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ON SPLIT FEASIBILITY PROBLEMS WITH ZERO-CONVEX REPRESENTATIONS

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Abstract. The split feasibility problem deals with finding a point in a closed convex subset of a real Hilbert space and such that its image under a bounded linear operator subsequently belongs to a closed convex subset of another Hilbert space. This paper concentrates on the split feasibility problem in which the closed convex subsets can be written as the sublevel sets of a zero-convex function. We propose an adaptive subgradient projection algorithm for solving this considered problem and analyze its convergence. Furthermore, we discuss some related problems.

1. Introduction

The celebrated split feasibility problem (in short, **SFP**), was initially introduced by Censor and Elfving [7], which can be mathematically formulated as the problem of finding a point

$$(\mathbf{SFP}) \qquad x^* \in C \quad \text{such that} \quad Ax^* \in Q,$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix. Subsequently, under the consistency of the problem, Byrne [4] presented a new iterative algorithm for solving the (**SFP**), namely CQ-algorithm, which is defined by the following iterative step:

$$x_{k+1} = P_C(x_k + \gamma A^\top (P_Q - I)Ax_k), \quad \forall k \geq 1,$$

where an initial $x_1 \in \mathbb{R}^n$, $\gamma \in (0, 2/\|A\|^2)$ and P_C and P_Q denote the metric projections onto C and Q, respectively. The convergence result of the generated sequence to a solution of the considered (**SFP**) was presented. Further, Byrne also proposed an application to dynamic emission tomographic image reconstruction. A few years later, the companion publications of Censor *et al.* [8] and Censor *et al.* [6] proposed an incredible application of (**SFP**), that is it can be modeled to the inverse problem of intensity-modulated radiation therapy treatment planning.

Recently, Xu [12] considered (**SFP**) in the context of infinite-dimensional Hilbert space and established a following CQ-algorithm: let H_1 and H_2 be real Hilbert

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spaces, $A : H_1 \to H_2$ be a bounded linear operator. For given $x_1 \in H_1$, and generated iterative scheme

$$x_{k+1} = P_C(x_k + \gamma A^*(P_Q - I)Ax_k), \qquad \forall k \geq 1,$$

where $\gamma \in (0, 2/\|A\|^2)$, and A^* is the adjoint operator of A. He also proved the weak convergence of the sequence via the above procedure to the solution of (**SFP**) as well.

It is worth noting that many real-world situations, the considered closed convex subset C, for instance as in (**SFP**), may be represented by a sublevel set of a given convex continuous function $f : H \to \mathbb{R}$ corresponding to a real constant λ. In this case we will assume without loss of generality that $C := \{x \in H : f(x) \leq 0\}$. More recently, the (**SFP**), or more generally the convex feasibility problems, in this case have been investigated by many authors, for instance [2], [3] [10], [13] and [12] and references therein.

Even if the convex representation plays an important role in mathematical models, there are some situations in which the representing functions lack of the convexity. Very recently, Censor and Reem [9] introduced a new class of functions which is called by zero-convex. This class is very wide and rich since the functions belonging to this class need not be convex and differentiable. They considered a convex feasibility problem in the case when the considered closed convex subsets represented by sublevel sets of given zero-convex functions. The advantage of this consideration is that, sometimes, the metric projection onto any closed convex set may not have a closed-form expression or the computation of such metric projection is very difficult. However, if the such closed convex set is of the particular structure, i.e. sublevel set of a zero-convex function, then we can use a subgradient projection instead of the metric projection which is easier to compute than the metric projection.

In this work, motivated by these all above theoretical and practical reasons, we will concentrate on a split feasibility problem in the case when the considered closed convex sets are represented by the zero sublevel sets of zero-convex functions, which we will call it by split zero-convex feasibility problem. We propose a method for solving this considered problem. We examine some imposed assumptions utilized for its convergence and prove that under these assumptions the sequence generated by the proposed algorithmic method converges weakly to a solution of the considered problem. We also discuss some related results, that is, the zero-convex feasibility problem introduced by Censor and Reem [9] and the split common fixed point problem considered by Wang and Xu [11].

