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CONVERGENCE THEOREMS FOR SOME CLASSES OF NONLINEAR MAPPINGS IN HILBERT SPACES

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Abstract. In this article, we introduce the concept of k-acute points of a mapping T in the Hilbert space setting, where $k \in [0, 1]$. Then, we have some properties of k-acute points and relations among k–acute points, attractive points and fixed points. Further, we apply these to rearrange proofs of some known convergence theorems and to prove new convergence theorems for nonlinear mappings.

1. Introduction

In 1967, Browder and Petryshyn [4] initiated the study of fixed points of strictly pseudo-contractions. In 1974, Ishikawa [9] made an impact on this study area. On the other hand, in 2011, Takahashi and Takeuchi [27] introduced the concept of attractive points and apply it to have an extension of the Baillon type ergodic theorem due to Kocourek, Takahashi, and Yao [12] without convexity. Motivated by these works, in the Hilbert space setting, we introduce the concept of k–acute points of a mapping T, where $k \in [0, 1]$. Then, we study some properties of k-acute points and relations among k–acute points, attractive points and fixed points. In other words, we rearrange properties of some nonlinear mappings by using the concept of k–acute points and relations between T and $S = kI + (1 - k)T$, where I is the identity mapping. In this direction, we have some results and apply these to rearrange proofs of known convergence theorems and to prove new convergence theorems for such nonlinear mappings in Hilbert spaces.

2. Preliminaries and basic concepts

In this article, we denote by R the set of real numbers and by N the set of positive integers. We denote by E a real Banach space and by H a real Hilbert space. For simplicity's sake, we remove "real". Of course, a Hilbert space is a Banach space.

Let C be a non–empty subset of a Banach space E and T be a mapping of C into E. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. $A(T)$ denotes the set of attractive points [27] of T, that is,

$$A(T) = \{\ v \in E :\ \|Tx - v\| \leq \|x - v\| \quad \text{for all} \quad x \in C\ \}.$$

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In the Banach space setting, there is another definition of the attractive points set which is different from our definition; see Lin and Takahashi [17]. T is said to be L-Lipschitzian if $\|Tx - Ty\| \leq L\|x - y\|$ for any $x, y \in C$, where $L \in [0, \infty)$. In particular, T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. For each $x \in E$, $x_0 \in C$ is called a nearest point of C to x if $\|x - x_0\| = \inf\{\|x - z\| : z \in C\}$. Here, we present typical properties of $A(T)$; see [27].

Lemma 2.1. *Let C be a non–empty subset of a Hilbert space H and T be a mapping of C into H. Then, $A(T)$ is closed and convex.*

Lemma 2.2. *Let C be a non–empty subset of a Banach space E and let T be a self–mapping on C with $A(T) \neq \emptyset$. Let $x \in A(T)$. Suppose there is the unique nearest point x_0 of C to x. Then, $x_0 \in F(T)$.*

Proof. By $x \in A(T)$, it is obvious that

$$\|x - Tx_0\| \leq \|x - x_0\| = \inf\{\|x - z\| : z \in C\}.$$

Since x_0 is the unique nearest point of C to x, we have $Tx_0 = x_0$ and $x_0 \in F(T)$. □

From now on, we discuss in the Hilbert space setting. Note the followings: We denote by C a subset of a Hilbert (Euclidean) space. C is always non–empty unless otherwise noted. Then, normally, "non–empty" is not described.

Let C be a subset of a Hilbert space H and T be a mapping of C into H. We denote by I the identity mapping on H. T is said to be quasi–nonexpansive if

$$(1)\ F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\| \leq \|x - v\| \quad \text{for } \ x \in C,\ v \in F(T).$$

Then T is quasi–nonexpansive if and only if $\emptyset \neq F(T) \subset A(T)$. That is, the concept of attractive points is closely related to quasi–nonexpansive mappings. However, we can easily find a non–increasing and continuous self–mapping T on a closed interval C in R such that $F(T) \neq \emptyset$ and $A(T) \cap C = \emptyset$. Here, we give an example.

Example 2.3. Let $C = [-1, 2] \subset R$ and T be the non–increasing and continuous self–mapping on C defined by

$$Tx = -2x \quad \text{if} \quad x \in [-1, 0], \qquad Tx = -x/2 \quad \text{if} \quad x \in (0, 2].$$

Then, one can easily see that $A(T) \cap C = \emptyset$ and $F(T) = \{0\}$.

Motivated by these facts as above and Takahashi and Takeuchi [27], we introduce the concept of k–acute points. Let $k \in [0, 1]$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. We define a set $\mathscr{A}_k(T)$ by

$$\mathscr{A}_k(T) = \{\ v \in H :\ \|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for all } \ x \in C\ \}.$$

We call $\mathscr{A}_k(T)$ the set of k–acute points of T. Because, in the 2–dimensional Euclidean space setting, $\angle v\, x\, Tx$ is not an obtuse angle for $x \in C$ and $v \in \mathscr{A}(T)$.

Note $\mathscr{A}_0(T) = A(T)$. We denote $\mathscr{A}_1(T)$ by $\mathscr{A}(T)$, that is,

$$\mathscr{A}(T) = \{\ v \in H :\ \|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for all } \ x \in C\ \}.$$

It is obvious that $A(T) \subset \mathscr{A}_{k_1}(T) \subset \mathscr{A}_{k_2}(T) \subset \mathscr{A}(T)$ for $k_1, k_2 \in [0, 1]$ with $k_1 \leq k_2$.

Let $k \in [0, 1)$. T is said to be k–demi–contractive if

$$(1)\ F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for } \ x \in C,\ v \in F(T).$$

Also, T is said to be hemi–contractive if

$$(1)\ F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for} \ \ x \in C,\ v \in F(T).$$

Then, for $k \in [0, 1)$, T is k–demi–contractive if and only if $\emptyset \neq F(T) \subset \mathscr{A}_k(T)$. Also, T is hemi–contractive if and only if $\emptyset \neq F(T) \subset \mathscr{A}(T)$.

T is said to be k–strictly pseudo–contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \qquad \text{for} \quad x, y \in C.$$

T is said to be pseudo–contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \ \|(I - T)x - (I - T)y\|^2 \qquad \text{for} \quad x, y \in C.$$

It is easy to see that T is k–demi–contractive if T is k–strictly pseudo–contractive and $F(T) \neq \emptyset$. Also, T is hemi–contractive if T is pseudo–contractive and $F(T) \neq \emptyset$.

Let C be a closed and convex subset of a Hilbert space H. It is well known that, for each $x \in H$, there is the unique nearest point x_0 of C to x. The mapping P_C defined by $P_C x = x_0$ for $x \in H$ is called the metric projection of H onto C. It is also known that P_C satisfies the following conditions: For $x \in H$, $y \in C$,

$$0 \leq \langle x - P_C x, P_C x - y\rangle \quad \text{and} \quad \|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2.$$

3. Acute points and attractive points

We present an example to see that the concept of attractive points is natural.

Example 3.1. Let C be the subset of a 2–dimensional Euclidean space R^2 defined by $C = \{(x_1, x_2) \in R^2 : 1 < x_1^2 + x_2^2 < 4\}$. Then C is neither closed nor convex.

(1) Let T be the nonexpansive self–mapping on C defined by

$$T(x_1, x_2) = (-x_1, x_2) \qquad\qquad \text{for} \quad (x_1, x_2) \in C.$$

Then $A(T) = \{(x_1, x_2) \in R^2 : x_1 = 0\}$ is the symmetric axis of this transformation. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and

$$v_{n+1} = T^n v_n, \qquad b_n = \frac{1}{n}\sum\nolimits_{i=1}^{n} v_i \qquad\qquad \text{for} \quad n \in N.$$

It is obvious that $\{b_n\}$ converges strongly to some $u \in A(T)$. However, u is not always a fixed point of T. Note that $\{b_n\}$ need not be a sequence in C.

(2) Let $\alpha \in (0, 2\pi)$ and S be the nonexpansive self–mapping on C defined by

$$S(x_1, x_2) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha) \quad \text{for} \ \ (x_1, x_2) \in C.$$

Then, $A(S)$ consists of the center of this rotation, that is, $A(S) = \{(0, 0)\}$. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and

$$v_{n+1} = S^n v_n, \qquad b_n = \frac{1}{n}\sum\nolimits_{i=1}^{n} v_i \qquad\qquad \text{for} \quad n \in N.$$

It is easy to see that $\{b_n\}$ converges strongly to $(0, 0) \in A(S)$. However, C does not contain $(0, 0)$. Then, $(0, 0)$ is not a fixed point of S.

We study properties of $\mathscr{A}_k(T)$ and some relations among $F(T)$, $A(T)$ and $\mathscr{A}(T)$. To prove Theorem 3.3, we need the following well–known lemma.

Lemma 3.2. *Let H be a Hilbert space. Let $x, y, z \in H$ and $c \in [0,1]$. Then,*

$$\|cx + (1-c)y - z\|^2 = c\|x-z\|^2 + (1-c)\|y-z\|^2 - c(1-c)\|x-y\|^2.$$

Theorem 3.3. *Let $k \in [0,1]$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. Then the followings hold.*

(1) *$\mathscr{A}_k(T)$ is closed and convex.*
(2) *If C is closed then $\mathscr{A}_k(T) \cap C$ is closed.*
(3) *If C is convex then $\mathscr{A}_k(T) \cap C$ is convex.*
(4) *If $k \in [0,1)$ and $v \in \mathscr{A}_k(T) \cap C$ then $v \in F(T)$.*

Proof. We prove (1). We show that $\mathscr{A}_k(T)$ is closed. Suppose a sequence $\{z_n\}$ in $\mathscr{A}_k(T)$ converges to some $z \in H$. Let $x \in C$. Then, we have that, for $n \in N$,

$$\|Tx - z_n\|^2 \leq \|x - z_n\|^2 + k\|x - Tx\|^2.$$

Since $\|\cdot\|^2$ is continuous and $\{z_n\}$ converges strongly to z, we have

$$\|Tx - z\|^2 \leq \|x - z\|^2 + k\|x - Tx\|^2.$$

Then, $z \in \mathscr{A}_k(T)$. We have that $\mathscr{A}_k(T)$ is closed. We show that $\mathscr{A}_k(T)$ is convex. Let $c \in (0,1)$ and $u, v \in \mathscr{A}_k(T)$. Let $x \in C$ and set $\mathscr{N} = \|Tx - (cu + (1-c)v)\|^2$. By Lemma 3.2, we have that

$$\begin{aligned}
\mathscr{N} &= c\|Tx-u\|^2 + (1-c)\|Tx-v\|^2 - c(1-c)\|u-v\|^2 \\
&\leq c(\|x-u\|^2 + k\|Tx-x\|^2) \\
&\quad + (1-c)(\|x-v\|^2 + k\|Tx-x\|^2) - c(1-c)\|u-v\|^2 \\
&= (c\|x-u\|^2 + (1-c)\|x-v\|^2 - c(1-c)\|u-v\|^2) + k\|Tx-x\|^2 \\
&= \|x - (cu + (1-c)v)\|^2 + k\|Tx-x\|^2.
\end{aligned}$$

Then we have $cu + (1-c)v \in \mathscr{A}_k(T)$. That is, $\mathscr{A}_k(T)$ is convex. Thus we have (1). Note that we do not claim $\mathscr{A}_k(T) \neq \emptyset$. By (1), it is obvious that (2) and (3) hold.

We prove (4). Suppose $v \in \mathscr{A}_k(T) \cap C$. Then,

$$\|Tv - v\|^2 \leq \|v - v\|^2 + k\|v - Tv\|^2 = k\|v - Tv\|^2.$$

By $k \in [0,1)$, we have $Tv = v$. That is, we have $v \in F(T)$. □

Remark 3.4. Let C be a subset of a Hilbert space H and T be a mapping of C into H. By $A(T) = \mathscr{A}_0(T)$ and Theorem 3.3 (1), we have Lemma 2.1 due to Takahashi and Takeuchi [27]. By Theorem 3.3 (4), $\mathscr{A}_k(T) \cap C \subset F(T)$ for $k \in [0,1)$. Then, $F(T) = \mathscr{A}_k(T) \cap C$ if $F(T) \subset \mathscr{A}_k(T) \cap C$. In other words, $F(T)$ is closed and convex if T is k–demi–contractive and C is closed and convex. However, $v \in \mathscr{A}(T) \cap C$ does not imply $v \in F(T)$. Let $B_x = \{v \in H : \langle Tx - x, x - v\rangle \leq 0\}$ for each $x \in C$. For $v \in H$ and $x \in C$, $\langle Tx - x, x - v\rangle \leq 0$ and $\|Tx - v\|^2 \leq \|x - v\|^2 + \|Tx - x\|^2$ are equivalent. Then,

$$\mathscr{A}(T) = \{\ v \in H :\ \langle Tx - x, x - v\rangle \leq 0 \quad \text{for all} \quad x \in C\ \} = \cap_{x \in C} B_x.$$

Since each B_x is closed and convex, we have again that $\mathscr{A}(T)$ is closed and convex.

