $a, b, c, d \in [0,1]$. Note that, by substituting $c = 1$ and $d = 0$,

$$\sphericalangle(\varphi_a, \varphi_b) = |a-b|\sphericalangle(\varphi_1, \varphi_0) = 2|a-b|\arccos t.$$

The above relation of φ shows that each $a, b, c \in [0,1]$ with $|a-b| = |a-c|$ induce $\blacktriangle\varphi_a\varphi_b\varphi_c$ on D_t, and hence, by setting $p = \frac{1}{2}(a+b)$ and $q = \frac{1}{2}(a+c)$,

$$\blacktriangle\varphi_a\varphi_b\varphi_c \cap \blacktriangle\varphi_p\varphi_a\varphi_b = \varphi_a\varphi_b \quad \text{and} \quad \blacktriangle\varphi_a\varphi_b\varphi_c \cap \blacktriangle\varphi_q\varphi_a\varphi_c = \varphi_a\varphi_c.$$

Let $\mathscr{D} = \{\frac{m}{2^n} \in [0,1] : m, n \in \mathbb{N} \cup \{0\}\}$. Then $\overline{\mathscr{D}} = [0,1]$ and $\overline{\varphi(\mathscr{D})} = \mathrm{S}_t$. For convenience, write $\blacktriangle_n^m$ for $\blacktriangle\varphi_{\frac{2m-1}{2^n}}\varphi_{\frac{m-1}{2^{n-1}}}\varphi_{\frac{m}{2^{n-1}}}$ for all $m, n \in \mathbb{N}$ with $m \leq n$ (see Figure 3). Thus

$$\overline{\bigcup\{\blacktriangle_n^m : n, m \in \mathbb{N}, m \leq n\}} = D_t.$$

For each $m, n \in \mathbb{N}$ with $m \leq n$, let r_n^m be the roof map defined on $\blacktriangle_n^m$, and write $\blacktriangle_n = \bigcup_{m \leq n} \blacktriangle_n^m$ for any $n \in \mathbb{N}$. Then $r_n := (\bigcup_{m \leq n} r_n^m : \blacktriangle_n \to \blacktriangle_n)$ maps each $\blacktriangle_n^m$, $m \leq n$, onto its legs. By letting P_1 be the projection $P_{\mathrm{span}\{(1,0)\}}$, it is straightforward to verify the following properties for every $(x,s) \in \blacktriangle_n$:

(1) $P_1 \circ r_n(x,s) = -P_1 \circ r_n(-x,s)$;
(2) $\|P_1(x,s)-(0,t)\| = \sqrt{|x|^2+t^2} \leq \sqrt{|P_1\circ r_n(x,s)|^2+t^2} = \|P_1\circ r_n(x,s)-(0,t)\|$;
(3) $\nu(P_1(x,s)) = \nu(x,0) = \nu(P_1\circ r_n(x,s))$.

For each $n \in \mathbb{N}$, define $f_n : D_t \to D_t$ by

$$f_n := \bigcup_{k\leq n} (r_n \circ r_{n-1} \circ \cdots \circ r_k) \cup \mathrm{id}_{D_t - \bigcup_{m\leq n} \blacktriangle_m}$$

(see Figure 4). That is, f_n maps $\bigcup_{m\leq n} \blacktriangle_m$ continuously onto all legs of $\blacktriangle_n$, but fixes $D_t - \bigcup_{m\leq n} \blacktriangle_m$. Observe that $\bigcup_{m\leq n} \blacktriangle_m$ is convex. By defining a $(\frac{1}{2\sqrt{1-t^2}})$-lipschitzian homeomorphism

$$\psi : [-\sqrt{1-t^2}, \sqrt{1-t^2}] \times \{t\} \to [0,1], \quad (x,t) \mapsto \frac{x+\sqrt{1-t^2}}{2\sqrt{1-t^2}},$$

each f_n satisfies $f_n \circ \psi^{-1}(\frac{m}{2^n}) = \varphi(\frac{m}{2^n})$ for all $m = 0, \ldots, 2^n$.

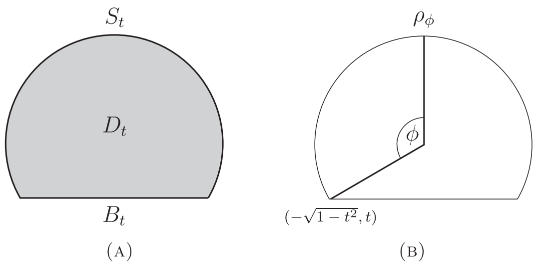

FIGURE 2. B_t, D_t, S_t in $\mathbb{R}^2$ and ρ_ϕ.