Notations. Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We denote the identity operator on H by I. The weak convergence of a sequence $\{x_k\}_{k=1}^{\infty}$ to $x \in H$ are denoted by $x_k \rightharpoonup x$. For any bounded linear operator A from a Hilbert space H_1 into a Hilbert space H_2, we denote its adjoint by A^*. For a subset $D \subset H_2$, we denote a subset $A^{-1}(D)$ of H_1 by $A^{-1}(D) := \{x \in H_1 : Ax \in D\}$. Let $f : H \to \mathbb{R}$ be a function, for each $\lambda \in \mathbb{R}$, we denote the sublevel sets of f corresponding to λ by $S^f_{\leq,\lambda} := \{x \in H : f(x) \leq \lambda\}$. Moreover, we denote the positive part of a function f by f_+, that is $f_+(x) := \max\{f(x), 0\}$, for all $x \in H$.

2. Zero-convex functions and their useful properties

In this section, we will recall a definition and some useful properties of a class of functions introduced by Censor and Reem [9]. We also investigate a usable property for our convergence result.

A function $f : H \to \mathbb{R}$ is called *zero-convex* at a point $x \in H$ if there exists $t \in H$ such that

$$f(x) + \langle t, y - x\rangle \leq 0, \quad \text{for all } y \in S^f_{\leq,0}.$$

The vector t is called *zero-subgradient* of f at x and, further, the set of all zero-subgradient of f at x is called the zero-subdifferential of f at x and is denoted by $\partial^0 f(x)$. If f is zero-convex at all point in H, then we call it by zero-convex.

It is worth noting that every convex function having at least one point of continuity on H is zero-convex. In particular, if H is finite dimensional, we can removed the requirement that at least one of continuity holds since every convex function is continuous. Furthermore, one can observe that if f is zero-convex, then the zero-subdiiferential $\partial^0 f(x)$ is nonempty for all $x \in H$.

The following proposition collects some useful properties from [9, Proposition 1, Proposition 2] involving the zero-convexity.

Proposition 2.1. *Let H be a real Hilbert space and $f : H \to \mathbb{R}$ be a given function. Then,*

(i) *If f is zero-convex, then its zero sublevel set $S^f_{\leq,0}$ is convex.*
(ii) *If the zero sublevel set $S^f_{\leq,0}$ is convex and closed, then f is a zero-convex function. In fact, if $x \in S^f_{\leq,0}$, then $0 \in \partial^0 f(x)$, and if $x \notin S^f_{\leq,0}$, then there exists $t \in \partial^0 f(x)$ with $t := \frac{f(x)}{\|x-m\|^2}(x - m) \neq 0$ where $m \in M$ is the orthogonal projection of x onto a closed hyperplane M strictly separating x from $S^f_{\leq,0}$.*
(iii) *If $f_i : H \to \mathbb{R}$ is a zero-convex function at x, for all $i = 1, \ldots, m$, then function f defined by $f(x) := \max\{f_i(x) : i = 1, \ldots, m\}$ is also zero-convex at x.*
(iv) *If f is zero-convex with $S^f_{\leq,0} \neq \emptyset$, then for any $x \in H$ such that $f(x) > 0$ we have any zero-subgradient $t \in \partial^0 f(x)$ satisfies $t \neq 0$.*

3. Problem formulation and its convergence

In this section, we begin by proposing a split feasibility problem which we will call it by a split zero-convex feasibility problem. We subsequently present a generic iterative method for solving the proposed problem.

Problem 3.1. Let H_1 and H_2 be real Hilbert spaces, $f : H_1 \to \mathbb{R}$, $g : H_2 \to \mathbb{R}$ be zero-convex functions, and $A : H_1 \to H_2$ be a bounded linear operator. The *split zero-convex feasibility problem* (in short, **SZFP**) is to find

$$x^* \in S^f_{\leq,0} \text{ such that } Ax^* \in S^g_{\leq,0}, \tag{3.1}$$

where $S^f_{\leq,0} := \{x \in H_1 : f(x) \leq 0\}$ and $S^g_{\leq,0} := \{y \in H_2 : g(y) \leq 0\}$.

Throughout this work, we may assume the consistency of Problem 3.1 so that $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ is nonempty.

It is easy to see that the solution set $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ is a convex set. We note that Problem 3.1 quite includes that of Xu [12, Section 4] since a convex function which is subdifferentiable on a zero sublevel set is a zero-convex one, see [9, Example 1].

On the other hand, by integrating ideas those of the Byrne' s forward-backward method for solving **SFP** and Censor-Reem [9, Algorithm 1]' s sequential subgradient projections algorithm for solving the zero-convex feasibility problem, we are in position to perform an algorithm for solving **SZFP** as follows.