We present an example to see that $v \in \mathscr{A}(T) \cap C$ does not imply $v \in F(T)$.

Example 3.5. Let $C = [0,1] \subset R$. Define a self–mapping T on C by

$$Tx = -x/2 + 1 \quad \text{if} \quad x \in [0, 1/2], \qquad Tx = -x/2 + 1/2 \quad \text{if} \quad x \in (1/2, 1].$$

It is obvious that $F(T) = \emptyset$ and $\langle Tx - x, x - y \rangle = (Tx - x)(x - y)$. We show $y \notin \mathscr{A}(T)$ if $y \in [0, 1/2)$. Let $y \in [0, 1/2)$ and set $x = y/2 + 1/4$. Then, we have $y < x < 1/2$ and $Tx > 3/4$. That is, $x - y > 0$ and $Tx - x > 1/4$. This implies $(Tx - x)(x - y) > 0$. We also show $z \notin \mathscr{A}(T)$ if $z \in (1/2, 1]$. Let $z \in (1/2, 1]$ and set $x = z/2 + 1/4$. Then, we have $1/2 < x < z$ and $Tx < 1/4$. That is, $x - z < 0$ and $Tx - x < -1/4$. This implies $(Tx - x)(x - z) > 0$. Furthermore, we can see that $(Tx - x)(x - \frac{1}{2}) \le 0$ for $x \in [0, 1]$. Thus we have $\mathscr{A}(T) \cap C = \{1/2\}$ and $F(T) = \emptyset$.

Let C be a bounded, closed and convex subset of R, that is, C is a closed interval. For example, we can easily see that the condition $\emptyset \ne F(T) = A(T) \cap C \subset \mathscr{A}(T) \cap C$ holds if T is nonexpansive. Also, non–increasing and continuous self–mappings on C are typical examples which satisfy the condition $\emptyset \ne F(T) = \mathscr{A}(T) \cap C$. In particular, we already have a non–increasing and continuous self–mapping T on C such that $A(T) \cap C = \emptyset$ and $\emptyset \ne F(T) = \mathscr{A}(T) \cap C$. Here, we give a self–mapping T on $C = [0, 1]$ satisfying $\emptyset \ne A(T) \cap C \ne \mathscr{A}(T) \cap C$.

Example 3.6. Let $C = [0,1] \subset R$. Let T be the self–mapping on C defined by

$$\begin{aligned} &Tx = -2x + 1 \quad \text{if} \;\; x \in [0, 1/3), \qquad Tx = x \quad \text{if} \;\; x \in [1/3, 2/3], \\ &Tx = -2x + 2 \quad \text{if} \;\; x \in (2/3, 1]. \end{aligned}$$

Then, one can easily see the facts that $F(T) = [1/3, 2/3] \subset \mathscr{A}(T) \cap C$, $1/2 \in A(T) \cap C$ and $1/3 \in (\mathscr{A}(T) \cap C) \backslash (A(T) \cap C)$.

4. Lemmas

We prepare lemmas needed in the sequel. Lemma 4.1 is due to Tan and Xu [30].

Lemma 4.1. *Let $\{a_n\}$ be a sequence of non–negative real numbers and $\{b_n\}$ be a sequence of non–negative real numbers with $\sum_{j=1}^{\infty} b_j < \infty$. Suppose $a_{n+1} \le a_n + b_n$ for $n \in N$. Then, $\{a_n\}$ converges to some $c \in [0, \infty)$.*

Many researchers take the following assertion or a similar assertion in their articles; for example, see Weng [31], Xu [33] and Aoyama et.al. [1].

Lemma 4.2. *Let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of non–negative real numbers and let $\{b_n\}$ be a sequence of real numbers which satisfies $\limsup_n b_n \le 0$. Let $\{c_n\}$ be a sequence of non–negative real numbers with $\sum_{n=1}^{\infty} c_n < \infty$. Suppose $a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n + c_n$ for all $n \in N$. Then $\lim_n a_n = \limsup_n a_n = 0$.*

Lemma 4.3. *A Hilbert space H has the Opial property [22]. That is, if $\{u_n\}$ is a sequence in H which converges weakly to $u \in H$, then, for $v \in H$ with $v \ne u$,*

$$\liminf_n \|u_n - u\| < \liminf_n \|u_n - v\|.$$

Lemmas 4.4–4.7 and Lemma 7.3 play important roles in this article.

Lemma 4.4. *Let $k \in [0,1)$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. Let S be the mapping defined by*

$$Sx = (kI + (1-k)T)x \qquad \text{for} \quad x \in C.$$

Then, $F(T) = F(S)$ and $A(S) = \mathscr{A}_k(T)$.

Proof. In our setting, it is known that $F(T) = F(S)$ holds. We show $A(S) = \mathscr{A}_k(T)$.

Let $x \in C$ and $v \in H$. Set $\mathscr{N} = \|x-v\|^2 - \|Sx - v\|^2$. Using Lemma 3.2, we have

$$\begin{aligned}\mathscr{N} &= \|x-v\|^2 - \|(kx + (1-k)Tx) - v\|^2 \\ &= \|x-v\|^2 - (k\|x-v\|^2 + (1-k)\|Tx-v\|^2 - k(1-k)\|x-v-(Tx-v)\|^2) \\ &= (1-k)(\|x-v\|^2 + k\|x-Tx\|^2 - \|Tx-v\|^2).\end{aligned}$$

By $k \in [0,1)$, this equality implies that $v \in A(S)$ if and only if $v \in \mathscr{A}_k(T)$.

Note that we claim neither $F(T) \neq \emptyset$ nor $\mathscr{A}_k(T) \neq \emptyset$. □

Lemma 4.5. *Let $k \in [0,1)$ and $c \in [k,1)$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. Let S and S' be mappings defined by*

$$Sx = (kI + (1-k)T)x, \quad S'x = (cI + (1-c)T)x \qquad \text{for} \quad x \in C.$$

Then, the followings hold.

(1) *T is k–demi–contractive if and only if S is quasi–nonexpansive.*
(2) *Suppose T is k–demi–contractive. Then S' is quasi–nonexpansive and*

$$\mathscr{A}_k(T) \cap C = F(T) = F(S') = A(S') \cap C.$$

Proof. We prove (1). Assume that T is k–demi–contractive. Then we know that $\emptyset \neq F(T) \subset \mathscr{A}_k(T) \cap C$. By Theorem 3.3 (4), we have $F(T) = \mathscr{A}_k(T) \cap C$. By Lemma 4.4, we have $\emptyset \neq F(S) = F(T) = \mathscr{A}_k(T) \cap C = A(S) \cap C$. This implies that S is quasi–nonexpansive. Assume that S is quasi–nonexpansive. Then we know $\emptyset \neq F(S) \subset A(S) \cap C$. By Theorem 3.3 (4) and Lemma 4.4, we have $\emptyset \neq F(T) = F(S) = A(S) \cap C = \mathscr{A}_k(T) \cap C$. This implies that T is k–demi–contractive.

We prove (2). If T is k–demi–contractive then T is c–demi–contractive. Then, by (1), we have that S' is quasi–nonexpansive. Also, we have $\mathscr{A}_c(T) \cap C = F(T) = F(S') = A(S') \cap C$ and $\mathscr{A}_c(T) \cap C = F(T) = \mathscr{A}_k(T) \cap C$. □

The following lemma is closely connected with Zhou's result [34].

Lemma 4.6. *Let $k \in [0,1)$ and $c \in [k,1)$. Let C be a subset of a Hilbert space H. Let T be a mapping of C into H. Let S and S' be mappings defined by*

$$Sx = (kI + (1-k)T)x, \quad S'x = (cI + (1-c)T)x \qquad \text{for} \quad x \in C.$$

Then, the followings hold.

(1) *T is k–strictly pseudo–contractive if and only if S is nonexpansive.*
(2) *Suppose T is k–strictly pseudo–contractive. Then S' is nonexpansive and*

$$\mathscr{A}_k(T) \cap C = F(T) = F(S') = A(S') \cap C.$$

Proof. Let $x, y \in C$ and set $\mathscr{N} = \|x-y\|^2 - \|Sx - Sy\|^2$. Using Lemma 3.2, we have

$$\begin{aligned}\mathscr{N} &= \|x-y\|^2 - \|(kx + (1-k)Tx) - (ky + (1-k)Ty)\|^2 \\ &= \|x-y\|^2 - \|k(x-y) + (1-k)(Tx - Ty)\|^2 \\ &= \|x-y\|^2 - (k\|x-y\|^2 + (1-k)\|Tx - Ty\|^2 - k(1-k)\|x - y - (Tx - Ty)\|^2) \\ &= (1-k)(\|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2 - \|Tx - Ty\|^2).\end{aligned}$$

We prove (1). By $k \in [0, 1)$ and this equality, T is k–strictly pseudo–contractive if and only if S is nonexpansive. We prove (2). Since T is k–strictly pseudo–contractive, T is c–strictly pseudo–contractive. Then, S' is nonexpansive, that is, $F(T) = F(S') \subset A(S')$. In the case of $F(T) \neq \emptyset$, by Lemma 4.5, we know $\mathscr{A}_k(T) \cap C = F(T) = F(S') = A(S') \cap C$. We note that this equality holds in the case $F(T) = \emptyset$. Note that we do not claim $F(T) \neq \emptyset$. □

Let C be a subset of H and T be a mapping of C into H. Let $\{x_n\}$ be a sequence in C which converges weakly to $u \in C$ and satisfies $\lim_n \|Tx_n - x_n\| = 0$. $I - T$ is said to be demiclosed at 0 if $u \in F(T)$ always holds for such $\{x_n\}$ and u.

Let $\{y_n\}$ be a sequence in C. Then, $\lim_n \|Ty_n - y_n\| = 0$ and $\lim_n \|Ty_n - y_n\|^2 = 0$ are equivalent. In the sequel, we use this fact without notice.

Lemma 4.7. *Let $k \in [0, 1)$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. Let S be the mapping defined by*

$$Sx = (kI + (1-k)T)x \qquad \text{for} \quad x \in C.$$

Then, for any sequence $\{u_n\}$ in C, the following holds.

$$\lim_n \|Tu_n - u_n\| = 0 \quad \text{if and only if} \quad \lim_n \|Su_n - u_n\| = 0.$$

Furthermore, $I - T$ is demiclosed at 0 if and only if $I - S$ is demiclosed at 0.

Proof. It is easy to see that, for $n \in N$,

$$\|Su_n - u_n\| = \|(ku_n + (1-k)Tu_n) - u_n\| = (1-k)\|Tu_n - u_n\|.$$

By $k \in [0, 1)$, $\lim_n \|Tu_n - u_n\| = 0$ and $\lim_n \|Su_n - u_n\| = 0$ are equivalent. By Lemma 4.4, we know $F(T) = F(S)$. These imply that $I - T$ is demiclosed at 0 if and only if $I - S$ is demiclosed at 0. □

Proofs of Lemmas 4.4–4.7 are easy. However, we think that these assertions are so interesting. In the sequel, sometimes we use these lemmas without notice.

The following lemma due to Marino and Xu [21] is a version of Browder's demiclosed principle in the Hilbert space setting.

Lemma 4.8. *Let $k \in [0, 1)$. Let C be a subset of a Hilbert space H and let T be a k–strictly pseudo–contractive mapping of C into H. Let S be the mapping defined by $Sx = (kI + (1-k)T)x$ for $x \in C$. Suppose $\{u_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Tu_n - u_n\| = 0$. Then, $u \in F(T) = F(S)$.*

Proof. By $k \in [0, 1)$, $\lim_n \|Tu_n - u_n\| = 0$ and $\lim_n \|Su_n - u_n\| = 0$ are equivalent. By Lemma 4.6, S is nonexpansive and $F(T) = F(S)$. Then, we prove $u \in F(S)$.