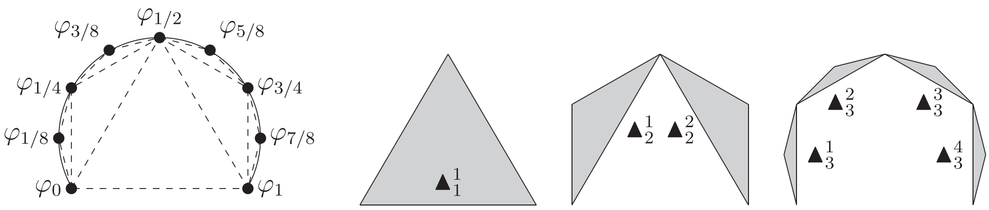

FIGURE 3. φ_n's and $\blacktriangle_n^m$'s.

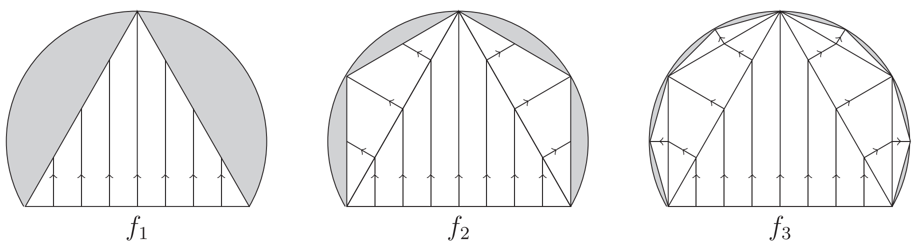

FIGURE 4. Maps f_1, f_2 and f_3 where the gray areas are fixed.

Denote by h_n and l_n the (common) height and the (common) legs' length of $\blacktriangle_n^1, \ldots, \blacktriangle_n^n$, respectively. For each $n \in \mathbb{N}$, since all $\blacktriangle_n^m$'s are defined on the unit ball,

$$h_n < l_n = |\varphi_0 \varphi_{\frac{1}{2^n}}| \leq \sphericalangle(\varphi_0, \varphi_{\frac{1}{2^n}}) = 2\left|0 - \frac{1}{2^n}\right| \arccos t = \frac{\arccos t}{2^{n-1}}.$$

Lemma 2.2. *The sequence (f_n) converges uniformly.*

Proof. For each $n \in \mathbb{N}$, observe that $f_n(x) \neq f_{n+1}(x)$ only if $x \in \blacktriangle_n$. Then

$$\|f_n - f_{n+1}\|_\infty = \sup_{x \in \blacktriangle_n} \|f_n(x) - f_{n+1}(x)\| = h_n < \frac{\arccos t}{2^{n-1}},$$

and the result follows immediately from the fact that

$$\|f_n - f_m\|_\infty \leq \sum_{i=m}^{n-1} \|f_{i+1} - f_i\|_\infty \leq \sum_{i=k+1}^{\infty} \frac{\arccos t}{2^{i-1}} = \frac{\arccos t}{2^{k-1}},$$

for each $n \geq m > k$. $\square$

Throughout this work, we let $f = \lim_{n\to\infty} f_n$. Then f is continuous by the previous lemma, and clearly, $f(D_t) = \bigcap_{n\in\mathbb{N}} f_n(D_t) = S_t$. The followings are some properties of f :

Properties 2.3.

(i) $P_1 \circ f(x, s) = -P_1 \circ f(-x, s)$ *for all* $(x, s) \in D_t$*;*
(ii) $\|P_1(x, s) - (0, t)\| \leq \|P_1 \circ f(x, s) - (0, t)\|$ *for all* $(x, s) \in D_t$*;*
(iii) $\nu(P_1(x, s)) = \nu(x, 0) = \nu(P_1 \circ f(x, s))$ *for all* $(x, s) \in D_t$*;*
(iv) $f|_{\mathrm{B}_t} = \varphi \circ \psi$*, which is a homeomorphism;*
(v) *each* $x \in D_t$ *has its unique associated base point* $x_0 \in \mathrm{B}_t$ *in sense that* $f(x) = f(x_0)$*;*
(vi) $\|x_0 - y_0\| \leq \|x - y\|$ *for all* $x, y \in D_t$*.*

Proof. The properties of r_n's imply (i)-(iii) while the properties of f_n's imply (iv). (v) follows from (iv), and (vi) follows from (ii) and (v). $\square$

3. Main results

We will give a new upper bound of $\kappa(t)$ that simultaneously yields the precise formula of $\kappa(t)$ for a finite-dimensional Hilbert space, and a sharper upper bound of $\kappa(t)$ for an infinite-dimensional Hilbert space.