Algorithm 3.2. Initialization: Choose $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}, \{\gamma_k\}_{k=1}^{\infty} \subset (0, +\infty)$. Take $x_1 \in H_1$ be arbitrary.
Iterative step: Given the current iterate $x_k \in H_1$, calculate $z_k \in H_2$ by

$$z_k := Ax_k - \beta_k \frac{g_+(Ax_k)}{\|d_k\|^2} d_k, \quad \text{where } d_k \in \partial^0 g(Ax_k) \setminus \{0\}.$$

Define $y_k \in H_1$ by

$$y_k := x_k + \gamma_k A^*(z_k - Ax_k),$$

and evaluate $x_{k+1} \in H_1$ by

$$x_{k+1} := y_k - \alpha_k \frac{f_+(y_k)}{\|c_k\|^2} c_k, \quad \text{where } c_k \in \partial^0 f(y_k) \setminus \{0\}.$$

Update $k := k + 1$.

Remark 3.3. Note that if there exists $k_0 \in \mathbb{N}$ in which both $f(x_{k_0})$ and $g(Ax_{k_0})$ are nonpositive, then the Algorithm 3.2 terminates and the iteration x_{k_0} subsequently is a solution of the **SZFP**. So in the rest of this work, we may assume that the Algorithm 3.2 does not terminate in any finite iteration $k \geq 1$.

We will use the following control condition throughout this work.

Condition 3.4. The real sequences $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}, \{\gamma_k\}_{k=1}^{\infty} \subset (0, +\infty)$ are satisfying

(I) $\underline{\varepsilon} \leq \alpha_k \leq 2 - \bar{\varepsilon}$ and $\underline{\zeta} \leq \beta_k \leq 1 - \bar{\zeta}$, with some arbitrary small positive real numbers $\underline{\varepsilon}, \bar{\varepsilon}, \underline{\zeta}$, and $\bar{\zeta}$;

(II) $0 < \underline{\gamma} := \inf_{k\geq 1} \gamma_k \leq \bar{\gamma} := \sup_{k \geq 1} \gamma_k < 2/\|A\|^2$.

To investigate our convergence result, we start with an important key lemma.

Lemma 3.5. *For a sequence $\{x_k\}_{k=1}^{\infty}$ generated by Algorithm 3.2, we have*

$$\begin{aligned} \|x_{k+1} - q\|^2 \leq\ & \|x_k - q\|^2 - \gamma_k(2 - \gamma_k\|A\|^2)\|z_k - Ax_k\|^2 \\ & -\alpha_k(2-\alpha_k)\frac{f_+(y_k)^2}{\|c_k\|^2} - 2\gamma_k\beta_k(1-\beta_k)\frac{g_+(Ax_k)^2}{\|d_k\|^2}, \end{aligned} \tag{3.2}$$

for $k \geq 1$ and $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$.

Proof. For $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ and $k \geq 1$, we observe that

$$\begin{aligned}\langle Ax_k - Aq, z_k - Ax_k\rangle &= -\|z_k - Ax_k\|^2 + \langle z_k - Aq, z_k - Ax_k\rangle \\ &= -\|z_k - Ax_k\|^2 + \beta_k^2 \frac{g_+(Ax_k)^2}{\|d_k\|^2} \\ &\quad -\beta_k \frac{g_+(Ax_k)}{\|d_k\|^2} \langle Ax_k - Aq, d_k\rangle .\end{aligned}$$

Furthermore, for $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ and $k \geq 1$, we also have

$$\begin{aligned}\|y_k - q\|^2 &= \|x_k - q\|^2 + \gamma_k^2 \|A^*(z_k - Ax_k)\|^2 + 2\gamma_k \langle x_k - q, A^*(z_k - Ax_k)\rangle \\ &\leq \|x_k - q\|^2 + \gamma_k^2 \|A\|^2 \|z_k - Ax_k\|^2 + 2\gamma_k \langle Ax_k - Aq, z_k - Ax_k\rangle.\end{aligned}$$