Arguing by contradiction, assume $Su \neq u$. Then, by the Opial property, we have

$$\begin{aligned}\liminf_n \|u_n - u\| &< \liminf_n \|u_n - Su\| \\ &\leq \liminf_n(\|u_n - Su_n\| + \|Su_n - Su\|) \\ &\leq \liminf_n \|u_n - u\|.\end{aligned}$$

This is a contradiction. Thus we have $u \in F(S) = F(T)$. $\square$

Lemma 4.9. *Let C be a subset of a Hilbert space H. Let $\{u_n\}$ be a sequence in H such that $\{\|u_n - w\|\}$ converges for each $w \in C$. Suppose $\{u_{n_i}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converge weakly to $u, v \in C$, respectively. Then $u = v$.*

Proof. Let $w \in C$. Then, since $\{\|u_n - w\|\}$ converges, any subsequence of $\{\|u_n - w\|\}$ converges to the same real number. Arguing by contradiction, we assume $u \neq v$. Then, by $u, v \in C$ and the Opial property, we have the followings:

$$\begin{aligned}\liminf_i \|u_{n_i} - u\| &< \liminf_i \|u_{n_i} - v\| = \liminf_j \|u_{n_j} - v\|, \\ \liminf_j \|u_{n_j} - v\| &< \liminf_j \|u_{n_j} - u\| = \liminf_i \|u_{n_i} - u\|.\end{aligned}$$

That is, we have $\liminf_i \|u_{n_i} - u\| < \liminf_i \|u_{n_i} - u\|$. This is a contradiction. $\square$

Lemma 4.10. *Let $a \in [0, 1]$. Let C be a subset of a Hilbert space H and T be a mapping of C into H. Let $x \in C$ and set $Sx = ax + (1-a)Tx$. Then, the followings hold.*

(1) *Suppose $u \in \mathscr{A}(T)$. Then,*
$$\|Sx - u\|^2 \leq \|x - u\|^2 + (1-a)^2\|Tx - x\|^2.$$
(2) *Suppose $v \in A(T)$. Then,*
$$\|Sx - v\|^2 \leq \|x - v\|^2 - a(1-a)\|Tx - x\|^2.$$

Proof. Let $x \in C$ and $z \in H$. Then, by Lemma 3.2, we have

$$\begin{aligned}\|Sx - z\|^2 &= \|ax + (1-a)Tx - z\|^2 \\ &= a\|x - z\|^2 + (1-a)\|Tx - z\|^2 - a(1-a)\|Tx - x\|^2.\end{aligned}$$

Suppose $u \in \mathscr{A}(T)$ and $v \in A(T)$. Then, by this equality, we have the followings:

$$\begin{aligned}\|Sx - u\|^2 &\leq a\|x - u\|^2 + (1-a)(\|Tx - x\|^2 + \|x - u\|^2) - a(1-a)\|Tx - x\|^2 \\ &= \|x - u\|^2 + (1-a)^2\|Tx - x\|^2, \\ \|Sx - v\|^2 &\leq a\|x - v\|^2 + (1-a)\|x - v\|^2 - a(1-a)\|Tx - x\|^2 \\ &= \|x - v\|^2 - a(1-a)\|Tx - x\|^2.\end{aligned}$$

$\square$

Lemma 4.11. *Let $\{a_n\}$ be a sequence in $[0, 1]$. Let C be a subset of a Hilbert space H and T be a self–mapping on C. Assume that there is a sequence $\{u_n\}$ in C such that $u_{n+1} = a_n u_n + (1 - a_n)Tu_n$ for $n \in N$. Then, the followings hold.*

(1) *Suppose $v \in A(T)$. Then, $\{\|u_n - v\|^2\}$ is non–increasing and converges.*
(2) *Suppose $A(T) \neq \emptyset$, $a, b \in (0, 1)$ with $a \leq b$, and $\{a_n\}$ is a sequence in $[a, b]$. Then $\sum_{n=1}^{\infty} \|Tu_n - u_n\|^2 < \infty$.*

(3) *Suppose* $A(T) \neq \emptyset$, $\{a_n\}$ *satisfies* $\sum_{n=1}^{\infty} a_n(1-a_n) = \infty$, *and*

$$(\mathrm{O_S}) \quad \|Tu_{n+1} - Tu_n\| \leq (1-a_n)\|u_n - Tu_n\| \quad \text{for} \quad n \in N.$$

Then, $\lim_n \|Tu_n - u_n\| = 0$.

Proof. We prove (1). Assume $v \in A(T)$. By Lemma 4.10, we have

$$\|u_{n+1} - v\|^2 \leq \|u_n - v\|^2 - a_n(1-a_n)\|Tu_n - u_n\|^2 \quad \text{for} \quad n \in N.$$

Then $\{\|u_n - v\|^2\}$ is non–increasing and converges. By this inequality, we have

$$a_n(1-a_n)\|Tu_n - u_n\|^2 \leq \|u_n - v\|^2 - \|u_{n+1} - v\|^2 \quad \text{for} \quad n \in N. \tag{4.1}$$

We prove (2). We know $a(1-b) \leq a_n(1-a_n)$ for $n \in N$. Then, by (4.1), we have

$$a(1-b)\sum_{j=1}^{n} \|Tu_j - u_j\|^2 \leq \|u_1 - v\|^2 - \|u_{n+1} - v\|^2 \leq \|u_1 - v\|^2$$

for $n \in N$. By $0 < a(1-b)$, we have $\sum_{n=1}^{\infty} \|Tu_n - u_n\|^2 < \infty$.

We prove (3). By (4.1), we have that, for $n \in N$,

$$\sum_{j=1}^{n} a_j(1-a_j)\|Tu_j - u_j\|^2 \leq \|u_1 - v\|^2 - \|u_{n+1} - v\|^2 \leq \|u_1 - v\|^2.$$

Then we have $\sum_{j=1}^{\infty} a_j(1-a_j)\|Tu_j - u_j\|^2 < \infty$. By $\sum_{j=1}^{\infty} a_j(1-a_j) = \infty$, we have $\liminf_n \|Tu_n - u_n\|^2 = 0$. On the other hand, by $(\mathrm{O_S})$, we have that, for $n \in N$,

$$\begin{aligned}
\|Tu_{n+1} - u_{n+1}\| &\leq \|Tu_{n+1} - Tu_n\| + \|Tu_n - u_{n+1}\| \\
&\leq (1-a_n)\|Tu_n - u_n\| + a_n\|Tu_n - u_n\| \\
&= \|Tu_n - u_n\|.
\end{aligned}$$

Then $\{\|Tu_n - u_n\|^2\}$ is non–increasing and converges. Thus we have

$$\lim_n \|Tu_n - u_n\|^2 = \liminf_n \|Tu_n - u_n\|^2 = 0.$$

We note that T satisfies the condition $(\mathrm{O_S})$ if T is nonexpansive; see [14]. □

5. Convergence theorems I

Let C be a subset of a Hilbert space H and S be a mapping of C into H. Under the condition $A(S) \neq \emptyset$, we prove some convergence theorems.

Theorem 5.1. *Let* $a, b \in (0,1)$ *with* $a \leq b$ *and* $\{a_n\}$ *be a sequence in* $[a,b]$. *Let* C *be a compact subset of a Hilbert space* H. *Let* S *be a continuous self-mapping on* C *satisfying* $F(S) \subset \mathscr{A}(S)$ *and* $A(S) \neq \emptyset$. *Suppose there is a sequence* $\{u_n\}$ *in* C *such that*

$$u_{n+1} = a_n u_n + (1-a_n)Su_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ *converges strongly to some* $u \in F(S)$.

Proof. By $A(S) \neq \emptyset$ and Lemma 4.11 (2), we have $\sum_{n=1}^{\infty} \|Su_n - u_n\|^2 < \infty$. Then $\lim_n \|Su_n - u_n\|^2 = 0$. Since C is compact, $\{u_n\}$ has a convergent subsequence. Let $\{u_{n_j}\}$ be a subsequence of $\{u_n\}$ which converges strongly to some $u \in C$. Then we know $\lim_j \|u_{n_j} - u\| = 0$ and $\lim_j \|Su_{n_j} - u_{n_j}\| = 0$. It is easy to see that

$$\|Su - u\| \leq \|Su - Su_{n_j}\| + \|Su_{n_j} - u_{n_j}\| + \|u_{n_j} - u\|$$

for $j \in N$. Since S is continuous at u, we have $Su = u$, that is, we have $u \in F(S)$.

By $u \in F(S) \subset \mathscr{A}(S)$ and Lemma 4.10, we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 + \|Su_n - u_n\|^2$$

for $n \in N$. Then, by $\sum_{n=1}^{\infty} \|Su_n - u_n\|^2 < \infty$ and Lemma 4.1, $\{\|u_n - u\|^2\}$ converges. Since $\{\|u_{n_j} - u\|^2\}$ converges to 0, $\{\|u_n - u\|^2\}$ itself converges to 0. Thus $\{u_n\}$ converges strongly to $u \in F(S)$. □

Under the assumptions of Theorem 5.1, we had $\emptyset \neq F(S) \subset \mathscr{A}(S)$. However, we cannot apply Shauder's theorem to have this condition.

By Lemmas 4.5 and 4.7, the following theorem is derived from Theorem 5.1.

Theorem 5.2. *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a compact subset of a Hilbert space H and T be a continuous self–mapping on C. Assume that one of the followings holds.*

(1) *T is hemi–contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.*
(2) *T is k–demi–contractive. S is the mapping defined by $S = kI + (1-k)T$.*
(3) *T is quasi–nonexpansive. S is the mapping defined by $S = T$.*

Suppose S is a self–mapping on C and there is a sequence $\{u_n\}$

$$u_{n+1} = a_n u_n + (1-a_n)Su_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ converges strongly to some $u \in F(T)$.

Let R^2 be a 2-dimensional Euclidean space. Let C be the compact and convex subset defined by $C = \{(x_1, x_2) \in R^2 : x_1, x_2 \in [0,1],\ x_1 + x_2 \leq 1\}$. Define a self–mapping T on C by

$$T(x_1, x_2) = \left(\frac{1}{2}(1 + x_1 - x_2)\,,\, x_2\right) \quad \text{for} \quad (x_1, x_2) \in C.$$

Let $u_1 \in C$ and $\{u_n\}$ be the sequence generated by $u_{n+1} = (u_n + Tu_n)/2$ for $n \in N$.

Under this setting, we can easily see the followings:

$$F(T) = \{(x_1, x_2) \in C : x_1 + x_2 = 1\},$$
$$A(T) = \{(x_1, x_2) \in R^2 : x_1 \geq 1\}, \qquad A(T) \cap C = \mathscr{A}(T) \cap C = \{(1,0)\}.$$

Since $F(T) \not\subset \mathscr{A}(T) \cap C$, T is not hemi–contractive. However, it is obvious that $\{u_n\}$ converges to a fixed point of T. For such T, maybe we do not have a convergence theorem. Then we give a convergence theorem for such mappings.

Theorem 5.3. *Let $a, b \in (0,1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a,b]$. Let C be a compact and convex subset of a Hilbert space H. Let T be a continuous self–mapping on C with $A(T) \neq \emptyset$. Let $u_1 \in C$ and $\{u_n\}$ be the sequence defined by*

$$u_{n+1} = a_n u_n + (1-a_n)Tu_n \qquad \text{for} \quad n \in N.$$

Suppose $F(T) \subset P_C(A(T))$, where P_C is the metric projection of H onto C. Then $\{u_n\}$ converges strongly to some $u \in F(T)$.

Proof. In the same way as in the proof of Theorem 5.1, we know that $\{u_n\}$ has a subsequence $\{u_{n_j}\}$ which converges strongly to some $u \in F(T)$. By $F(T) \subset$

$P_C(A(T))$, there is $v \in A(T)$ such that $u = P_C v$. By Lemma 4.11 (1), $\{\|u_n - v\|^2\}$ converges. Then,

$$\lim_n \|u_n - v\|^2 = \lim_j \|u_{n_j} - v\|^2 = \|u - v\|^2.$$

By $\{u_n\} \subset C$ and $u = P_C v$, we have

$$\|v - u\|^2 + \|u - u_n\|^2 \leq \|v - u_n\|^2 \qquad \text{for} \quad n \in N.$$

Then $\limsup_n \|u - u_n\|^2 \leq 0$. Thus $\{u_n\}$ converges strongly to $u \in F(T)$.

Note that, in this setting, by Lemma 2.2, $F(T) = P_C(A(T))$ holds. □

We consider weak convergence theorems in the case $A(S) \neq \emptyset$ and $F(S) \subset \mathscr{A}(S)$. To have the following results, we have to assume demicloseness at 0 of $I - S$.

Theorem 5.4. *Let $a, b \in (0, 1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a, b]$. Let C be a weakly closed subset of a Hilbert space H. Let S be a self–mapping on C such that $F(S) \subset \mathscr{A}(S)$, $A(S) \neq \emptyset$, and $I - S$ is demiclosed at 0. Suppose there is a sequence $\{u_n\}$ in C such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(S)$.

