Linear and Nonlinear Analysis
Volume 2, Number 1, 2016, 101–111

Yokohama Publishers
ISSN 2188-8167

CONSTRAINT QUALIFICATIONS FOR KKT OPTIMALITY CONDITION IN CONVEX OPTIMIZATION WITH LOCALLY LIPSCHITZ INEQUALITY CONSTRAINTS

SHUNSUKE YAMAMOTO AND DAISHI KUROIWA

Abstract. We study constraint qualifications for the Karush-Kuhn-Tucker (KKT) optimality condition in a convex optimization problem whose constraint functions are locally Lipschitz but not necessarily convex, which was observed by Dutta and Lalitha in 2013. We give several constraint qualifications for the KKT optimality condition, which are modifications of well-known constraint qualifications of convex or nonlinear optimization, the Basic constraint qualification (BCQ), Guignard's constraint qualification, Abadie's constraint qualification, Cottle's constraint qualification and the linearly independent constraint qualification. We discuss all relations among these constraint qualifications, especially, we show that two of them are necessary and sufficient constraint qualifications for the KKT optimality condition. In addition, we remark that the Slater condition is not a constraint qualification for the optimality in this convex optimization problem.

1. Introduction

In this paper, we consider the following convex optimization problem:

$$(P)\begin{cases} \min & f(x) \\ \text{s.t.} & x \in S, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $S \subseteq \mathbb{R}^n$ is a non-empty convex set. Throughout this paper we assume that the feasible set S is given as

$$S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I = \{1, \dots, m\}$, are locally Lipschitz functions, and assume that for all $x \in S$ and $i \in I(x) = \{i \in I \mid g_i(x) = 0\}$, g_i are regular at x. Constraint functions g_i are usually assumed convex in the usual convex optimization, however, g_i are locally Lipschitz but not necessarily convex or differentiable in this paper.

In general, constraint qualifications are essential to solve optimization problems, because these assures the existence of Karush-Kuhn-Tucker multipliers (KKT optimality condition) when an element is a solution of an optimization problem. In

2010 *Mathematics Subject Classification.* 90C25, 90C46.

Key words and phrases. Convex optimization, constraint qualification, the KKT optimality condition, locally Lipschitz inequality constraints.

The authors are grateful to an anonymous referee for many comments and suggestions improved the quality of the paper. The author's work was partially supported by JSPS KAKENHI Grant Number 25400205.

2008, it was shown that the basic constraint qualification (BCQ), which was introduced in [4], is a necessary and sufficient constraint qualification for the KKT optimality condition in a convex optimization problem, whose constraint set S is described by the above and every constraint functions are convex, by Li, Ng and Pong, see [7].

Recently, the KKT optimality conditions for a convex optimization problem, whose constraint set S is described by the above but every constraint functions are not necessarily convex, was studied. In 2010, a convex optimization problem, whose objective function is differentiable convex and constraint functions are differentiable but not necessarily convex, was discussed and a constraint qualification for the optimality condition was given by Lasserre, see [6]. In 2013, a convex optimization problem, whose objective function is convex not necessarily differentiable and constraint functions are locally Lipschitz but not necessarily convex or differentiable, was discussed, and a constraint qualification for the optimality condition was given by Dutta and Lalitha, see [2].

In this paper, we investigate several constraint qualifications, which are modifications of well-known constraint qualifications, for the KKT optimality in condition the convex optimization problem (P), which was discussed by Dutta and Lalitha in [2], and compare our results and previous ones. The paper is organized as follows. In Section 2, we describe our notation and present preliminary results. In Section 3, we propose several constraint qualifications for the KKT optimality condition, which are modifications of well-known constraint qualifications of convex or nonlinear optimization, the BCQ, Guignard's constraint qualification, Abadie's constraint qualification, Cottle's constraint qualification and the linearly independent constraint qualification, see [7, 10]. Also, we discuss the relations among all constraint qualifications which are introduced by the authors, and Dutta and Lalitha. In addition, we remark that the Slater condition is not a constraint qualification for the KKT optimality condition in the convex optimization problem (P). Finally, we summarize our results in Section 4.