By using the above equality and inequality, we get that

$$\begin{aligned}\|x_{k+1} - q\|^2 &= \|y_k - q\|^2 + \alpha_k^2 \frac{f_+(y_k)^2}{\|c_k\|^2} - 2\alpha_k \frac{f_+(y_k)}{\|c_k\|^2} \langle y_k - q, c_k\rangle \\ &\leq \|x_k - q\|^2 + \gamma_k^2 \|A\|^2 \|z_k - Ax_k\|^2 + 2\gamma_k \langle Ax_k - Aq, z_k - Ax_k\rangle \\ &\quad +\alpha_k^2 \frac{f_+(y_k)^2}{\|c_k\|^2} - 2\alpha_k \frac{f_+(y_k)}{\|c_k\|^2} \langle y_k - q, c_k\rangle \\ &= \|x_k - q\|^2 - \gamma_k(2 - \gamma_k \|A\|^2) \|z_k - Ax_k\|^2 \\ &\quad +2\gamma_k \beta_k^2 \frac{g_+(Ax_k)^2}{\|d_k\|^2} - 2\gamma_k \beta_k \frac{g_+(Ax_k)}{\|d_k\|^2} \langle Ax_k - Aq, d_k\rangle \\ &\quad +\alpha_k^2 \frac{f_+(y_k)^2}{\|c_k\|^2} - 2\alpha_k \frac{f_+(y_k)}{\|c_k\|^2} \langle y_k - q, c_k\rangle \\ &\leq \|x_k - q\|^2 - \gamma_k(2 - \gamma_k \|A\|^2) \|z_k - Ax_k\|^2 \\ &\quad -\alpha_k(2 - \alpha_k) \frac{f_+(y_k)^2}{\|c_k\|^2} - 2\gamma_k \beta_k (1 - \beta_k) \frac{g_+(Ax_k)^2}{\|d_k\|^2},\end{aligned}$$

which the last one holds because $c_k \in \partial^0 f(y_k) \setminus \{0\}$ and $d_k \in \partial^0 g(Ax_k) \setminus \{0\}$, respectively. Further, by applying Condition 3.4, we can obtain that

$$\begin{aligned}\|x_{k+1} - q\|^2 &\leq \|x_k - q\|^2 - \underline{\gamma}(2 - \bar{\gamma}\|A\|^2) \|z_k - Ax_k\|^2 \\ &\quad -\underline{\varepsilon}\bar{\varepsilon} \frac{f_+(y_k)^2}{\|c_k\|^2} - 2\underline{\gamma\zeta}\bar{\zeta} \frac{g_+(Ax_k)^2}{\|d_k\|^2},\end{aligned} \tag{3.3}$$

for $k \geq 1$ and $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. □

For our convergence result we will assume the following assumptions.

Assumption 3.6. The representative function f and g are weakly lower semicontinuous.

Assumption 3.7. The sequences of selected zero-subgradients $\{c_k\}_{k=1}^\infty$, $\{d_k\}_{k=1}^\infty$ are bounded.

Remark 3.8. (i) Notice that the sublevel sets $S^f_{\leq,0}$ and $S^g_{\leq,0}$ in Problem 3.1 may lack of closedness, since the zero-convexity need not be continuous. Thus, when we have to restrict our consideration to a special class of functions; as Assumption 3.6;

the closedness of such sublevel sets also holds immediately. Moreover, we also get that $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ is a closed set.

(ii) Recalling that to consider Assumption 3.7, Censor and Reem [9] defined the property of bondedeness on bounded subset for the zero-subgradients as: a zero-convex function $f : H \to \mathbb{R}$ is said to be *partial bounded on bounded sets of the zero-subgrdients*, if for any bounded subset $B \subset H$, there exists a $M > 0$ such that for all $x \in B$, there exist at least one zero-subgradient $t \in \partial^0 f(x)$ satisfying $\|t\| \leq M$. Let us notice that if we assume further the representative functions f and g are partial bounded on bounded sets of the zero-subgrdients and the sequence $\{x_k\}_{k=1}^\infty$ is bounded, then Assumption 3.7 holds. To do so, note that, if $\{x_k\}_{k=1}^\infty$ is a bounded sequence, then so is $\{Ax_k\}_{k=1}^\infty$ and consequently implies the boundedness of $\{d_k\}_{k=1}^\infty$. Further, observe that, according to the lines proof of Lemma 3.5, it holds that

$$\|y_k - q\|^2 \leq \|x_k - q\|^2 - 2\underline{\gamma\zeta}\bar{\zeta}\frac{g_+(Ax_k)^2}{\|d_k\|^2},$$

for $k \geq 1$ and $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. So, we also have $\{y_k\}_{k=1}^\infty$ is bounded and hence the boundedness of $\{c_k\}_{k=1}^\infty$ is obtained.