Proof. By $A(S) \neq \emptyset$ and Lemma 4.11 (2), we have $\sum_{n=1}^{\infty} \|Su_n - u_n\|^2 < \infty$ and $\lim_n \|Su_n - u_n\|^2 = 0$. Also, by Lemma 4.11 (1), we know that $\{u_n\}$ is bounded. So, $\{u_n\}$ has a weakly convergent subsequence. Let $\{u_{n_j}\}$ be a subsequence of $\{u_n\}$ which converges weakly to some $u \in C$. We know $\lim_j \|Su_{n_j} - u_{n_j}\|^2 = 0$. Then, since $I - S$ is demiclosed at 0, we have $u \in F(S)$. Let $v \in F(S) \subset \mathscr{A}(S)$. By Lemma 4.10, we have

$$\|u_{n+1} - v\|^2 \leq \|u_n - v\|^2 + \|Su_n - u_n\|^2$$

for $n \in N$. By $\sum_{n=1}^{\infty} \|Su_n - u_n\|^2 < \infty$ and Lemma 4.1, $\{\|u_n - v\|^2\}$ converges.

We confirmed that $\{\|u_n - v\|\}$ converges for each $v \in F(S)$. We also confirmed that every weakly convergent subsequence of $\{u_n\}$ converges weakly to a point of $F(S)$. Then, by Lemma 4.9, every weakly convergent subsequence of $\{u_n\}$ converges weakly to $u \in F(S)$. Thus $\{u_n\}$ itself converges weakly to $u \in F(S)$. □

By Lemmas 4.5 and 4.7, the following theorem is derived from Theorem 5.4.

Theorem 5.5. *Let $a, b \in (0, 1)$ with $a \leq b$ and $\{a_n\}$ be a sequence in $[a, b]$. Let C be a weakly closed subset of a Hilbert space H and T be a self–mapping on C such that $I - T$ is demiclosed at 0. Assume that one of the followings hold.*

(1) *T is hemi–contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.*
(2) *T is k–demi–contractive. S is the mapping defined by $S = kI + (1 - k)T$.*
(3) *T is quasi–nonexpansive. S is the mapping defined by $S = T$.*

Suppose S is a self–mapping on C and there is a sequence $\{u_n\}$

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

6. Convergence theorems II

We begin this section with presenting Theorem 6.1 due to Takahashi and Takeuchi [27]. Then, by Lemmas 4.4, 4.6, and Theorem 6.1, we will have Theorem 6.2.

Theorem 6.1. *Let C be a bounded subset of a Hilbert space H. Let S be a non-expansive self–mapping on C. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and*

$$v_{n+1} = Sv_n, \quad b_n = \frac{1}{n}\sum\nolimits_{t=1}^{n} v_t \qquad \text{for} \quad n \in N.$$

Then the followings hold.

(1) *$A(S)$ is non-empty, closed and convex.*
(2) *$\{b_n\}$ converges weakly to some $u \in A(S)$.*

Theorem 6.2. *Let $k \in [0, 1)$. Let C be a bounded subset of a Hilbert space H. Let T be a k–strictly pseudo–contractive self–mapping on C. Let S be the mapping defined by $Sx = (kI + (1-k)T)x$ for $x \in C$. Assume that S is a self–mapping on C. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by $v_1 \in C$ and*

$$v_{n+1} = Sv_n, \quad b_n = \frac{1}{n}\sum\nolimits_{t=1}^{n} v_t \qquad \text{for} \quad n \in N.$$

Then the followings hold.

(1) *$\mathscr{A}_k(T)$ is non-empty, closed and convex.*
(2) *$\{b_n\}$ converges weakly to some $u \in \mathscr{A}_k(T)$.*

Furthermore, if C is closed and convex then the followings hold.

(3) *$F(T)$ is non-empty, closed and convex.*
(4) *$\{b_n\}$ converges weakly to $u \in F(T)$.*

Proof. We show (1) and (2). By Lemmas 4.4 and 4.6, S is nonexpansive and $F(T) = F(S) \subset \mathscr{A}_k(T) = A(S)$. By Theorem 6.1, we have that $\mathscr{A}_k(T)$ is non–empty, closed and convex. We also have that $\{b_n\}$ converges weakly to some $u \in \mathscr{A}_k(T)$.

We show (3) and (4). Since both $\mathscr{A}_k(T)$ and C are closed and convex, so is $F(T) = \mathscr{A}_k(T) \cap C$. We can easily have $\{b_n\} \subset C$. Then, since C is weakly closed, we have $u \in \mathscr{A}_k(T) \cap C = F(T)$. In this setting, the assumption that S is a self–mapping on C is unnecessary. Furthermore, (3) is known. □

Remark 6.3. In Theorem 6.2 (1) and (2), convexity of C is not always necessary. Then, we give an example. We denote by Q the set of rational numbers. Let $k \in [0, 1) \cap Q$ and $C = (0, 1) \cap Q$. Then, for any self–mapping T on C, S is also a self–mapping. However, C is not convex.

Theorem 6.4 is Suzuki's theorem in the Hilbert space setting; see Suzuki [24]. We note that Chidume and Chidume [7] also proved the theorem independently. By Lemma 4.6 and Theorem 6.4, we will have Theorem 6.5 closely connected with Zhou's another result [34].

Theorem 6.4. *Let $c \in (0, 1)$. Let $\{a_n\}$ be a sequence in $[0, 1]$ satisfying $\lim_n a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Let C be a closed and convex subset of a Hilbert space H. Let*

S be a nonexpansive self–mapping on C with $F(S) \neq \emptyset$. Let $z, u_1 \in C$ and $\{u_n\}$ be the sequence defined by

$$u_{n+1} = a_n z + (1 - a_n)(cu_n + (1 - c)Su_n) \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to the point of $F(S)$ nearest to z.

Theorem 6.5. *Let $k \in [0, 1)$, $b \in (0, 1)$ and $c \in [k, 1)$. Let $\{a_n\}$ be a sequence in $[0, 1]$ satisfying $\lim_n a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Let C be a closed and convex subset of a Hilbert space H. Let T be a k–strictly pseudo–contractive self–mapping on C with $F(T) \neq \emptyset$ and S be the mapping defined by $Sx = (cI + (1 - c)T)x$ for $x \in C$. Let $z, u_1 \in C$ and $\{u_n\}$ be the sequence defined by*

$$u_{n+1} = a_n z + (1 - a_n)(bu_n + (1 - b)Su_n) \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to the point of $F(T)$ nearest to z.

Proof. Since T is k–strictly pseudo–contractive and $F(T) \neq \emptyset$, S is a nonexpansive self–mapping on C with $\emptyset \neq F(T) = F(S) = A(S) \cap C$. By Theorem 6.4, $\{u_n\}$ converges strongly to the point of $F(T) = F(S)$ nearest to z. $\square$

Recently, some researchers considered minimal norm problems for some nonlinear mappings. They presented some iterations to find the fixed point nearest to 0 for k–strictly pseudo–contractive mappings or k–demi–contractive mappings, in the special setting $C = H$. It seems that the condition $0 \in C = H$ is essential for their arguments. Let $k \in [0, 1)$ and $a \in (k, 1)$. Then, there is $b \in (0, 1)$ such that

$$a = k + b(1 - k) = b + (1 - b)k, \quad 1 - a = (1 - b)(1 - k). \tag{6.1}$$

In the same setting as Theorem 6.5, assume $0 \in C$ and set $c = k$ and $z = 0$. For $a \in (k, 1)$, we can take $b \in (0, 1)$ satisfying (6.1). Then under the conditions as above, the iteration in Theorem 6.5 becomes as follows:

$$\begin{aligned} u_{n+1} &= a_n 0 + (1 - a_n)(bu_n + (1 - b)(ku_n + (1 - k)Tu_n)) \\ &= (1 - a_n)(au_n + (1 - a)Tu_n) \qquad\qquad \text{for} \quad n \in N. \end{aligned}$$

Thus we have the following Halpern type convergence theorem which is a corollary of Theorem 6.5. It is obvious that we can apply Theorem 6.6 in the case of $C = H$.

Theorem 6.6. *Let $k \in [0, 1)$ and $a \in (k, 1)$. Let $\{a_n\}$ be a sequence in $[0, 1]$ satisfying $\lim_n a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$. Let C be a closed and convex subset of a Hilbert space H with $0 \in C$. Let T be a k–strictly pseudo–contractive self–mapping on C with $F(T) \neq \emptyset$. Let $u_1 \in C$ and $\{u_n\}$ be the sequence defined by*

$$u_{n+1} = (1 - a_n)(au_n + (1 - a)Tu_n) \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to the point of $F(T)$ nearest to 0.

In 2008, Maingé and Măruşter proved the following theorem (Theorem 4.1 in [19]).

Theorem 6.7. *Let $k \in [0, 1)$ and $b \in (0, 1 - k)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1)$ and $\{\beta_n\}$ be a sequence in $(0, b]$ such that*

$$\lim_n \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \lim_n \frac{\alpha_n}{\beta_n} = 0.$$

Let H be a Hilbert space and T be a k–demi–contractive mapping on H. Assume that $I - T$ is demiclosed at 0. Let $x_1 \in H$ and $\{x_n\}$ be the sequence defined by

$$y_n = (1 - \alpha_n)x_n, \quad x_{n+1} = (1 - \beta_n)y_n + \beta_n T y_n \qquad \text{for} \quad n \in N.$$

Then $\{x_n\}$ converges strongly to the point of $F(T)$ nearest to 0.

Motivated by Maingé and Măruşter [19], we present the following results.

Theorem 6.8. *Let $\{a_n\}$ be a sequence in $[0, 1)$ and $\{b_n\}$ be a sequence in $(0, 1)$ such that*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \lim_n \frac{a_n}{b_n(1 - b_n)} = 0.$$

Let C be a closed and convex subset of a Hilbert space H with $0 \in C$. Let S be a quasi–nonexpansive self–mapping on C such that $I - S$ is demiclosed at 0. For each $n \in N$, let U_n be the mapping defined by $U_n x = (b_n I + (1 - b_n)S)x$ for $x \in C$. Let $u_1 \in C$ and $\{u_n\}$ be the sequence defined by

$$u_{n+1} = (1 - a_n)U_n u_n \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to the point of $F(S)$ nearest to 0.

Proof. It is easy to see that each U_n is a self–mapping on C. By $0 \in C$, each $(1 - a_n)U_n$ is also a self–mapping on C. Then $\{u_n\}$ is well–defined. By the quasi–nonexpansiveness of S and Lemma 4.5, each U_n is also quasi–nonexpansive and

$$\emptyset \neq F(S) = A(S) \cap C = A(U_n) \cap C = F(U_n).$$

By Theorem 3.3, $F(S)$ is closed and convex. Then we can set $v = P_{F(S)}0 \in F(S)$, where $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Set $D = \{x \in C : \|x - v\| \leq \|u_1 - v\| + \|v\|\}$. Then we easily see that $0, v, u_1 \in D$ and D is bounded, closed and convex. For each $n \in N$, it is obvious that

$$\begin{aligned} \|U_n x - v\| &\leq \|x - v\| \leq \|u_1 - v\| + \|v\|, \\ \|aU_n x - v\| &\leq a\|U_n x - v\| + (1 - a)\|v\| \\ &\leq a(\|u_1 - v\| + \|v\|) + (1 - a)\|v\| \\ &\leq \|u_1 - v\| + \|v\| \end{aligned}$$

for $a \in [0, 1]$ and $x \in D$. Then each U_n and each $(1 - a_n)U_n$ are self–mappings on D. We have that $\{u_n\}$ and $\{U_n u_n\}$ are sequences in the weakly compact set D.

It is obvious that, for $n \in N$,

$$\|U_n u_n - u_n\| = \|(b_n u_n + (1 - b_n)Su_n) - u_n\| = (1 - b_n)\|Su_n - u_n\|. \tag{6.2}$$

We know $v \in F(S) \subset A(S)$ and $U_n u_n = b_n u_n + (1 - b_n)Su_n$. Then, by Lemma 4.10, we have the following:

$$\|U_n u_n - v\|^2 \leq \|u_n - v\|^2 - b_n(1 - b_n)\|Su_n - u_n\|^2 \qquad \text{for} \quad n \in N. \tag{6.3}$$

Let $\{u_k\}$ be a subsequence of $\{u_n\}$ such that a term u_i of $\{u_n\}$ is a term of $\{u_k\}$ if $a_i > 0$. We denote by P the index set of $\{u_k\}$. Also, let $\{u_l\}$ be a subsequence of $\{u_n\}$ such that a term u_i of $\{u_n\}$ is a term of $\{u_l\}$ if $a_i = 0$. We denote by V the index set of $\{u_l\}$. It is possible that $\{u_l\}$ has at most finite terms. However, by

$\sum_{n=1}^{\infty} a_n = \infty$, $\{u_k\}$ must have countable infinite terms. We can consider P and V as subsequences $\{k\}$ and $\{l\}$ of $N = \{n\}$, respectively.