2. Preliminaries

In this section, we describe our notation and present preliminary results. A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for each $x \in \mathbb{R}^n$, there exist $M > 0$ and $r > 0$ such that $|g(y) - g(z)| \leq M\|y - z\|$ for each $y, z \in B(x, r)$ where $B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$. For a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$, the Clarke directional derivative of g at $x \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$, denoted by $g^\circ(x, d)$, is given by

$$g^\circ(x, d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y + td) - g(y)}{t}.$$

For each $x \in \mathbb{R}^n$, the function $g^\circ(x, \cdot)$ is a positively homogeneous convex function. The Clarke subdifferential of g at x, denoted by $\partial^\circ g(x)$, is defined by

$$\partial^\circ g(x) = \{\xi \in \mathbb{R}^n \mid \langle \xi, d \rangle \leq g^\circ(x, d), \forall d \in \mathbb{R}^n\}.$$

The set $\partial^\circ g(x)$ is a non-empty, convex and compact subset of $\mathbb{R}^n$. Moreover the Clarke directional derivative is the support function of the Clarke subdifferential, that is,

$$g^\circ(x,d) = \max_{\xi \in \partial^\circ g(x)} \langle \xi, d\rangle.$$

When g is convex, that is,

$$g((1-\alpha)x + \alpha y) \leq (1-\alpha)g(x) + \alpha g(y) \text{ for each } x, y \in \mathbb{R}^n \text{ and } \alpha \in (0,1),$$

then g is locally Lipschitz, $g^\circ(x,\cdot) = g'(x,\cdot)$ and $\partial^\circ g(x) = \partial g(x)$ for each $x \in \mathbb{R}^n$, where

$$g'(x,d) = \lim_{t\downarrow 0} \frac{g(x+td) - g(x)}{t},$$

$$\partial g(x) = \{\xi \in \mathbb{R}^n \mid \langle \xi, d\rangle \leq g'(x,d), \forall d \in \mathbb{R}^n\}.$$

In general, a locally Lipschitz function g is said to be regular at x if g is directionally differentiable at x in the all directions d and $g^\circ(x,\cdot) = g'(x,\cdot)$, see [1].

Let C be a set in $\mathbb{R}^n$. We denote the closure, the interior, the conical hull and the convex hull of C by $\operatorname{cl} C$, $\operatorname{int} C$, $\operatorname{cone} C$ and $\operatorname{co} C$, respectively. The negative polar cone of C, denoted by C^-, is defined by

$$C^- = \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 0, \forall x \in C\}.$$

It is well-known that C^- is a closed convex cone, and

$$C^{--} = (C^-)^- = \operatorname{cl}\operatorname{cone}\operatorname{co} C.$$

For any $x \in C$, the tangent cone of C at x, denoted by $T_C(x)$, is defined by

$$T_C(x) = \{y \in \mathbb{R}^n \mid \exists\{(x_k, \alpha_k)\} \subseteq C \times \mathbb{R}_+ \text{ s.t. } x_k \to x, \alpha_k(x_k - x) \to y\},$$

where $\mathbb{R}_+ = [0, +\infty)$. The set $T_C(\bar{x})$ is a closed cone. The normal cone of C at x, denoted by $N_C(x)$, is defined by $N_C(x) = (T_C(x))^-$. When C is a convex set, it is well-known that

$$T_C(x) = \operatorname{cl}\operatorname{cone}(C - x) = N_C(x)^-, \text{ and}$$

$$N_C(x) = (C-x)^- = \{\xi \in \mathbb{R}^n \mid \langle \xi, y - x\rangle \leq 0, \forall y \in C\}.$$

Next the following result is used in one of our results.