Next, we will ensure that a sequence generated by Algorithm 3.2 converges weakly to a solution of **SZFP** as the following theorem.

Theorem 3.9. *If the Condition 3.4, Assumption 3.6, and Assumption 3.7 hold, then the sequence $\{x_k\}_{k=1}^\infty$ generated by Algorithm 3.2 converges weakly to a point in $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$.*

Proof. By utilizing Lemma 3.5, we obtain that the sequence $\{\|x_k - q\|\}_{k=1}^\infty$ is monotone decreasing and bounded form below, therefore, $\lim_{k\to\infty} \|x_k - q\|$ exists, for all $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. Subsequently, we obtain that

$$\text{(3.4)} \qquad \lim_{k\to\infty} \|z_k - Ax_k\| = 0; \qquad \lim_{k\to\infty} g_+(Ax_k) = 0; \qquad \lim_{k\to\infty} f_+(y_k) = 0.$$

By the definition of y_k and using $\lim_{k\to\infty} \|z_k - Ax_k\| = 0$, we also have

$$\lim_{k\to\infty} \|x_k - y_k\| = 0.$$

Since the sequence $\{x_k\}_{k=1}^\infty$ is bounded, there exists a subsequence $\{x_{k_l}\}_{l=1}^\infty$ of $\{x_k\}_{k=1}^\infty$ such that $x_{k_l} \rightharpoonup x^*$ for some $x^* \in H_1$. We observe, by using the weakly lower semicontinuity of f and g, that

$$f(x^*) \leq \lim_{l\to\infty} f(y_{k_l}) = \lim_{l\to\infty} f_+(y_{k_l}) = 0,$$

and, similarly,

$$g(Ax^*) \leq \lim_{l\to\infty} g(Ax_{k_l}) = \lim_{l\to\infty} g_+(Ax_{k_l}) = 0.$$

These imply that $x^* \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. Finally, it remains to show that $x_k \rightharpoonup x^* \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. Suppose that there is another subsequence $\{x_{k_j}\}_{j=1}^\infty$ of $\{x_k\}_{k=1}^\infty$

such that $x_{k_j} \rightharpoonup v \in H_1$. Then, we also obtain that $v \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. Suppose to contrary that $v \neq x^*$, by applying the well known Opial's lemma, we have

$$\begin{aligned}
\lim_{k\to\infty} \|x_k - v\| &= \lim_{j\to\infty} \|x_{k_j} - v\| \\
&< \lim_{j\to\infty} \|x_{k_j} - x^*\| \\
&= \lim_{k\to\infty} \|x_k - x^*\| \\
&= \lim_{l\to\infty} \|x_{k_l} - x^*\| \\
&< \lim_{l\to\infty} \|x_{k_l} - v\| \\
&= \lim_{k\to\infty} \|x_k - v\|,
\end{aligned}$$

which is a contradiction. Thus, $v = x^*$. This implies that every subsequence of $\{x_k\}_{k=1}^\infty$ converges weakly to the same point in $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. Therefore, the sequence $\{x_k\}_{k=1}^\infty$ converges weakly to a point $x^* \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$. This proof is accomplished. □

The following remark gives some sufficient conditions on the convergence in norm of the generated sequence.

Remark 3.10. Of course, the weak convergence result in Theorem 3.9 can be strong if the considered space H_1 is either finite dimensional or compact. Alternatively, the convergence in norm of the sequence $\{x_k\}_{k=1}^\infty$ in Theorem 3.9 also holds whenever the interior of $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ is nonempty, see Bauschke and Borwein [1, Theorem 2.16(iii)] for more details.

The following remark gives another technique on the boundedness of the generated sequence.

Remark 3.11. Alternatively, if a starting point $x_1 \in H_1$ in the Algorithm 3.2 is specifically chosen that it closes to $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0}) \neq \emptyset$ enough, then we can guarantee the boundedness of the sequence $\{x_k\}_{k=1}^\infty$. Indeed, let $\epsilon > 0$ be a fixed positive real number such that $\mathrm{dist}(x_1, S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})) < \epsilon$. We can choose $\eta \in [\epsilon, 2\epsilon]$ and there subsequently exists $q' \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$ such that $\|x_1 - q'\| \leq \eta$. Since (3.3) holds for $k \geq 1$ and $q \in S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$, we can obtain the boundedness of the sequence $\{x_k\}_{k=1}^\infty$.

4. Some related results

In this section, we will discuss some implementations and related problems of the split zero-convex feasibility problem.