As usual, let $(H, \langle\cdot,\cdot\rangle)$ be a (real) Hilbert space, $H = E \oplus \mathrm{span}\{\mathrm{e}\} = E \oplus \mathbb{R}$, where E is the orthogonal complement of e. Each element in H can be uniquely represented as $x \oplus y$, for some $x \in E$ and $y = \langle x, \mathrm{e}\rangle \in \mathbb{R}$, and hence $\|x \oplus y\|^2 = \|x\|^2 + |y|^2$. Recall that every n-dimensional Hilbert space is isometrically isomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$. Also, it is straightforward to verify the following proposition :

Proposition 3.1. *For each $\phi \in (0, \pi)$ and $s > 0$, the map $(0, 1) \to \mathbb{R}$ defined by $r \mapsto \frac{\sqrt{2-2\cos(r\phi)}}{rs}$ is decreasing with $\sup_{r\in(0,1)} \frac{\sqrt{2-2\cos(r\phi)}}{rs} = \frac{\phi}{s}$.*

Lemma 3.2. *Let $\triangle APP'$, $\triangle BQQ'$, $\triangle CRR'$ and $\triangle CSS'$ be pairwise similar isosceles triangles on parallel planes in $\mathbb{R}^n$, where $n \geq 3$, with relations:*

(i) $[A, B, C]$ *are collinear and perpendicular to each triangles;*
(ii) $\{Q, R, S\} \subseteq \operatorname{span}\{B - A, P - A\}$ *and* $\{Q', R', S'\} \subseteq \operatorname{span}\{B - A, P' - A\}$*;*
(iii) $\|R - R'\| \leq \|S - S'\|$*;*
(iv) $\|P - P'\| \leq \|Q - Q'\|$ *(so that* $\|A - P\| \leq \|B - Q\|$*).*

Set $\phi =$ *the base-angle of each triangles (i.e.* $\phi = \hat{P} = \hat{P'} = \hat{Q} = \hat{Q'} = \hat{S} = \hat{S'} = \angle CRR' = \angle CR'R$*),*

$$\begin{array}{lll} p = \|P - P'\|, & q = \|Q - Q'\|, & l = \|P - Q\| = \|P' - Q'\|, \\ h = \|A - B\|, & k = \|B - Q\| - \|A - P\|, & n = \|P - Q'\| = \|P' - Q\|, \\ r = \|R - R'\|, & s = \|R - S\| = \|R' - S'\|, & t = \|R - S'\| = \|R' - S\| \end{array}$$

(see Figure 5). If either $\frac{p}{r}, \frac{l}{s} \leq \alpha$ *or* $\frac{q}{r}, \frac{l}{s} \leq \alpha$ *for some* $\alpha > 0$*, so is* $\frac{n}{t}$*.*

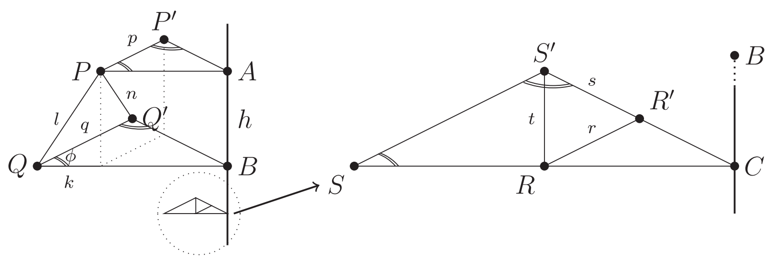

FIGURE 5. Relations of isosceles triangle in Lemma 3.2.

Proof. Note that $\cos\phi \geq 0$ because ϕ is the base-angle of the isosceles triangle. Consider the following cases.
<u>Case I</u>: $\frac{p}{r}, \frac{l}{s} \leq \alpha$ for some $\alpha > 0$. Then

$$\begin{aligned} \frac{n^2}{t^2} &= \frac{h^2 + (k^2 + p^2 - 2kp\cos(\pi - \phi))}{s^2 + r^2 - 2sr\cos(\pi - \phi)} \\ &= \frac{(h^2 + k^2) + p^2 + 2kp\cos\phi}{s^2 + r^2 + 2sr\cos\phi} \\ &\leq \frac{l^2 + p^2 + 2lp\cos\phi}{s^2 + r^2 + 2sr\cos\phi}. \end{aligned}$$

<u>Case II</u>: $\frac{q}{r}, \frac{l}{s} \leq \alpha$ for some $\alpha > 0$. Then

$$\frac{n^2}{t^2} = \frac{h^2 + (k^2 + q^2 - 2kq\cos\phi)}{s^2 + r^2 - 2sr\cos(\pi - \phi)} \leq \frac{(h^2 + k^2) + q^2 + 2kq|\cos\phi|}{s^2 + r^2 + 2sr\cos\phi} \leq \frac{l^2 + q^2 + 2lq\cos\phi}{s^2 + r^2 + 2sr\cos\phi}.$$