Consider the subsequence $\{u_l\}$ of $\{u_n\}$. By (6.3) and $a_l = 0$, for $l \in V$, we have

$$\|u_{l+1} - v\|^2 = \|U_l u_l - v\|^2 \leq \|u_l - v\|^2 - b_l(1 - b_l)\|Su_l - u_l\|^2 \leq \|u_l - v\|^2. \tag{6.4}$$

For each $l \in V$, we set $K_l = 0$. Then we can rewrite (6.4) as follows:

$$\|u_{l+1} - v\|^2 \leq (1 - a_l)\|u_l - v\|^2 + a_l K_l \quad \text{for} \quad l \in V. \tag{6.5}$$

Consider the subsequence $\{u_k\}$ of $\{u_n\}$. For $k \in P$, set $\mathscr{N}_k = \|u_{k+1} - v\|^2$. Then, by (6.3), we have that, for $k \in P$,

$$\begin{aligned} \mathscr{N}_k &= \|(1 - a_k)(U_k u_k - v) - a_k v\|^2 \\ &\leq (1 - a_k)\|U_k u_k - v\|^2 + a_k^2\|v\|^2 - 2a_k(1 - a_k)\langle U_k u_k - v, v\rangle \\ &\leq \left((1 - a_k)\|u_k - v\|^2 - (1 - a_k) \times \frac{a_k}{a_k} \times b_k(1 - b_k)\|Su_k - u_k\|^2\right) \\ &\qquad + a_k^2\|v\|^2 + 2a_k(1 - a_k)\langle U_k u_k - v, -v\rangle. \end{aligned} \tag{6.6}$$

For each $k \in P$, we set $y_k = U_k u_k$ and

$$K_k = -\frac{1}{a_k}(1 - a_k)b_k(1 - b_k)\|Su_k - u_k\|^2 + a_k\|v\|^2 + 2(1 - a_k)\langle y_k - v, -v\rangle. \tag{6.7}$$

Then we can rewrite (6.6) as follows:

$$\|u_{k+1} - v\|^2 \leq (1 - a_k)\|u_k - v\|^2 + a_k K_k \quad \text{for} \quad k \in P. \tag{6.8}$$

Let $k \in P$. Then, by (6.7) and $a_k, b_k \in (0, 1)$, it is obvious that

$$K_k \leq a_k\|v\|^2 + 2(1 - a_k)\langle y_k - v, -v\rangle \leq \|v\|^2 + 2\|y_k - v\|\,\|v\|. \tag{6.9}$$

Since D is weakly compact, we know $\limsup_k K_k < \infty$.

We show $\limsup_k K_k \leq 0$. Arguing by contradiction, assume $0 < \limsup_k K_k$. Then there is a subsequence $\{k_j\}$ of $\{k\}$ satisfying

$$0 < \limsup_k K_k = \lim_j K_{k_j}.$$

Let $\{u_{k_j}\}$ be the subsequence of $\{u_k\}$ corresponding to $\{k_j\}$. Since D is weakly compact, $\{u_{k_j}\}$ has a subsequence which converges weakly to some $u \in D$. Without loss of generality, we can assume that $\{u_{k_j}\}$ converges weakly to $u \in D$.

It is obvious that $0 < K_{k_j}$ for sufficiently large j. By (6.7), we can rewrite $0 < K_{k_j}$ as follows:

$$\frac{1}{a_{k_j}}(1 - a_{k_j})b_{k_j}(1 - b_{k_j})\|Su_{k_j} - u_{k_j}\|^2 < a_{k_j}\|v\|^2 + 2(1 - a_{k_j})\langle y_{k_j} - v, -v\rangle.$$

Then, since D is bounded and $\lim_j a_{k_j} = 0$, there is $L > 0$ such that

$$\frac{b_{k_j}(1 - b_{k_j})}{a_{k_j}}\|Su_{k_j} - u_{k_j}\|^2 < \frac{a_{k_j}}{(1 - a_{k_j})}\|v\|^2 + 2\|y_{k_j} - v\|\|v\| < L$$

for sufficiently large j. That is, for sufficiently large j, the following holds:

$$\|Su_{k_j} - u_{k_j}\|^2 < \frac{a_{k_j}}{b_{k_j}(1 - b_{k_j})}L. \tag{6.10}$$

Recall $\lim_j a_{k_j}/b_{k_j}(1-b_{k_j})=0$. Then, by (6.10) and (6.2), we have

$$\lim_j \|Su_{k_j}-u_{k_j}\|^2=0, \quad \lim_j \|U_{k_j}u_{k_j}-u_{k_j}\|^2=\lim_j \|y_{k_j}-u_{k_j}\|^2=0.$$

Since $I-S$ is demiclosed at 0, we have $u\in F(S)$. It is obvious that $\{y_{k_j}\}$ also converges weakly to u. Confirm that $\lim_j a_{k_j}=0$, $v=P_{F(S)}0$, and $\{y_{k_j}\}$ converges weakly to $u\in F(S)$. Then, by (6.9), we have

$$\lim_j K_{k_j}\le \lim_j\left(a_{k_j}\|v\|^2+2(1-a_{k_j})\langle y_{k_j}-v,-v\rangle\right)=2\langle u-v,0-v\rangle\le 0.$$

Thus we have $0<\limsup_k K_k=\lim_j K_{k_j}\le 0$. This is a contradiction.

By (6.5) and (6.8), we have

$$\|u_{n+1}-v\|^2\le(1-a_n)\|u_n-v\|^2+a_nK_n \quad \text{for} \quad n\in N. \tag{6.11}$$

On the other hand, we know $\limsup_k K_k\le 0$ and $K_l=0$ for all $l\in V$. These imply $\limsup_n K_n\le 0$. Thus, by (6.11), $\sum_{n=1}^{\infty}a_n=\infty$, and Lemma 4.2, we have $\lim_n \|u_n-v\|^2=0$. Hence, $\{u_n\}$ converges strongly to $v=P_{F(S)}0$. □

Theorem 6.9. *Let $k\in[0,1)$ and $c\in[k,1)$. Let $\{a_n\}$ be a sequence in $[0,1)$ and $\{b_n\}$ be a sequence in $(0,1)$ such that*

$$\lim_n a_n=0, \qquad \sum\nolimits_{n=1}^{\infty}a_n=\infty, \qquad \lim_n\frac{a_n}{b_n(1-b_n)}=0.$$

Let C be a closed and convex subset of a Hilbert space H with $0\in C$. Let T be a k–demi–contractive self–mapping on C such that $I-T$ is demiclosed at 0. Let S be the mapping defined by $Sx=(cI+(1-c)T)x$ for $x\in C$. For each $n\in N$, let U_n be the mapping defined by $U_nx=(b_nI+(1-b_n)S)x$ for $x\in C$. Let $u_1\in C$ and $\{u_n\}$ be the sequence defined by

$$u_{n+1}=(1-a_n)U_nu_n \qquad \text{for} \quad n\in N.$$

Then $\{u_n\}$ converges strongly to the point of $F(T)$ nearest to 0.

Proof. Recall Lemmas 4.5 and 4.7. Then, since T is k–demi–contractive, S is quasi–nonexpansive and $\emptyset\ne F(T)=F(S)=A(S)\cap C$. Since $I-T$ is demiclosed at 0, $I-S$ is also demiclosed at 0. Then, by Theorem 6.8, we have the result. □

We can deduce the following assertion from Theorem 6.9.

Theorem 6.10. *Let $k\in[0,1)$ and $b\in(0,1-k)$. Let $\{\alpha_n\}$ be a sequence in $[0,1)$ and $\{\beta_n\}$ be a sequence in $(0,b]$ such that*

$$\lim_n \alpha_n=0, \qquad \sum\nolimits_{n=1}^{\infty}\alpha_n=\infty, \qquad \lim_n\frac{\alpha_n}{\beta_n}=0.$$

Let C be a closed and convex subset of a Hilbert space H with $0\in C$. Let T be a k–demi–contractive self–mapping on C. Assume that $I-T$ is demiclosed at 0. Let $\{y_n\}$ and $\{x_n\}$ be sequences defined by $y_1\in C$ and

$$x_n=(1-\beta_n)y_n+\beta_nTy_n, \quad y_{n+1}=(1-\alpha_n)x_n \qquad \text{for} \quad n\in N.$$

Then $\{y_n\}$ converges strongly to the point of $F(T)$ nearest to 0.

Proof. We can rewrite the iteration as follows:

$$y_{n+1} = (1-\alpha_n)((1-\beta_n)y_n + \beta_n T y_n) \qquad \text{for} \quad n \in N. \tag{6.12}$$

Set $a_n = \alpha_n$ for $n \in N$ and $c = k$. Then $\lim_n a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$.

Also set $d = b/(1-k)$. By $0 < \beta_n \leq b < 1-k$, we have

$$0 < \frac{\beta_n}{1-k} \leq \frac{b}{1-k} = d < 1 \qquad \text{for} \quad n \in N.$$

Set $b_n = 1 - \beta_n/(1-k)$ for $n \in N$. Then $\{b_n\}$ is a sequence in $(0,1)$. Note that $1 - b_n = \beta_n/(1-k) \in (0, d\,]$ and $b_n \in [1-d, 1)$ hold for $n \in N$. We easily see that

$$0 < \frac{a_n}{(1-b_n)b_n} \leq \frac{1-k}{\beta_n} \times \frac{a_n}{(1-d)} = \frac{1-k}{1-d}\frac{\alpha_n}{\beta_n} \qquad \text{for} \quad n \in N.$$

By $\lim_n \alpha_n/\beta_n = 0$, we have $\lim_n a_n/b_n(1-b_n) = 0$. Thus $\{a_n\}$ and $\{b_n\}$ satisfy the conditions in Theorem 6.9. On the other hand, it is easy to see that

$$\beta_n = (1-b_n)(1-k), \quad (1-\beta_n) = b_n + (1-b_n)k \qquad \text{for} \quad n \in N.$$

Let $u_1 = y_1$. Then consider the iteration in Theorem 6.9, in the setting as above. Recall $c = k$. Then it is easy to see that the iteration becomes as follows:

$$\begin{aligned} u_{n+1} &= (1-a_n)U_n u_n \\ &= (1-a_n)(b_n u_n + (1-b_n)S u_n) \\ &= (1-a_n)(b_n u_n + (1-b_n)(k u_n + (1-k)T u_n)) \\ &= (1-\alpha_n)((1-\beta_n)u_n + \beta_n T u_n) \qquad \text{for} \quad n \in N. \end{aligned}$$

So, this iteration corresponds to (6.12). By Theorem 6.9, $\{y_n\}$ converges strongly to $v = P_{F(T)}0$. Since $\lim_n \alpha_n = 0$ and $\|y_{n+1}\| = (1-\alpha_n)\|x_n\|$, $\{x_n\}$ is bounded. By $\|x_n - y_{n+1}\| = \alpha_n\|x_n\|$, $\{x_n\}$ also converges strongly to $v = P_{F(T)}0$. □

Remark 6.11. From our point of view, iterations in Theorems 6.8, 6.9 and 6.10 are a kind of Halpern type iteration in structure. It is obvious that Theorem 6.10 is closely connected with Maingé and Măruşter's theorem. In the argument on Theorem 4.1 [19], their accounts of the sequence $\{\alpha_n\}$ is not helpful. They do not refer to the case that $\{\alpha_n\}$ has a null constant subsequence. Furthermore, it seems that the part of their proof connected with Lemma 4.3 [19] contains a circular argument as a matter of form.

7. Convergence theorems III

We begin this section with presenting the following theorem due to Ishikawa [9]. The iterative procedure in the theorem is called Ishikawa iteration.

Theorem 7.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[0,1]$ such that*

$$(1) \quad a_n \leq b_n, \quad (2) \quad \lim_n b_n = 0, \quad (3) \quad \sum_{n=1}^{\infty} a_n b_n = \infty.$$

Let C be a compact and convex subset of a Hilbert space H and T be a Lipschitzian pseudo-contractive self–mapping on C. Let $\{u_n\}$ be the sequence defined by $u_1 \in C$ and

$$\begin{cases} v_n = b_n T u_n + (1-b_n)u_n \\ u_{n+1} = a_n T v_n + (1-a_n)u_n \qquad \text{for} \quad n \in N. \end{cases}$$

Then $\{u_n\}$ converges strongly to some fixed point of T.