4.1. **Zero-convex feasibility problem.** Let us consider **SZFP** as a following form. Let $H := H_1 = H_2$ be a real Hilbert space and the operator A be the identity operator, **SZFP** is nothing else than the *two-sets zero-convex feasibility problem* of finding

$$x^* \in S^f_{\leq,0} \cap S^g_{\leq,0}, \tag{4.1}$$

and Algorithm 3.2 reduces to the following *relaxed alternating subgradient projections method*: Given the current iterate $x_k \in H$, compute

$$\begin{aligned} z_k &:= x_k - \beta_k \frac{g_+(x_k)}{\|d_k\|^2} d_k, \quad \text{where } d_k \in \partial^0 g(x_k) \setminus \{0\}; \\ y_k &:= (1-\gamma_k)x_k + \gamma_k z_k; \\ x_{k+1} &:= y_k - \alpha_k \frac{f_+(y_k)}{\|c_k\|^2} c_k, \quad \text{where } c_k \in \partial^0 f(y_k) \setminus \{0\}, \end{aligned}$$

for all $k \geq 1$, where x_1 is arbitrary taken in H.

By this considering we can obtain a convergence result of the relaxed alternating subgradient projections method for the Problem (4.1).

Corollary 4.1. *Assume that the sequences of selected zero-subgradients* $\{c_k\}_{k=1}^{\infty}$, $\{d_k\}_{k=1}^{\infty}$ *are bounded. If the real sequences* $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}, \{\gamma_k\}_{k=1}^{\infty}$ *are satisfying* $\underline{\varepsilon} \leq \alpha_k \leq 2 - \bar{\varepsilon}$ *and* $\underline{\zeta} \leq \beta_k \leq 1 - \bar{\zeta}$*, with some arbitrary small positive real numbers* $\underline{\varepsilon}, \bar{\varepsilon}, \underline{\zeta}$*, and* $\bar{\zeta}$ *and* $0 < \underline{\gamma} := \inf_{k\geq 1} \gamma_k \leq \bar{\gamma} := \sup_{k \geq 1} \gamma_k < 2$*, then the sequence* $\{x_k\}_{k=1}^{\infty}$ *generated by the relaxed alternating subgradient projections method weakly converges to a point in* $S^f_{\leq,0} \cap S^g_{\leq,0}$ *provided that it is nonempty.*

Proof. It is clearly that the two-sets zero-convex feasibility problem is **SZFP** and the method (4.2) is a specialization of Algorithm 3.2. Since A is the identity operator, we have that $\|A\| = 1$. Thanks to Theorem 3.9, we immediately obtain the required result. ☐

Remark 4.2. Note that the Problem (4.1) is an exactly specialization of the zero-convex feasibility problem introduced by Censor and Reem [9], nevertheless, this relaxed alternating subgradient projections method does not coincide with their sequential subgradient projection method [9, Algorithm 1].

Consider the zero-convex feasibility problem of finding

$$x^* \in \bigcap_{i=1}^{m} S^{f_i}_{\leq,0}, \tag{4.2}$$

where $f_i : H \to \mathbb{R}$ is zero-convex and weakly lower semicontinuous, for all $i = 1, \ldots, m$, one may notice by Proposition 2.1(iii) that it is equivalent to finding a point

$$x^* \in S^f_{\leq,0}, \tag{4.3}$$

where $f : H \to \mathbb{R}$ is defined by $f(x) := \max\{f_i(x) : i = 1, \ldots, m\}$. In this case, it easily applies Corollary 4.1 to obtain a convergence result for Problem (4.2).

However, let us consider a network system consisting of m users. Each user i ($i \in \{1, \ldots, m\}$) has, in hand, a possible constrained function $f_i : H \to \mathbb{R}$ and can not get the explicit forms of other users possible constrained functions. This means that it is impossible to obtain the explicit form of $\max_{1\leq i \leq m} f_i$. Based on these ideas, we present a *simultaneous subgradient projections method* for Problem (4.2) as: Given $x_1 \in H$ arbitrary and $\{\alpha_k\}_{k=1}^{\infty} \subset (0, +\infty)$. For the current iterate

$x_k \in H$, calculate the next iterate $x_{k+1} \in H$ as

$$x_{k+1} := x_k - \alpha_k \sum_{i=1}^{m} \omega_i \frac{(f_i)_+(x_k)}{\|c_{i,k}\|^2} c_{i,k}, \tag{4.4}$$

where $c_{i,k} \in \partial^0 f_i(x_k) \setminus \{0\}$ and $\{\omega_i\}_{i=1}^m \in (0,1)$ with $\sum_{i=1}^m \omega_i = 1$.