Recall that for each $a, b, c, d, k > 0$, $\frac{a+b}{c+d} \leq k$ if $\frac{a}{c}, \frac{b}{d} \leq k$. Therefore, since $\frac{l^2}{s^2} \leq \alpha^2$, $\frac{p^2}{r^2} \leq \alpha^2$ (or $\frac{q^2}{r^2} \leq \alpha^2$) and $\frac{2lp\cos\phi}{2sr\cos\phi} \leq \alpha^2$ (or $\frac{2lq\cos\phi}{2sr\cos\phi} \leq \alpha^2$), it follows that $\frac{n}{t} \leq \alpha$. $\square$

Theorem 3.3. *For each* $-1 < t < 1$,

$$\kappa(t) \leq \frac{\arccos t}{\sqrt{1 - t^2}}.$$

Proof. Let $-1 < t < 1$, and write $k_t = \frac{\arccos t}{\sqrt{1-t^2}}$. If a k_t-lipschitzian retraction $F : D_t \to \mathrm{S}_t$ exists, $F \circ P_{D_t} : \mathrm{B} \to \mathrm{S}_t$ is also a k_t-lipschitzian retraction because $P_{D_t} : \mathrm{B} \to D_t$ is nonexpansive, and so, the proof is complete. Thus it suffices to show the existence of such a map F. Consider the following cases.

Case I: $\dim H = 2$.

The map f defined in the previous section plays the role of F in this case. To see this, it suffices, by Properties 2.3(iii)-(v), to prove that $f|_{\mathrm{B}_t} = \varphi\circ\psi$ is k_t-lipschitzian. Let $x \neq y \in \mathrm{B}_t$. Then $x = \psi^{-1}(a)$ and $y = \psi^{-1}(b)$ for some $a, b \in [0,1]$. Since ψ is $(\frac{1}{2\sqrt{1-t^2}})$-lipschitzian, $|a-b| = |\psi(x) - \psi(y)| \leq \frac{1}{2\sqrt{1-t^2}}\|x-y\|$, which implies that

$$\begin{aligned}\frac{\|f(x)-f(y)\|}{\|x-y\|} &= \frac{\|\varphi_a - \varphi_b\|}{\|x-y\|} \leq \frac{\sqrt{2-2\cos\sphericalangle(\varphi_a,\varphi_b)}}{2|a-b|\sqrt{1-t^2}} \\ &= \frac{\sqrt{2-2\cos(2|a-b|\arccos t)}}{2|a-b|\sqrt{1-t^2}}.\end{aligned}$$

Note that $\arccos t \in (0,\pi)$ and $2\sqrt{1-t^2} > 0$. By Proposition 3.1, we obtain

$$\frac{\|f(x)-f(y)\|}{\|x-y\|} \leq \lim_{2|r-s|\to 0^+} \frac{\sqrt{2-2\cos(2|r-s|\arccos t)}}{2|r-s|\sqrt{1-t^2}} = \frac{\arccos t}{\sqrt{1-t^2}} = k_t.$$

Case II: $\dim H > 2$.

Observe that $(\mathrm{span}\{\mathrm{e},x\}\cap D_t)\cap(\mathrm{span}\{\mathrm{e},y\}\cap D_t) = \mathrm{span}\{\mathrm{e}\}\cap D_t$ for any $x \neq y \in \mathrm{B}_t \cap \mathrm{S}_t$. Let $F = \bigcup_{p\in \mathrm{B}_t\cap \mathrm{S}_t} f_p : D_t \to \mathrm{S}_t$, where f_p is the map f defined on $\mathrm{span}\{\mathrm{e},p\}\cap D_t$. Then F is a k_t-lipschitzian retraction. To see this, without loss of generality, let $A = x\oplus r, C = y \oplus s \in H - \mathrm{span}\{\mathrm{e}\}$. If $A \in \mathrm{span}\{\mathrm{e}, C\}$, the proof follows from Case I. Assume $A \notin \mathrm{span}\{\mathrm{e},C\}$. Write $FA = x_f \oplus r_f$ and $FC = y_f \oplus s_f$. Set $P = \frac{\|y_f\|}{\|x_f\|}x_f \oplus s_f = z_f \oplus s_f$ and $Q = \frac{\|x_f\|}{\|y_f\|}y_f \oplus r_f = w_f \oplus r_f$. By Properties 2.3(v), there are $A_0 = x_t \oplus t, P_0 = z_t \oplus t \in \mathrm{span}\{\mathrm{e}, x\oplus 0\}\cap \mathrm{B}_t$ and $C_0 = y_t \oplus t, Q_0 = w_t \oplus t \in \mathrm{span}\{\mathrm{e}, y \oplus 0\} \cap \mathrm{B}_t$ such that $FA = FA_0$, $FC = FC_0$, $P = FP = FP_0$ and $Q = FQ = FQ_0$. Since the isometric isomorphism among $\mathrm{span}\{\mathrm{e},A\}$, $\mathrm{span}\{\mathrm{e},C\}$ and $\mathbb{R}^2$ yields $\|z_t\| = \|y_t\|$, $\|z_f\| = \|y_f\|$, $\|w_t\| = \|x_t\|$ and $\|w_f\| = \|x_f\|$, this shows that $\triangle(FA, 0\oplus r_f, Q)$, $\triangle(FC, 0\oplus s_f, P)$, $\triangle(A_0, 0\oplus t, Q_0)$ and $\triangle(C_0, 0\oplus t, P_0)$ form isosceles triangles as required by Lemma 3.2. Recall from Properties 2.3(iii) that $\nu(z_f) = \nu(z_t)$ and $\nu(y_f) = \nu(y_t)$. Then $\sphericalangle(y_t,z_t) = \sphericalangle(y_f,z_f)$, and so,