Ishikawa [9] made an impact on the study of pseudo-contractions. By studying [9], we can easily verify that his proof is also effective for Lipschitzian hemi–contractive mappings. Furthermore, we can have Lemma 7.3. From now on, we replace the form $aI + (1-a)T$ by Ishikawa's form $bT + (1-b)I$, where $b = (1-a)$.

Before proving Lemma 7.3, we prepare the following trivial lemma.

Lemma 7.2. *Let L be a positive real number and s be the positive solution of the equation $1 - 2x - (Lx)^2 = 0$. Let $b \in (0, s)$, $c \in (0, b]$ and $d = 1 - 2b - (bL)^2$. Let C be a convex subset of a Hilbert space H and T be an L-Lipschitzian self–mapping on C. Let S be the mapping defined by $Sx = (cT + (1-c)I)x$ for $x \in C$. Then $bL, d \in (0,1)$ and the following holds:*

$$\|TSx - Tx\| \le cL\|Tx - x\| \quad \text{for} \quad x \in C.$$

Proof. We know $c > 0$ and $s = 1/(\sqrt{L^2+1}+1)$. Then we have

$$0 < cL \le bL < sL = \frac{1}{\sqrt{L^2+1}+1}L < \frac{1}{L+1}L < \frac{1}{L}L = 1.$$

By $d = 1 - 2b - (bL)^2$ and $b \in (0, s)$, it is obvious that

$$0 = 1 - 2s - (sL)^2 < 1 - 2b - (bL)^2 = d < 1.$$

It is easy to see that, for $x \in C$,

$$\|TSx - Tx\| \le L\|(cTx + (1-c)x) - x\| = cL\|Tx - x\|.$$

□

Lemma 7.3. *Let L be a positive real number and $s = 1/(\sqrt{L^2+1}+1)$. Let a and b be positive real numbers satisfying $a, b \in (0, s)$ and $a \le b$. Set $d = 1-2b-(bL)^2$. Let C be a convex subset of a Hilbert space H and T be an L-Lipschitzian self–mapping on C. Define mappings S and U by*

$$Sx = bTx + (1-b)x, \quad Ux = aTSx + (1-a)x \qquad \text{for} \quad x \in C.$$

Then there are $k \in (0,1)$ and $K \in (0,\infty)$ such that

$$k\|Tx - x\| \le \|TSx - x\|, \qquad \|Ux - x\| \le K\|Tx - x\| \qquad \text{for} \quad x \in C.$$

Furthermore, suppose $\emptyset \ne F(T) \subset \mathscr{A}(T)$. Then the followings hold:

(1) $\|Ux - v\|^2 \le \|x - v\|^2 - dab\|Tx - x\|^2 \qquad$ for $\quad x \in C$ *and* $v \in \mathscr{A}(T)$,

(2) $F(T) = F(S) = F(TS) = F(U) = \mathscr{A}(T) \cap C = A(U) \cap C$,
$\mathscr{A}_{1-a}(TS) \cap C = A(U) \cap C$.

Proof. We show the first assertion. Let $x \in C$. By Lemma 7.2, we have

$$\|Tx - x\| \leq \|TSx - Tx\| + \|TSx - x\| \leq bL\|Tx - x\| + \|TSx - x\|.$$

By $\|Ux - x\| = \|(aTSx + (1-a)x) - x\|$, we also have

$$\begin{aligned}\|Ux - x\| &= a\|TSx - x\| \\ &\leq a(\|T(bTx + (1-b)x) - Tx\| + \|Tx - x\|) \\ &\leq a(bL + 1)\|Tx - x\|.\end{aligned}$$

By setting $k = 1 - bL \in (0, 1)$ and $K = a(bL + 1) \in (0, \infty)$, we have the result.

We prove (1). By our assumptions, we know $d, dab \in (0, 1)$ and $\emptyset \neq F(T) \subset \mathscr{A}(T)$. Let $v \in \mathscr{A}(T)$ and $x \in C$. Recall T is L-Lipschitzian and $S = bT + (1-b)I$.

Then, by $a \leq b$ and Lemmas 7.2, 4.10 and 3.2, we have the followings:

$$\begin{aligned}\|TSx - v\|^2 &\leq \|Sx - v\|^2 + \|TSx - Sx\|^2, \\ \|Sx - v\|^2 &\leq \|x - v\|^2 + b^2\|Tx - x\|^2, \\ \|TSx - Sx\|^2 &= \|TSx - (bTx + (1-b)x)\|^2 \\ &= b\|TSx - Tx\|^2 + (1-b)\|TSx - x\|^2 - b(1-b)\|Tx - x\|^2 \\ &\leq b(bL)^2\|Tx - x\|^2 + (1-a)\|TSx - x\|^2 - b(1-b)\|Tx - x\|^2.\end{aligned}$$

Recall $U = aTS + (1-a)I$ and set $\mathscr{N} = \|Ux - v\|^2$. Then we have

$$\begin{aligned}\mathscr{N} &= a\|TSx - v\|^2 + (1-a)\|x - v\|^2 - a(1-a)\|TSx - x\|^2 \\ &\leq a(\|x - v\|^2 + b^2\|Tx - x\|^2 \\ &\qquad + b(bL)^2\|Tx - x\|^2 + (1-a)\|TSx - x\|^2 - b(1-b)\|Tx - x\|^2) \\ &\qquad + (1-a)\|x - v\|^2 - a(1-a)\|TSx - x\|^2 \\ &= \|x - v\|^2 + ab(b + (bL)^2 - (1-b))\|Tx - x\|^2 \\ &= \|x - v\|^2 - dab\|Tx - x\|^2.\end{aligned} \tag{7.1}$$

Thus we have $\|Ux - v\|^2 \leq \|x - v\|^2 - dab\|Tx - x\|^2$ for $v \in \mathscr{A}(T)$ and $x \in C$.

We prove (2). By (1) and Lemma 4.4, we have $\mathscr{A}(T) \subset A(U) = \mathscr{A}_{1-a}(TS)$. Then, by Theorem 3.3 (4), we have

$$\emptyset \neq F(T) \subset \mathscr{A}(T) \cap C \subset \mathscr{A}_{1-a}(TS) \cap C = A(U) \cap C \subset F(U). \tag{7.2}$$

By the definitions of S and U, it is obvious that

$$F(T) = F(S) \subset F(TS) = F(U). \tag{7.3}$$

Let $u \in F(U) \subset C$ and $v \in \mathscr{A}(T)$. By (1), we have

$$\|u - v\|^2 = \|Uu - v\|^2 \leq \|u - v\|^2 - dab\|Tu - u\|^2.$$

Then, by $dab > 0$, we have $\|Tu - u\| = 0$. Thus $F(U) \subset F(T)$. Hence, by (7.2) and (7.3), we have

$$F(T) = F(S) = F(TS) = F(U) = \mathscr{A}(T) \cap C = \mathscr{A}_{1-a}(TS) \cap C = A(U) \cap C.$$

□

Remark 7.4. Techniques in the proof of Lemma 7.3 are essentially prepared by Ishikawa [9]. For mappings in Lemma 7.3 (1) and (2), we know $F(T) = F(U) = A(U) \cap C$ and $F(T) = F(TS) = \mathscr{A}_{1-a}(TS) \cap C$. That is, U is a quasi–nonexpansive mapping with $F(T) = F(U)$ and TS is a $(1-a)$–demi–contractive mapping with $F(T) = F(TS)$.

The following is a version of Theorem 7.1.

Theorem 7.5. *Let L be a positive real number and $s = 1/(\sqrt{L^2+1}+1)$. Let b be a real number satisfying $b \in (0, s)$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[0, b]$ such that*

$$(1) \quad a_n \le b_n, \qquad (2) \quad \sum_{n=1}^{\infty} a_n b_n = \infty.$$

Let C be a compact and convex subset of a Hilbert space H and T be an L-Lipschitzian self–mapping on C satisfying $F(T) \subset \mathscr{A}(T)$. For each $n \in N$, define mappings S_n and U_n by

$$S_n x = b_n T x + (1-b_n)x, \quad U_n x = a_n T S_n x + (1-a_n)x \qquad \text{for} \quad x \in C.$$

Let $\{u_n\}$ be the sequence defined by $u_1 \in C$ and

$$u_{n+1} = U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to some $u \in F(T)$.

Proof. By Shauder's theorem, we have $\emptyset \neq F(T) \subset \mathscr{A}(T)$. Set $d = 1 - 2b - (bL)^2$ and $d_n = 1 - 2b_n - (b_n L)^2$ for $n \in N$. Then $0 < d \le d_n$ for $n \in N$.

Let $v \in \mathscr{A}(T)$. By $a_n \le b_n$, it is obvious that $u_{n+1} = U_n u_n = u_n$ if $a_n b_n = 0$. Then, by Lemma 7.3 (1), we have that, for $n \in N$,

$$\|u_{n+1} - v\|^2 = \|U_n u_n - v\|^2 \le \|u_n - v\|^2 - d a_n b_n \|Tu_n - u_n\|^2.$$

This implies that $\{\|u_n - v\|^2\}$ converges. Furthermore, we have that, for $n \in N$,

$$\sum_{i=1}^{n} d a_i b_i \|Tu_i - u_i\|^2 \le \|u_1 - v\|^2 - \|u_{n+1} - v\|^2 \le \|u_1 - v\|^2.$$

Then we have $\sum_{i=1}^{\infty} d a_i b_i \|Tu_i - u_i\|^2 < \infty$. By $\sum_{i=1}^{\infty} d a_i b_i = \infty$, we have $\liminf_n \|Tu_n - u_n\|^2 = 0$. We know that there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ satisfying $\lim_j \|Tu_{n_j} - u_{n_j}\|^2 = \liminf_n \|Tu_n - u_n\|^2 = 0$. Since C is compact, $\{u_{n_j}\}$ has a subsequence converging strongly to some $u \in C$. Without loss of generality, we can assume that $\{u_{n_j}\}$ converges strongly to $u \in C$. Since T is continuous, we have $\|Tu - u\|^2 = \lim_j \|Tu_{n_j} - u_{n_j}\|^2 = 0$. Then $u \in F(T) \subset \mathscr{A}(T)$. This implies that $\{\|u_n - u\|^2\}$ converges. Since $\{\|u_{n_j} - u\|^2\}$ converges to 0, $\{\|u_n - u\|^2\}$ itself converges to 0. Thus $\{u_n\}$ converges strongly to $u \in F(T)$. $\square$

By theoretical interest, we present the following theorem.

Theorem 7.6. *Let $a, b, c \in (0, 1)$. Let C be a compact and convex subset of a Hilbert space H and T be a continuous self–mapping on C. Define mappings S and U by*

$$Sx = bTx + (1-b)x, \quad Ux = aTSx + (1-a)x \qquad \text{for} \quad x \in C.$$

Assume $F(U) \subset A(U)$. Let $\{u_n\}$ be the sequence defined by $u_1 \in C$ and

$$u_{n+1} = cUu_n + (1-c)u_n \quad \text{for} \quad n \in N.$$

Suppose there are $k \in (0,1)$ *and a subsequence* $\{u_{n_s}\}$ *of* $\{u_n\}$ *which satisfy either of the following conditions:*

$$\text{(L1)} \qquad k\|Tu_{n_s} - u_{n_s}\| \le \|TSu_{n_s} - u_{n_s}\| \qquad \text{for} \quad s \in N,$$

$$\text{(L2)} \qquad k\|Tu_{n_s} - u_{n_s}\| \le \|TTSu_{n_s} - Tu_{n_s}\| \qquad \text{for} \quad s \in N.$$

Then $\{u_n\}$ *converges strongly to some* $u \in F(T)$*.*

Proof. By Shauder's theorem, we have $\emptyset \ne F(T)$, that is,

$$\emptyset \ne F(T) = F(S) \subset F(TS) = F(U) \subset A(U).$$

Let $v \in A(U)$. Then, by Lemma 4.10, we have that, for $n \in N$,

$$\|u_{n+1} - v\|^2 = \|(cUu_n + (1-c)u_n) - v\|^2 \le \|u_n - v\|^2 - c(1-c)\|Uu_n - u_n\|^2.$$

By this inequality, $\{\|u_n - v\|^2\}$ converges. Furthermore, we have that, for $n \in N$,

$$c(1-c)\|Uu_n - u_n\|^2 \le \|u_n - v\|^2 - \|u_{n+1} - v\|^2.$$

By $c \in (0,1)$, we have $\lim_n \|Uu_n - u_n\|^2 = 0$. Since C is compact, there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges strongly to some $u \in C$. Since U is continuous, we have $\|Uu - u\|^2 = \lim_j \|Uu_{n_j} - u_{n_j}\|^2 = 0$. Thus $u \in F(U) \subset A(U)$. Hence, $\{\|u_n - u\|^2\}$ converges. Since $\{\|u_{n_j} - u\|^2\}$ converges to 0, $\{\|u_n - u\|^2\}$ itself converges to 0. Thus $\{u_n\}$ converges strongly to $u \in F(U)$.