Theorem 2.1. *([9]) Let f be a real-valued convex function on $\mathbb{R}^n$. If there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) < 0$, then we have $\{x \in \mathbb{R}^n \mid f(x) < 0\} = \operatorname{int}\{x \in \mathbb{R}^n \mid f(x) \leq 0\}$.*

Proof. The proof is shown by using Theorem 11 and Remark 1 in [9]. □

The following theorem is shown by Dutta and Lalitha in [2].

Theorem 2.2. *([2]) Let $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I = \{1, \ldots, m\}$, be locally Lipschitz functions, and let $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I\}$. Assume that S is a convex set, all g_i are regular at $\bar{x}$, the Slater condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for each $i \in I$, and $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$. Then for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:*

(i) *for each $x \in S$, $f(\bar{x}) \leq f(x)$,*
(ii) *there exists $\lambda \in \mathbb{R}^I_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i\in I} \lambda_i \partial^\circ g_i(\bar{x})$ and for each $i \in I$, $\lambda_i g_i(\bar{x}) = 0$.*

Condition (ii) of this theorem is called the KKT optimality condition. In the convex case, the basic constraint qualification (BCQ) and the Slater condition are famous constraint qualifications which are conditions for the KKT optimality condition to be fulfilled by any convex objective function f having $\bar{x}$ as a local minimum on the convex feasible set S, and the BCQ is a necessary and sufficient condition for the KKT optimality condition. In the differentiable case, it is well-known that the Guignard's constraint qualification, the Abadie's constraint qualification, the Cottle's constraint qualification and the linearly independent constraint qualification are conditions for the KKT optimality condition to be fulfilled by any differentiable objective function f having $\bar{x}$ as a local minimum on the feasible set S, and the Guignard's constraint qualification is a necessary and sufficient condition for the KKT optimality condition. In this paper, we present several constraint qualifications which are conditions for the KKT optimality condition to be fulfilled by any convex objective function f having $\bar{x}$ as a local minimum on the convex feasible set S, and discuss these constraint qualifications.

3. Constraint qualifications for KKT optimality condition in convex optimization under locally Lipschitz inequality constraints

In this paper, we consider the following convex optimization problem:

$$(P)\begin{cases} \min\ f(x) \\ \text{s.t. } x \in S, \end{cases}$$

where f is a real-valued convex function on $\mathbb{R}^n$ and S is a convex set. Throughout this paper we assume that the feasible set S is given as

$$S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\},$$

where g_i, $i \in I = \{1, \ldots, m\}$, are real-valued locally Lipschitz functions on $\mathbb{R}^n$ and g_i is regular at every $x \in S$ and every $i \in I(x)$, where $I(x) = \{i \in I \mid g_i(x) = 0\}$.

In this study, we discuss the following conditions:

- (A) $N_S(\bar{x}) = \operatorname{cone}\operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$,
- (B) $T_S(\bar{x}) = \bigcap_{i\in I(\bar{x})}(\partial^\circ g_i(\bar{x})^-)$ and $\operatorname{cone}\operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed,
- (C) there exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0\rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$,
- (D) the Slater condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for each $i \in I$, and $0 \notin \partial^\circ g_i(\bar{x})$, for each $i \in I(\bar{x})$,
- (E) $0 \notin \operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$,
- (F) $\operatorname{int} S \neq \emptyset$ and $0 \notin \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$,
- (G) for each $y_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, $\{y_i\}_{i\in I(\bar{x})}$ is linearly independent.

At first, we provide the following lemma, which is important to show our results:

Lemma 3.1. *Let $\bar{x} \in S$. Then for each $i \in I(\bar{x})$, $\xi_i \in \partial^\circ g_i(\bar{x})$ and $x \in S$,*

$$\langle \xi_i, x - \bar{x}\rangle \leq 0.$$

That is, $\partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ for each $i \in I(\bar{x})$.