$$\begin{aligned}\|FC_0 - FP_0\|^2 = \|y_f - z_f\|^2 &= (2-2\cos\sphericalangle(y_f,z_f))\|y_f\|^2 \\ &\leq (2-2\cos\sphericalangle(y_t,z_t))\|FC - \mathrm{e}\|^2 \\ &= (2-2\cos\sphericalangle(y_t,z_t))\|f_C C_0 - f_C(0\oplus t)\|^2 \\ &\leq k_t^2(2-2\cos\sphericalangle(y_t,z_t))\|C_0 - (0\oplus t)\|^2 \\ &\leq k_t^2(2-2\cos\sphericalangle(y_t,z_t))\|y_t\|^2 \\ &= k_t^2\|(y_t\oplus t) - (z_t \oplus t)\|^2 = k_t^2\|C_0 - P_0\|^2.\end{aligned}$$

Since f_A is k_t-lipschitzian, $\|FP_0 - FA_0\| \le k_t\|P_0 - A_0\|$. Apply Lemma 3.2 and Properties 2.3(vi), hence

$$\|FA - FC\| = \|FA_0 - FC_0\| \le k_t\|A_0 - C_0\| \le k_t\|A - C\|. \qquad \square$$

Corollary 3.4. *For every finite-dimensional Hilbert space and $-1 < t < 1$,*

$$\kappa(t) = \frac{\arccos t}{\sqrt{1-t^2}}.$$

Proof. Follows directly from Theorem 3.3 and the fact that $\kappa(t) \ge \frac{\arccos t}{\sqrt{1-t^2}}$ [4, Observation 3.5]. $\square$

We are now assume that H is infinite-dimensional. For convenience, we write $k_0 = k_0(H)$. Notice that, in this case, H is isometrically isomorphic to E because E has co-dimension one.

For $0 < \phi < \frac{\pi}{2}$ and $s \in \mathbb{R}$, define the cone C_ϕ, its boundary V_ϕ, and the parallel cone section $\mathrm{B}_s(C_\phi)$ of the cone C_ϕ respectively by

- $C_\phi = \{x \oplus r \in E \oplus \mathbb{R} : \|x\| \le r\cot\phi\}$;
- $V_\phi = \{x \oplus r \in E \oplus \mathbb{R} : \|x\| = r\cot\phi\}$;
- $\mathrm{B}_s(C_\phi) = C_\phi \cap E_s = C_\phi \cap (E + se)$.

We also let

- $-C_\phi = \{x \oplus r \in E \oplus \mathbb{R} : \|x\| \le -r\cot\phi\}$;
- $C_{\phi,s} = C_\phi + se$.

Notice that $r \ge 0$ for both C_ϕ and V_ϕ, while $r \le 0$ for $-C_\phi$, and $r \ge s$ for $C_{\phi,s}$.