We can perform a convergence result of this simultaneous subgradient projections method as follows.

Theorem 4.3. *Assume that the sequence of selected zero-subgradients $\{c_{i,k}\}_{k=1}^{\infty}$ is bounded for all $i = 1, \dots, m$. If the real sequences $\{\alpha_k\}_{k=1}^{\infty}$ is satisfying $\underline{\varepsilon} \le \alpha_k \le 2-\bar{\varepsilon}$, with some arbitrary small positive real numbers $\underline{\varepsilon}, \bar{\varepsilon}$, then the sequence $\{x_k\}_{k=1}^{\infty}$ generated by the simultaneous subgradient projections method (4.4) weakly converges to a point in $\bigcap_{i=1}^m S_{\le,0}^{f_i}$ provided that it is nonempty.*

Proof. Let $q \in \bigcap_{i=1}^m S_{\le,0}^{f_i}$ be given. Taking into account Lemma 3.5 with $H = H_1 = H_2$, $g \equiv 0$, viewing the term $\sum_{i=1}^m \omega_i \frac{(f_i)_+(x_k)}{\|c_{i,k}\|^2} c_{i,k}$ as $\frac{f_+(y_k)}{\|c_k\|^2} c_k$, and using the convexity of the function norm $\|\cdot\|$, we obtain that

$$\|x_{k+1} - q\|^2 \le \|x_k - q\|^2 - \alpha_k(2 - \alpha_k) \sum_{i=1}^{m} \omega_i \frac{(f_i)_+(x_k)}{\|c_{i,k}\|},$$

for $k \ge 1$. Then, we have $\lim_{k\to\infty} \|x_k - q\|^2$ exists, so that $\lim_{k\to\infty} \frac{(f_i)_+(x_k)}{\|c_{i,k}\|} = 0$ for all $i = 1, \dots, m$ and hence $\lim_{k\to\infty}(f_i)_+(x_k) = 0$ for all $i = 1, \dots, m$. By applying the proof of Theorem 3.9, we can obtain the remaining proof as desired. □

Remark 4.4. (i) It is worth noting that the idea of this simultaneous subgradient projections method and Proposition 4.3 can be applied easily to the case of multiple-set split zero-convex feasibility problem.

(ii) The problem of the form (4.3) in the finite dimensional setting and each f_i is convex was also investigated by Butnariu *et al.* [3] which their Algorithm 3.1 does not required the maximum-valued function.

4.2. **As a specialization of the split common fixed point problem.** In this subsection, we will show that the split zero-convex feasibility problem can be solving by utilizing the idea of Wang-Xu [11]' s split common fixed point problem.

Recall that, in a real Hilbert space H, the operator $U : H \to H$ with $\mathrm{Fix}(U) \neq \emptyset$ is called *cutter* if and only if it holds

$$\langle x - Ux, z - Ux \rangle \le 0$$

for all $x \in H$ and all $z \in \mathrm{Fix}(U)$. We note that the set of all fixed points of a cutter is closed and convex. For more details on a cutter and its properties, the reader may consult the book of Cegieski [5].

Let $f : H \to \mathbb{R}$ be a zero-convex function with nonempty zero sublevel set. That is, let $c_x \in \partial^0 f(x)$ for all $x \in H$. We define an operator $T : H \to H$ by

$$T(x) := \begin{cases} x - \frac{f(x)}{\|c_x\|^2} c_x & \text{if } x \notin S_{\le,0}^{f}, \\ x & \text{if } x \in S_{\le,0}^{f}. \end{cases}$$

Clearly that T is well defined by Proposition 2.1(iv). Next, we will present an important properties of such operator T.