We show $u \in F(T)$. Recall $F(U) = F(TS)$. By our assumptions, there are $k \in (0,1)$ and a subsequence $\{u_{n_s}\}$ of $\{u_n\}$ which satisfy (L1) or (L2).

Assume that k and $\{u_{n_s}\}$ satisfy (L1). Note that TS and T are continuous. Since $\{u_{n_s}\}$ converges strongly to $u \in F(TS)$, we have

$$\lim_s \|TSu_{n_s} - u_{n_s}\| = \|TSu - u\| = \|u - u\| = 0.$$

Then we have

$$k\|Tu - u\| = k \lim_s \|Tu_{n_s} - u_{n_s}\| \le \lim_s \|TSu_{n_s} - u_{n_s}\| = 0.$$

By $k \in (0,1)$, we have $\|Tu - u\| = 0$.

Assume that $k \in (0,1)$ and $\{u_{n_s}\}$ satisfy (L2). We know that TTS and T are continuous. Since $\{u_{n_s}\}$ converges strongly to $u \in F(TS)$, we have

$$\lim_s \|TTSu_{n_s} - Tu_{n_s}\| = \|T(TS)u - Tu\| = \|Tu - Tu\| = 0.$$

Then we have

$$k\|Tu - u\| = k \lim_s \|Tu_{n_s} - u_{n_s}\| \le \lim_s \|TTSu_{n_s} - Tu_{n_s}\| = 0.$$

By $k \in (0,1)$, we have $\|Tu - u\| = 0$. Thus, in both cases, we have $u \in F(T)$. $\square$

Remark 7.7. For simplicity's sake, we chose simple control sequences in Theorem 7.6. In the theorem, let T be an L–Lipschitzian hemi–contractive mapping and assume that a, b satisfy conditions in Lemma 7.3. Then U satisfies $F(U) \subset A(U)$. Since T is L–Lipschitzian, there is $k \in (0,1)$ such that k and $\{u_n\}$ satisfy condition (L1) by Lemma 7.3. So, Theorem 7.6 is slightly wider than Theorem 7.5.

To see relations between Theorem 7.5 and Theorem 7.6, we give an example.

Example 7.8. Let $C = [-1, 1] \subset R$ and T be the self–mapping on C defined by

$$Tx = x^2 \quad \text{if} \quad x \leq 0, \qquad Tx = -\sqrt{x} \quad \text{if} \quad x > 0.$$

Then C is compact and convex. It is obvious that T is a strictly decreasing continuous self–mapping on C. One can easily see $F(T) = \mathscr{A}(T) \cap C = \{0\}$ and $A(T) \cap C = \emptyset$. For convenience, we set $u = 0$. We show some properties of T. By $F(T) = \mathscr{A}(T) \cap C = \{0\}$, T is hemi–contractive. Let $x, y \in C$ with $x, y > 0$. Then we can easily see that T is not Lipschitzian by the following:

$$|Tx - Ty| = |-\sqrt{x} - (-\sqrt{y})| = |\sqrt{x} - \sqrt{y}| = \frac{1}{\sqrt{x} + \sqrt{y}} \; |x - y|.$$

Note $F(T) = \mathscr{A}(T) \cap C = \{u\} = \{0\}$. Let $k \in [0, 1)$ and $x \in (0, 1]$. Then we have $|Tx - u|^2 = (-\sqrt{x})^2 = x$ and

$$|x - u|^2 + k|Tx - x|^2 = x^2 + k(x + \sqrt{x})^2 = (1 + k)x^2 + 2kx\sqrt{x} + kx.$$

Thus we have

$$\lim_{x \downarrow 0} \frac{|x - u|^2 + k|Tx - x|^2}{|Tx - u|^2} = k < 1.$$

This implies that T is not k–demi–contractive.

We consider a sequence $\{u_n\}$ generated by the iteration in Theorem 7.6. For simplicity's sake, we consider mappings $S = (T + I)/2$ and $U = (TS + I)/2$. That is, $a = b = 1/2$. Note that $F(TS) \subset A(TS)$ implies $F(U) \subset A(U)$. In our setting, we can easily see that $F(T) = F(TS)$ and $F(TS)$ is singleton. Furthermore, $F(TS) = \{0\}$ and $TS(-1) = TS(0) = TS(1) = 0$. Then, to see $F(TS) \subset A(TS)$, we may assume $u_1 \in (-1, 0) \cup (0, 1)$.

Let $x \in (0, 1)$ and $y \in (-1, 0)$. Set $z = -y \in (0, 1)$. We confirm that $F(TS) \subset A(TS)$ holds. We can easily have the following calculation results:

$$0 < TSx = \frac{(x - \sqrt{x})^2}{4} < (-\sqrt{x})^2 = x, \quad 0 < TSy = \frac{(z^2 - z)^2}{4} < (-z)^2 < |y|.$$

That is, $|TSx - 0| \leq |x - 0|$ for $x \in (0, 1)$ and $|TSy - 0| \leq |y - 0|$ for $y \in (-1, 0)$.

We can also have the followings:

$$TTSx - Tx = -\frac{(\sqrt{x} - x)}{2} + \sqrt{x} = \frac{(\sqrt{x} + x)}{2} = \frac{1}{2} \, |Tx - x|,$$

$$TSy - y = \frac{(z^2 - z)^2}{4} + z > z > \frac{1}{2}(z^2 + z) = \frac{1}{2} \, |Ty - y|.$$

Then $\{u_n\}$ must have a subsequence $\{u_{n_s}\}$ such that $1/2$ and $\{u_{n_s}\}$ satisfy (L1) or (L2). Thus $\{u_n\}$ converges strongly to $0 \in F(T)$ by Theorem 7.6.

Motivated by [19], we show the following theorem for Lipschitzian hemi–contractive mappings. In an easily understood manner, we choose simple control sequences.

Theorem 7.9. *Let L be a positive real number and $s = 1/(\sqrt{L^2 + 1} + 1)$. Let a and b be real numbers satisfying $a, b \in (0, s)$ and $a \leq b$. Let C be a closed and convex subset of a Hilbert space H with $0 \in C$. Let T be an L–Lipschitzian self–mapping on C with $\emptyset \neq F(T) \subset \mathscr{A}(T)$. Assume that $I - T$ is demiclosed at 0. Define mappings S and U by*

$$Sx = bTx + (1 - b)x, \quad Ux = aTSx + (1 - a)x \qquad \text{for} \quad x \in C.$$

Let $\{a_n\}$ *be a sequence in* $(0,1)$ *satisfying* $\lim_n(1-a_n)=0$ *and* $\sum_{n=1}^{\infty}(1-a_n)=\infty$. *Let* $\{u_n\}$ *be the sequence defined by* $u_1 \in C$ *and*

$$u_{n+1} = a_n U u_n \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ *converges strongly to the point of* $F(T)$ *nearest to* 0.

Proof. It is easy to see that S, TS and U are self–mappings on C. By $0 \in C$, each a_nU is also a self–mapping on C. Then $\{u_n\}$ is well–defined. By Lemma 7.3 and Theorem 3.3, U is quasi–nonexpansive, $F(T) = F(U) = A(U) \cap C$, and $F(T)$ is closed and convex. Then we can set $v = P_{F(T)}0$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$. Set $D = \{x \in C : \|x-v\| \le \|u_1 - v\| + \|v\|\}$. Then we can easily see that $0, v, u_1 \in D$ and D is bounded, closed and convex. It is obvious that, for $a \in [0,1]$ and $x \in D$,

$$\begin{aligned}
\|Ux - v\| &\le \|x - v\| \le \|u_1 - v\| + \|v\|, \\
\|aUx - v\| &\le a\|Ux - v\| + (1-a)\|v\| \\
&\le a(\|u_1 - v\| + \|v\|) + (1-a)\|v\| \\
&\le \|u_1 - v\| + \|v\|.
\end{aligned}$$

Then U and each a_nU are self–mappings on D. We confirmed that $\{u_n\}$ and $\{Uu_n\}$ are sequences in the weakly compact set D.

Set $\mathscr{N}_n = \|u_{n+1} - v\|^2$ for $n \in N$. Then, by $v \in F(T) \subset \mathscr{A}(T)$ and Lemma 7.3 (1), we have that, for $n \in N$,

$$\begin{aligned}
\mathscr{N}_n &= \|a_n(Uu_n - v) - (1-a_n)v\|^2 \\
&\le a_n\|Uu_n - v\|^2 + (1-a_n)^2\|v\|^2 - 2a_n(1-a_n)\langle Uu_n - v, v\rangle \\
&\le a_n(\|u_n - v\|^2 - dab\|Tu_n - u_n\|^2) \\
&\qquad + (1-a_n)^2\|v\|^2 + 2a_n(1-a_n)\langle Uu_n - v, -v\rangle,
\end{aligned} \tag{7.4}$$

where $d = 1 - 2b - (bL)^2 \in (0,1)$. For each $n \in N$, we set $y_n = Uu_n$ and

$$K_n = -\frac{1}{1-a_n}\, a_n dab\|Tu_n - u_n\|^2 + (1-a_n)\|v\|^2 + 2a_n\langle y_n - v, -v\rangle. \tag{7.5}$$

Then we can rewrite (7.4) as follows:

$$\|u_{n+1} - v\|^2 \le a_n\|u_n - v\|^2 + (1-a_n)K_n \quad \text{for} \quad n \in N. \tag{7.6}$$

By (7.5) and $a_n \in (0,1)$, it is obvious that, for $n \in N$,

$$K_n \le (1-a_n)\|v\|^2 + 2a_n\langle y_n - v, -v\rangle \le \|v\|^2 + 2\|y_n - v\|\,\|v\|. \tag{7.7}$$

Then, since D is weakly compact, we have $\limsup_n K_n < \infty$.

We show $\limsup_n K_n \le 0$. Arguing by contradiction, assume $0 < \limsup_n K_n$. Then there is a subsequence $\{n_j\}$ of $\{n\}$ such that

$$0 < \limsup_n K_n = \lim_j K_{n_j}.$$

Let $\{u_{n_j}\}$ be the subsequence of $\{u_n\}$ corresponding to $\{n_j\}$. Since D is weakly compact, $\{u_{n_j}\}$ has a subsequence which converges weakly to some $u \in D$. Without loss of generality, we can assume that $\{u_{n_j}\}$ converges weakly to $u \in D$. It is obvious that $0 < K_{n_j}$ for sufficiently large j. By (7.5), we can rewrite $0 < K_{n_j}$ as follows:

$$\frac{1}{1-a_{n_j}} a_{n_j} dab \|Tu_{n_j} - u_{n_j}\|^2 < (1-a_{n_j})\|v\|^2 + 2a_{n_j}\langle y_{n_j} - v, -v\rangle.$$

Then, since D is bounded and $\lim_j (1-a_{n_j}) = 0$, there is $K > 0$ such that

$$\frac{dab}{1-a_{n_j}}\|Tu_{n_j} - u_{n_j}\|^2 < \frac{(1-a_{n_j})}{a_{n_j}}\|v\|^2 + 2\|y_{n_j} - v\|\|v\| < K$$

for sufficiently large j. That is, for sufficiently large j, the following holds:

$$\|Tu_{n_j} - u_{n_j}\|^2 < \frac{1-a_{n_j}}{dab} K.$$

By $\lim_j (1-a_{n_j}) = 0$ and Lemma 7.3, we have

$$\lim_j \|Tu_{n_j} - u_{n_j}\|^2 = 0, \quad \lim_j \|Uu_{n_j} - u_{n_j}\|^2 = \lim_j \|y_{n_j} - u_{n_j}\|^2 = 0.$$

Since $I - T$ is demiclosed at 0, we have $u \in F(T)$. It is obvious that $\{y_{n_j}\}$ also converges weakly to u. Confirm that $\lim_j (1-a_{n_j}) = 0$, $v = P_{F(T)}0$, and $\{y_{n_j}\}$ converges weakly to $u \in F(T)$. Then, by (7.7), we have

$$\lim_j K_{n_j} \le \lim_j \left((1-a_{n_j})\|v\|^2 + 2a_{n_j}\langle y_{n_j} - v, -v\rangle\right) = 2\langle u - v, 0 - v\rangle \le 0.$$

Thus we have $0 < \limsup_n K_n = \lim_j K_{n_j} \le 0$. This is a contradiction.