Proof. For each $i \in I(\bar{x})$, $\xi_i \in \partial^\circ g_i(\bar{x})$ and $x \in S$,

$$\langle \xi_i, x - \bar{x}\rangle \leq g_i^\circ(\bar{x}, x - \bar{x}).$$

From the regularity of g_i at $\bar{x}$,

$$\langle \xi_i, x - \bar{x} \rangle \leq g_i'(\bar{x}, x - \bar{x}) = \lim_{t \downarrow 0} \frac{g_i(\bar{x} + t(x - \bar{x})) - g_i(\bar{x})}{t}.$$

Since $\bar{x} + t(x - \bar{x}) \in S$ for each $t \in (0, 1)$ and $i \in I(\bar{x})$, we have $g_i'(\bar{x}, x - \bar{x}) \leq 0$, so $\langle \xi_i, x - \bar{x} \rangle \leq 0$. □

Now we show a result that conditions (A) and (B) are necessary and sufficient constraint qualifications for the optimality conditions in convex optimization problem (P).

Theorem 3.2. *Let $\bar{x} \in S$. Then the following statements are equivalent:*

(A) $N_S(\bar{x}) = \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$,

(B) $T_S(\bar{x}) = \bigcap_{i \in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-)$ *and* $\operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ *is closed,*

(O) *for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:*

  (i) $f(x) \geq f(\bar{x})$ *for each* $x \in S$,

  (ii) *there exists* $\lambda \in \mathbb{R}^I_+$ *such that* $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$ *and for each* $i \in I$, $\lambda_i g_i(\bar{x}) = 0$.

Proof. First, we prove (A)⇔(B). It is clear that (A) holds if and only if $N_S(\bar{x}) = \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ and $\operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed. From convexity of S, we have $N_S(\bar{x})^- = T_S(\bar{x})$. Therefore, it is enough to show that $(\bigcap_{i \in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-))^- = \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This equality is given by the following property:

$$\bigcap_{i \in I} (A_i^-) = (\bigcup_{i \in I} A_i)^- \text{ for any } A_i \subseteq \mathbb{R}^n (i \in I).$$

Next, we prove (A)⇒(O). Let f be a real-valued convex function on $\mathbb{R}^n$. The proof that (ii) implies (i) is easy and omitted. Conversely, assume (i). For each $x \in S$, since $\bar{x} + \alpha(x - \bar{x}) \in S$ for each $\alpha \in (0, 1)$,

$$f(\bar{x}) \leq f(\bar{x} + \alpha(x - \bar{x})),$$

that is,

$$0 \leq f'(\bar{x}, x - \bar{x}) = \max_{\xi \in \partial f(\bar{x})} \langle \xi, x - \bar{x} \rangle.$$

Therefore $0 \leq \inf_{x \in S} \max_{\xi \in \partial f(\bar{x})} \langle \xi, x - \bar{x} \rangle$. According to Sion's minimax theorem (see e.g. [8, 5]), we can invert the infimum and the maximum, and we get $0 \leq \max_{\xi \in \partial f(\bar{x})} \inf_{x \in S} \langle \xi, x - \bar{x} \rangle$. Then there exists $\eta \in \partial f(\bar{x})$ such that

$$\langle -\eta, x - \bar{x} \rangle \leq 0 \text{ for each } x \in S.$$

Thus, $-\eta \in N_S(\bar{x})$. From (A), $-\eta \in \operatorname{cone}\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$. Then there exist $\mu_i \geq 0$ and $\xi_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that $-\eta = \sum_{i \in I(\bar{x})} \mu_i \xi_i$. Put

$$\lambda_i = \begin{cases} \mu_i & \text{if } \quad i \in I(\bar{x}), \\ 0 & \text{if } \quad i \in I \setminus I(\bar{x}), \end{cases}$$

for each $i \in I$. Then it is clear that $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$. Moreover,

$$-\eta = \sum_{i \in I(\bar{x})} \lambda_i \xi_i = \sum_{i \in I} \lambda_i \xi_i \in \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x}).$$