Proposition 4.5. *Let f and T be defined as above. Then, T is a cutter with $\mathrm{Fix}(T) = S^f_{\leq,0}$. Furthermore, if f is weakly lower semicontinuous and partial bounded on bounded sets of the zero-subgrdients, then $T - I$ is demi-closed at zero, that is, if $x_k \rightharpoonup x^* \in H$ and $\|Tx_k - x_k\| \to 0$, then $Tx^* = x^*$.*

Proof. We first show that $\mathrm{Fix}(T) = S^f_{\leq,0}$. In fact, the reverse inclusion is obvious. We may show that $\mathrm{Fix}(T) \subset S^f_{\leq,0}$. Suppose that $x \notin S^f_{\leq,0}$. Thus, we have $f(x) > 0$ and then, by Proposition 2.1 (iv), that $c_x \neq 0$. That is, $\frac{f(x)}{\|c_x\|^2} c_x \neq 0$ which implies that $x \notin \mathrm{Fix}(T)$. Next, we will show that T is cutter. Let $x \notin S^f_{\leq,0}$ and $z \in \mathrm{Fix}(T)(= S^f_{\leq,0})$ be given. By the definition of zero-convexity, we have

$$\begin{aligned}\langle x - Tx, z - Tx\rangle &= \langle x - Tx, z - x\rangle + \|x - Tx\|^2 \\ &= \frac{f(x)}{\|c_x\|^2}\langle c_x, z - x\rangle + \left(\frac{f(x)}{\|c_x\|}\right)^2 \\ &\leq 0.\end{aligned}$$

Finally, we will show that $T - I$ is demi-closed at 0. Let $\{x_k\}_{k=1}^\infty \subset H$ be such that $x_k \rightharpoonup x \in H$ and $\|Tx_k - x_k\| \to 0$. Then, $\{x_k\}_{k=1}^\infty$ is bounded and so is the selected zero-subgradient $\{c_k := c_{x_k}\}_{k=1}^\infty$. Further, we have

$$\frac{f_+(x_k)}{\|c_k\|} = \|Tx_k - x_k\| \to 0,$$

which implies that $\lim_{k\to\infty} f_+(x_k) = 0$. Therefore, by the weak lower semicontinuity of f, we obtain that

$$f(x^*) \leq \varliminf_{k\to\infty} f(x_k) \leq \lim_{k\to\infty} f_+(x_k) = 0,$$

and hence $Tx^* = x^*$. □

Now, let us consider the Algorithm 3.2 in the case when $\alpha_k \equiv \beta_k \equiv 1$ for all $k \geq 1$ and define the operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$ by

$$T(x) := \begin{cases} x - \frac{f(x)}{\|c_x\|^2} c_x & \text{if } x \notin S^f_{\leq,0}, \\ x & \text{if } x \in S^f_{\leq,0}, \end{cases}$$

and

$$S(y) := \begin{cases} y - \frac{g(y)}{\|d_y\|^2} d_y & \text{if } y \notin S^g_{\leq,0}, \\ y & \text{if } y \in S^g_{\leq,0}. \end{cases}$$

Then, in this case, Algorithm 3.2 becomes

$$x_{k+1} := T(x_k + \gamma_k A^*(S(Ax_k) - Ax_k)), \quad \forall k \geq 1 \tag{4.5}$$

where $\{\gamma_k\}_{k=1}^\infty \subset (0, +\infty)$ and $x_1 \in H_1$ is arbitrary.

So, we can propose a convergence result of the sequence generated by (4.5) to a solution of Problem 3.1 as the following theorem.

Theorem 4.6. *Let* $\{x_k\}_{k=1}^{\infty}$ *be a sequence generated by (4.5) and assume that the functions* f *and* g *are weakly lower semicontinuous and partial bounded on bounded sets of the zero-subgrdients. If* $0 < \underline{\gamma} := \inf_{k\geq 1} \gamma_k \leq \bar{\gamma} := \sup_{k\geq 1} \gamma_k < 2/\|A\|^2$, *then the sequence* $\{x_k\}_{k=1}^{\infty}$ *converges weakly to a point in* $S^f_{\leq,0} \cap A^{-1}(S^g_{\leq,0})$.

Proof. We note by Proposition 4.5 that T and S are cutter and $T - I$ and $S - I$ are both demi-closed at zero. By applying Theorem 3.3 of Wang-Xu [11] in the case of $p = 1$, we can obtain our convergence result. □

Remark 4.7. We can notice that the convergence result of the Algorithm 3.2 and the method (4.5) or, in general, Wang-Xu [11] iterative procedure 3.7 are not identical. Indeed, the convergence result of Algorithm 3.2 is depended on two stepsizes α_k, β_k as in Condition 3.4(I) so that $\alpha_k \in (0, 2)$ and $\beta_k \in (0, 1)$, however, the convergence of the method (4.5) is merely depended on $\alpha_k \equiv \beta_k \equiv 1$.

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