We know $\limsup_n K_n \le 0$. Thus, by $\sum_{n=1}^{\infty}(1-a_n) = \infty$, (7.6), and Lemma 4.2, we have $\lim_n \|u_n - v\|^2 = 0$. That is, $\{u_n\}$ converges strongly to $v = P_{F(T)}0$. □

We present procedures finding a common fixed point of two Lipschitzian hemi–contractive mappings. For simplicity's sake, we choose simple control sequences.

Theorem 7.10. *Let L be a positive real number and $s = 1/(\sqrt{L^2+1}+1)$. Let a be a real number satisfying $a \in (0, s)$. Let C be a closed and convex subset of a Hilbert space H. For $j \in \{1, 2\}$, let T_j be an L–Lipschitzian self–mapping on C satisfying $F(T_j) \subset \mathscr{A}(T_j)$. Assume that each $I - T_j$ is demiclosed at 0 and $\cap_{j=1}^2 F(T_j) \neq \emptyset$. For $j \in \{1, 2\}$, define mappings S_j and U_j by*

$$S_j x = aT_j x + (1-a)x, \quad U_j x = aT_j S_j x + (1-a)x \qquad \text{for} \quad x \in C.$$

Generate sequences $\{u_n\}$ and $\{w_n\}$ in C by the following iterations, respectively.

(a) *Let $u_1 \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_{n+1} = U_2 U_1 u_n \qquad \text{for} \quad n \in N.$$

(b) *Let $w_1 \in C$ and define a sequence $\{w_n\}$ in C by*

$$w_{n+1} = \frac{1}{2}\sum_{j=1}^{2} U_j w_n \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ and $\{w_n\}$ converge weakly to some $u, w \in \cap_{j=1}^2 F(T_j)$, respectively.

Proof. Fix $v \in \cap_{j=1}^{2} F(T_j)$ arbitrarily. Set $D_{(a)} = \{x \in C : \|x - v\| \leq \|u_1 - v\|\}$. Then $v \in \cap_{j=1}^{2} F(T_j) \subset \cap_{j=1}^{2} \mathscr{A}(T_j)$ and $u_1, v \in D_{(a)}$. It is obvious that $D_{(a)}$ is bounded, closed and convex. By Lemma 7.3 (2), $F(T_j) = F(U_j) = A(U_j) \cap C$ for $j \in \{1, 2\}$. That is, for each j, U_j is a self–mapping on $D_{(a)}$. Then $U_2 U_1$ is also a self–mapping on $D_{(a)}$. Thus we can generate $\{u_n\}$ in $D_{(a)}$.

Set $D_{(b)} = \{x \in C : \|x - v\| \leq \|w_1 - v\|\}$. Then $w_1, v \in D_{(b)}$. We also know that $D_{(b)}$ is bounded, closed and convex. In the same way as above, for each j, U_j is a self–mapping on $D_{(b)}$. Then $\frac{1}{2}\sum_{j=1}^{2} U_j$ is also a self–mapping on $D_{(b)}$. Thus we can generate $\{w_n\}$ in $D_{(b)}$. Furthermore, we have

$$0 < 2 \sup\{\|w_n - v\| : n \in N\} + 1 \leq 2\|w_1 - v\| + 1 < \infty.$$

We show that $\{u_n\}$ converges weakly to a point of $\cap_{j=1}^{2} F(T_j)$. By $v \in \cap_{j=1}^{2} \mathscr{A}(T_j)$ and Lemma 7.3 (1), we have

$$\begin{aligned} \|u_{n+1} - v\|^2 &= \|U_2 U_1 u_n - v\|^2 \\ &\leq \|U_1 u_n - v\|^2 - da^2 \|T_2 U_1 u_n - U_1 u_n\|^2 \\ &\leq \|u_n - v\|^2 - da^2 \|T_1 u_n - u_n\|^2 - da^2 \|T_2 U_1 u_n - U_1 u_n\|^2 \end{aligned} \tag{7.8}$$

for $n \in N$, where $d = 1 - 2a - (aL)^2 \in (0, 1)$. Then $\{\|u_n - v\|^2\}$ and $\{\|u_n - v\|\}$ converge. We confirmed that $\{\|u_n - v\|\}$ converges for each $v \in \cap_{j=1}^{2} F(T_j)$.

By (7.8), we also have that, for $n \in N$,

$$da^2 (\|T_1 u_n - u_n\|^2 + \|T_2 U_1 u_n - U_1 u_n\|^2) \leq \|u_n - v\|^2 - \|u_{n+1} - v\|^2.$$

By $da^2 > 0$, we have $\lim_n \|T_1 u_n - u_n\|^2 = 0$ and $\lim_n \|T_2 U_1 u_n - U_1 u_n\|^2 = 0$. Since $\{u_n\}$ is bounded, $\{u_n\}$ has a weakly convergent subsequence. Let $\{u_{n_l}\}$ be a subsequence which converges weakly to some $u \in H$. Since C is weakly closed, we know $u \in C$. Furthermore, since $I - T_1$ is demiclosed at 0, $u \in F(T_1)$. By $\lim_n \|T_1 u_n - u_n\|^2 = 0$ and Lemma 7.3, we have $\lim_n \|U_1 u_n - u_n\|^2 = 0$. Then $\{U_1 u_{n_l}\}$ also converges weakly to u. We know $\lim_l \|T_2 U_1 u_{n_l} - U_1 u_{n_l}\|^2 = 0$. Since $I - T_2$ is demiclosed at 0, we have $u \in F(T_2)$. Then we have $u \in \cap_{j=1}^{2} F(T_j)$. We confirmed that every weakly convergent subsequence of $\{u_n\}$ converges weakly to a point of $\cap_{j=1}^{2} F(T_j)$. Thus, by Lemma 4.9, every weakly convergent subsequence of $\{u_n\}$ converges weakly to u. That is, $\{u_n\}$ converges weakly to $u \in \cap_{j=1}^{2} F(T_j)$.

We show that $\{w_n\}$ converges weakly to a point of $\cap_{j=1}^{2} F(T_j)$. By $v \in \cap_{j=1}^{2} \mathscr{A}(T_j)$ and Lemma 7.3 (1), we know that, for each j,

$$\|U_j w_n - v\|^2 \leq \|w_n - v\|^2 - da^2 \|T_j w_n - w_n\|^2 \tag{7.9}$$

for $n \in N$, where $d = 1 - 2a - (aL)^2 \in (0, 1)$. Set $K = 2\|w_1 - v\| + 1$. We need not know the value of K. We know $\|U_j w_n - v\| \leq \|w_n - v\| \leq \|w_1 - v\|$ for $n \in N$. For positive real numbers s, t, c, k, $kc^2 \leq s^2 - t^2$ and $kc^2 \leq (s - t)(s + t)$ are equivalent. Then, for each j, it follows from (7.9) that, for $n \in N$,

$$\|U_j w_n - v\| \leq \|w_n - v\| - \frac{da^2}{K} \|T_j w_n - w_n\|^2.$$

This inequality holds even if $w_n = v$. We can easily have

$$\begin{aligned}\|w_{n+1} - v\| &\le \frac{1}{2}\sum\nolimits_{j=1}^{2} \|U_j w_n - v\| \\ &\le \frac{1}{2}\sum\nolimits_{j=1}^{2} (\|w_n - v\| - \frac{da^2}{K}\|T_j w_n - w_n\|^2) \\ &= \|w_n - v\| - \frac{da^2}{2K}\sum\nolimits_{j=1}^{2} \|T_j w_n - w_n\|^2\end{aligned} \tag{7.10}$$

for $n \in N$. Then $\{\|w_n - v\|\}$ converges. We confirmed that $\{\|w_n - v\|\}$ converges for each $v \in \cap_{j=1}^{2} F(T_j)$. By (7.10), we also have that, for $n \in N$, $j \in \{1, 2\}$,

$$\frac{da^2}{2K}\|T_j w_n - w_n\|^2 \le \frac{da^2}{2K}\sum\nolimits_{i=1}^{2} \|T_i w_n - w_n\|^2 \le \|w_n - v\| - \|w_{n+1} - v\|.$$

By $(da^2)/(2K) > 0$, this implies $\lim_n \|T_j w_n - w_n\|^2 = 0$ for $j \in \{1, 2\}$. Since $\{w_n\}$ is bounded, $\{w_n\}$ has a weakly convergent subsequence. Let $\{w_{n_l}\}$ be a subsequence which converges weakly to some $w \in C$. Since $I - T_j$ is demiclosed at 0 for $j \in \{1, 2\}$, we have $w \in \cap_{j=1}^{2} F(T_j)$. We also confirmed that every weakly convergent subsequence of $\{w_n\}$ converges weakly to a point of $\cap_{j=1}^{2} F(T_j)$. Thus, by Lemma 4.9, every weakly convergent subsequence of $\{w_n\}$ converges weakly to w. That is, $\{w_n\}$ itself converges weakly to $w \in \cap_{j=1}^{2} F(T_j)$. $\square$

We show a strong convergence theorem corresponding to Theorem 7.10.

Theorem 7.11. *Let L be a positive real number and $s = 1/(\sqrt{L^2 + 1} + 1)$. Let a be a real number satisfying $a \in (0, s)$. Let C be a compact and convex subset of a Hilbert space H. For $j \in \{1, 2\}$, let T_j be an L–Lipschitzian self–mapping on C satisfying $F(T_j) \subset \mathscr{A}(T_j)$. Assume $\cap_{j=1}^{2} F(T_j) \neq \emptyset$. For $j \in \{1, 2\}$, define mappings S_j and U_j by*

$$S_j x = aT_j x + (1 - a)x, \quad U_j x = aT_j S_j x + (1 - a)x \qquad \text{for} \quad x \in C.$$

Generate sequences $\{u_n\}$ and $\{w_n\}$ in C by the following iterations, respectively.

(a) *Let $u_1 \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_{n+1} = U_2 U_1 u_n \qquad \text{for} \quad n \in N.$$

(b) *Let $w_1 \in C$ and define a sequence $\{w_n\}$ in C by*

$$w_{n+1} = \frac{1}{2}\sum\nolimits_{j=1}^{2} U_j w_n \qquad \text{for} \quad n \in N.$$

Then $\{u_n\}$ and $\{w_n\}$ converge strongly to some $u, w \in \cap_{j=1}^{2} F(T_j)$, respectively.

Proof. Refer to the proof of Theorem 7.10 to have the results.

We already know that we can generate $\{u_n\}$ and $\{w_n\}$. We show that $\{u_n\}$ converges strongly to a point of $\cap_{j=1}^{2} F(T_j)$. In the same way as in the proof of Theorem 7.10, we have that $\{\|u_n - v\|\}$ converges for each $v \in \cap_{j=1}^{2} F(T_j)$, $\lim_n \|T_1 u_n - u_n\|^2 = 0$ and $\lim_n \|T_2 U_1 u_n - U_1 u_n\|^2 = 0$. Since C is compact, $\{u_n\}$ has a subsequence $\{u_{n_l}\}$ which converges strongly to some $u \in C$. Since T_1 and $T_2 U_1$ are continuous, we have $\|T_1 u - u\|^2 = \lim_n \|T_1 u_n - u_n\|^2 = 0$ and $\|T_2 U_1 u - U_1 u\|^2 = \lim_n \|T_2 U_1 u_n - U_1 u_n\|^2 = 0$. We already know $F(U_1) = F(T_1)$.

Then we have $\|T_2u-u\|^2=0$. Thus we have $u \in \cap_{j=1}^2 F(T_j)$. This implies that $\{\|u_n-u\|\}$ converges. Since $\{\|u_{n_l}-u\|\}$ converges to 0, $\{\|u_n-u\|\}$ itself converges to 0. Thus $\{u_n\}$ converges strongly to $u \in \cap_{j=1}^2 F(T_j)$.

We show that $\{w_n\}$ converges strongly to a point of $\cap_{j=1}^2 F(T_j)$. In the same way as in the proof of Theorem 7.10, we have that $\{\|w_n-v\|\}$ converges for each $v \in \cap_{j=1}^2 F(T_j)$ and $\lim_n \|T_jw_n-w_n\|^2=0$ for $j \in \{1,2\}$. Since C is compact, $\{w_n\}$ has a subsequence $\{w_{n_l}\}$ which converges strongly to some $w \in C$. Since each T_j is continuous, $\|T_jw-w\|^2=\lim_l \|T_jw_{n_l}-w_{n_l}\|^2=0$ for $j \in \{1,2\}$. That is, $w \in \cap_{j=1}^2 F(T_j)$. So, $\{\|w_n-w\|\}$ converges. Since $\{\|w_{n_l}-w\|\}$ converges to 0, $\{\|w_n-w\|\}$ itself converges to 0. Thus $\{w_n\}$ converges strongly to $w \in \cap_{j=1}^2 F(T_j)$. $\square$

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