Lemma 3.5. *Let $0 < \phi < \frac{\pi}{2}$. Each $A = x \oplus r$, $B = y \oplus s$ and $P = \frac{r}{s}y \oplus r$ in C_ϕ with $r \ge s > 0$ satisfy*

$$2\langle P - A, P - B\rangle \le (\|P - A\|^2 + \|P - B\|^2)\cos\phi$$

Proof. Let $A, B, C \in C_\phi$ be as above. Set $Q = \|\frac{r}{s}y - x\|\frac{y}{\|y\|} \oplus 0$. Then $\|Q\| = \|P - A\|$ and

$$\begin{aligned}
\langle P - A, P - B\rangle &= \left\langle \left(\frac{r}{s}y - x\right) \oplus 0, \left(\frac{r}{s} - 1\right) y \oplus (r - s)\right\rangle \\
&= \left\langle \left(\frac{r}{s}y - x\right) \oplus 0, \left(\frac{r}{s} - 1\right) y \oplus 0\right\rangle \\
&\le \left\|\frac{r}{s}y - x\right\| \left(\frac{r}{s} - 1\right)\|y\| \\
&= \left\langle \left\|\frac{r}{s}y - x\right\| \frac{y}{\|y\|} \oplus 0, \left(\frac{r}{s} - 1\right) y \oplus 0\right\rangle \\
&= \left\langle \left\|\frac{r}{s}y - x\right\| \frac{y}{\|y\|} \oplus 0, \left(\frac{r}{s} - 1\right) y \oplus (r - s)\right\rangle = \langle Q, P - B\rangle.
\end{aligned}$$

Recall the following equivalence:

$$\|y\| \le s\cot\phi \iff \|y\|^2(1 - (\cos\phi)^2) \le (s\cos\phi)^2 \iff \frac{\|y\|}{\sqrt{\|y\|^2 + s^2}} \le \cos\phi.$$

Since $\|P-A\|\|P-B\| = \|Q\|\|P-B\| = \|\frac{r}{s}y-x\|(\frac{r}{s}-1)\sqrt{\|y\|^2+s^2}$ and $\|y\| \leq s\cot\phi$,

$$\frac{\langle P-A, P-B\rangle}{\|P-A\|\|P-B\|} \leq \frac{\langle Q, P-B\rangle}{\|P-A\|\|P-B\|} = \frac{\|y\|}{\sqrt{\|y\|^2+s^2}} \leq \cos\phi,$$

which implies that

$$2\langle P-A, P-B\rangle \leq 2\|P-A\|\|P-B\|\cos\phi \leq \left(\|P-A\|^2+\|P-B\|^2\right)\cos\phi. \quad \square$$

Lemma 3.6. *Let $g : \mathrm{B}_E \to \mathrm{S}_E$ be k-lipschitzian and $0 < \phi < \frac{\pi}{2}$. The map $G : C_\phi \to D_\phi$ defined by $G(0\oplus 0) = 0\oplus 0$, and $G(x\oplus r) = (r\cot\phi)g(\frac{x}{r\cot\phi})\oplus r$ if $r > 0$, is $\frac{\max\{k,\csc\phi\}}{\sqrt{1-\cos\phi}}$-lipschitzian.*

Proof. Firstly, let us observe that the maps $0\mapsto 0$, and $x\mapsto (r\cot\phi)g(\frac{x}{r\cot\phi})$ for $r>0$, on $\{x : \|x\| \leq r\cot\phi\} \subseteq E$ is k-lipschitzian. Without loss of generality, let $A = x\oplus r, B = y\oplus s \in C_\phi$ where $r \geq s$. The case $r = s$ is clear. The case $s = 0$ yields $y = 0$, which implies that

$$\begin{aligned}\|GA-GB\|^2 &= \|GA\|^2 = (r\cot\phi)^2 + r^2 = (r\csc\phi)^2\\ &\leq (\csc\phi)^2(\|x\|^2+r^2) = (\csc\phi)^2\|A-B\|^2.\end{aligned}$$

For case $r > s > 0$, let $P = \frac{r}{s}y\oplus r \in C_\phi$. Apply Lemma 3.5 to obtain

$$\begin{aligned}\|A-B\|^2 &= \|P-A\|^2+\|P-B\|^2 - 2\langle P-A, P-B\rangle\\ &\geq (1-\cos\phi)(\|P-A\|^2+\|P-B\|^2).\end{aligned}$$

Recall that $\langle g(z)\oplus 0, g(w)\oplus 0\rangle \leq 1 = \|g(z)\|^2$ for all $z, w \in \mathrm{B}$, $GA = (r\cot\phi)g(\frac{x}{r\cot\phi})\oplus r$, $GB = (s\cot\phi)g(\frac{y}{s\cot\phi})\oplus s$ and $GP = (r\cot\phi)g(\frac{y}{s\cot\phi})\oplus r$. Then

$$\begin{aligned}\langle GP-GA, GP-GB\rangle &= \left\langle (r\cot\phi)\left(g\left(\frac{y}{s\cot\phi}\right) - g\left(\frac{y}{r\cot\phi}\right)\right)\oplus 0,\right.\\ &\qquad\left. ((r-s)\cot\phi)g\left(\frac{y}{s\cot\phi}\right)\oplus(r-s)\right\rangle\\ &= r(r-s)(\cot\phi)^2\left(\left\|g\left(\frac{y}{s\cot\phi}\right)\right\|^2 - \left\langle g\left(\frac{y}{r\cot\phi}\right)\oplus 0, g\left(\frac{y}{s\cot\phi}\right)\oplus 0\right\rangle\right) \geq 0,\end{aligned}$$

and hence, again by Lemma 3.5,

$$\begin{aligned}\|GA-GB\|^2 &= \|GP-GA\|^2+\|GP-GB\|^2 - 2\langle GP-GA, GP-GB\rangle\\ &\leq \|GP-GA\|^2+\|GP-GB\|^2\\ &\leq k^2\|P-A\|^2 + \left\|(r-s)(\cot\phi)g\left(\frac{y}{s\cot\phi}\right)\oplus(r-s)\right\|^2\\ &= k^2\|P-A\|^2 + (1+\cot^2\phi)(r-s)^2\\ &\leq k^2\|P-A\|^2 + (\csc^2\phi)\|P-B\|^2\\ &\leq \max\{k^2,\csc^2\phi\}(\|P-A\|^2+\|P-B\|^2)\\ &\leq \frac{\max\{k^2,\csc^2\phi\}}{1-\cos\phi}\|A-B\|^2.\end{aligned}$$

$\square$

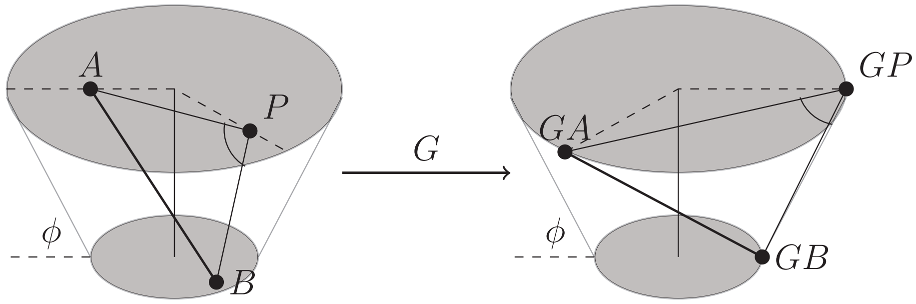

FIGURE 6. Points A, B, P, GA, GB and GP in Lemma 3.6.

Lemma 3.7. *Let $g : E \to E$ be k-lipschitzian where $k \geq 1$. The map $G : H \to H$ defined by $G(x \oplus r) = gx \oplus r$ is k-lipschitzian.*

Proof. Let $A = x \oplus r, B = y \oplus s \in H$. Then

$$\|GA - GB\|^2 = \|gx - gy\|^2 + |r - s|^2 \leq k^2 \left(\|x - y\|^2 + |r - s|^2\right) = k^2\|A - B\|^2. \quad \square$$

Theorem 3.8. *For every infinite-dimensional Hilbert space and $-1 \leq t \leq 1$,*

$$\kappa(t) \leq \frac{3\sqrt{3}}{2}k_0.$$

Proof. Recall that $\kappa(-1) = k_0$ and $\kappa(t) \leq \frac{\arccos t}{\sqrt{1-t^2}} < 11.2 < \frac{3\sqrt{3}}{2}k_0$ if $-\frac{1+\sqrt{3}}{2\sqrt{2}} < t \leq 1$. It suffices to assume that $-1 < t \leq -\frac{1+\sqrt{3}}{2\sqrt{2}}$. Let $\varepsilon > 0$. By the definition of k_0, there exists a $(k_0 + \varepsilon)$-lipschitzian retraction $g_\varepsilon : \mathrm{B} \to \mathrm{S}$. Fix $0 < \phi < \frac{\pi}{2}$. Construct two cones $C_1 = -C_\phi + \mathrm{e}$ and $C_2 = C_{\phi,(t-\sqrt{1-t^2}\tan\phi)}$. Let F_s be the common parallel cone section of C_1 and C_2, i.e., $F_s = \mathrm{B}_s(C_1) \cap \mathrm{B}_s(C_2) = \mathrm{B}_s(C_1) = \mathrm{B}_s(C_2)$.

Case I: $F_s \subseteq D_t$.

Let $P = \{x \oplus r \in C_1 : r \geq s\} \subseteq D_t$ and $Q = \{x \oplus r \in C_2 \cap D_t : r \leq s\}$ (see Figure 7(A)). Then $A = P \cup Q$ is convex.

Case II: $F_s \nsubseteq D_t$.

Let $a = \max\{r : \mathrm{B}_r(C_2) \subseteq D_t\}$. Set $P = \{x \oplus r \in C_1 : r \geq 1 - (\sqrt{1-a^2})\tan\phi\}$, $Q = \{x \oplus r \in C_2 \cap D_t : r \leq a\}$ and construct a cylinder $R = \{x \oplus r \in D_t : \|x\| \leq \sqrt{1-a^2}, a \leq r \leq 1 - (\sqrt{1-a^2})\tan\phi\}$ (see Figure 7(B)). Then $A = P \cup Q \cup R$ is convex with $P \cap Q = \emptyset$, $P \cap R = \mathrm{B}_{1-\sqrt{1-a^2}\tan\phi}(C_1)$ and $Q \cap R = \mathrm{B}_a(C_2)$.

Recall that $\mathrm{B}_E = \mathrm{B} \cap E$ and $\mathrm{S}_E = \mathrm{S} \cap E$. Since E and H are isometrically isomorphic, there is a $(k_0 + \varepsilon)$-lipschitzian retraction $g_\varepsilon : \mathrm{B}_E \to \mathrm{S}_E$. By applying Lemma 3.6 and Lemma 3.7 with the map g_ε to P, Q and R in both cases, there exists a $\frac{\max\{k_0+\varepsilon,\csc\phi\}}{\sqrt{1-\cos\phi}}$-lipschitzian retraction $G : A \to \partial A - \mathrm{B}_t^\circ$ (because $k_0 + \varepsilon > 1$, $\sqrt{1-\cos\phi}^{-1} > 1$ and A is convex).

The straightforward calculation shows that for each $-1 < t \leq -\frac{1+\sqrt{3}}{2\sqrt{2}}$,

$$\begin{aligned}\inf\{\|x \oplus r\| : x \oplus r \in \partial A - \mathrm{B}_t^\circ\} &= \inf\{\|x \oplus r\| : x \oplus r \in P\} \\ &= \inf\{\|x \oplus r\| : \|x\| = (1-r)\cot\phi\} \\ &= \inf\{\sqrt{((1-r)\cot\phi)^2 + r^2} : -1 < r < 1\} = \cos\phi.\end{aligned}$$

Denote by ρ the radial projection onto S. Then $\rho \circ G \circ P_{D_t} : \mathrm{B} \to \mathrm{S}_t$ is a $\frac{\max\{k_0+\varepsilon, \csc\phi\}}{(\sqrt{1-\cos\phi})\cos\phi}$-lipschitzian retraction. Finally, by minimizing such a Lipschitz constant, a $\frac{3\sqrt{3}}{2}(k_0+\varepsilon)$-lipschitzian retraction is obtained at $\phi = \arccos\frac{2}{3}$. □

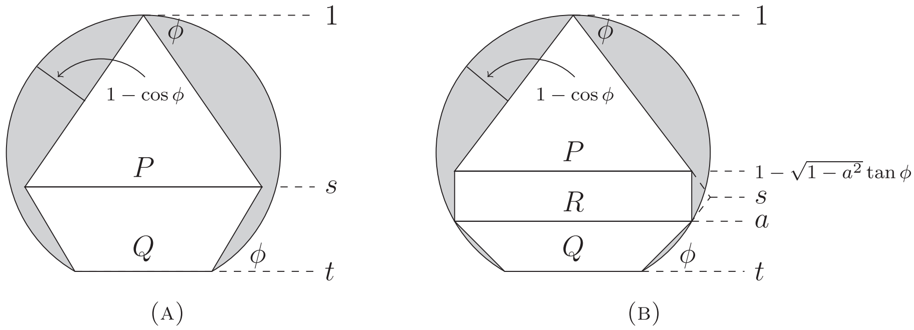

FIGURE 7. The set A (white area) in Theorem 3.8.

By Theorem 3.3, Theorem 3.8 and Observation 3.9 [4], we obtain :

Corollary 3.9. *For every infinite-dimensional Hilbert space and* $-1 \leq t \leq 1$,

$$\kappa(t) \leq \min\left\{\frac{2}{1+t}, \frac{\arccos t}{\sqrt{1-t^2}}, \frac{3\sqrt{3}}{2}k_0\right\}.$$

Moreover, by combining Theorem 3.8 and Observation 3.11 [4], we have the following better approximation results :

Corollary 3.10. *For every infinite-dimensional Hilbert space, there exists* $-1 < a < 1$ *such that*

$$\frac{2}{3\sqrt{3}}\kappa(t) \leq k_0 \leq \kappa(t),$$

or equivalently,

$$1 \leq \frac{\kappa(t)}{k_0} \leq \frac{3\sqrt{3}}{2} \approx 2.59808$$

for all $-1 < t < a$.

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Manuscript received 31 August 2015
revised 20 June 2016

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