Hence, $0 = \eta + (-\eta) \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$. Finally, we prove (O)⇒(A), $\operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ is shown by using Lemma 3.1. Conversely, let $\eta \in N_S(\bar{x})$. Then

$$\langle -\eta, \bar{x} \rangle \leq \langle -\eta, x \rangle \text{ for each } x \in S.$$

Put $f = \langle -\eta, \cdot \rangle$, then f is a convex function, and (i) of (O) holds. So, (ii) of (O) holds. Hence, there exists $\lambda \in \mathbb{R}^I_+$ such that

$$\begin{cases} 0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x}), \\ \lambda_i g_i(\bar{x}) = 0 \text{ for each } i \in I. \end{cases}$$

From $\partial f(\bar{x}) = \{-\eta\}$ and $0 \in \partial f(\bar{x}) + \sum_{i \in I} \partial^\circ g_i(\bar{x})$, $\eta \in \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$. Since $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$, we have

$$\sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x}) = \sum_{i \in I(\bar{x})} \lambda_i \partial^\circ g_i(\bar{x}) \subseteq \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x}).$$

Thus, $\eta \in \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This completes the proof. □

Remark 3.3. (1) We remark that Theorem 3.2 holds even if the index set I is infinite. In this case, (ii) of (O) is as follows: there exist a finite subset $J \subseteq I(\bar{x})$ and $\lambda \in \mathbb{R}^J_+$ such that $0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial^\circ g_i(\bar{x})$ and for each $i \in I, \lambda_i g_i(\bar{x}) = 0$.

(2) When all g_i are convex, then condition (A),

$$N_S(\bar{x}) = \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

is called basic constraint qualification (BCQ).

(3) When all g_i are continuously differentiable at $\bar{x}$ and S is not necessarily convex, then condition (A),

$$N_S(\bar{x}) = \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \{\nabla g_i(\bar{x})\},$$

which is equivalent to

$$\operatorname{cl} \operatorname{co} T_S(\bar{x}) = \{x \in \mathbb{R}^n \mid \langle \nabla g_i(\bar{x}), x \rangle \leq 0, \forall i \in I(\bar{x})\},$$

is called Guignard's constraint qualification, and condition (B),

$$T_S(\bar{x}) = \{x \in \mathbb{R}^n \mid \langle \nabla g_i(\bar{x}), x \rangle \leq 0, \forall i \in I(\bar{x})\},$$

is called Abadie's constraint qualification, see [10]. In this case, both Guignard's and Abadie's constraint qualifications are necessary and sufficient constraint qualifications for optimality condition of (P).

Next we show a result that condition (C) is a sufficient constraint qualification for the optimality conditions in convex optimization problem (P). When all g_i are continuously differentiable at $\bar{x}$, condition (C), that is,

$$\text{there exists } y_0 \in \mathbb{R}^n \text{ such that } \langle \nabla g_i(\bar{x}), y_0 \rangle < 0 \text{ for each } i \in I(\bar{x}),$$

is called Cottle's constraint qualification, see [10]. To show the result, we give the following lemma:

Lemma 3.4. *Let Λ be an index set, and let $A_\lambda \subseteq \mathbb{R}^n$, $\lambda \in \Lambda$, be non-empty convex sets. If $\bigcap_{\lambda\in\Lambda} \operatorname{int} A_\lambda \neq \emptyset$, then $\operatorname{cl} \bigcap_{\lambda\in\Lambda} \operatorname{int} A_\lambda = \bigcap_{\lambda\in\Lambda} \operatorname{cl} A_\lambda$.*

Proof. The equality $\operatorname{cl} \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} \operatorname{cl} A_\lambda$ is shown straightforwardly and omitted. Since $\operatorname{cl}\operatorname{int} A_\lambda = \operatorname{cl} A_\lambda$ for each $\lambda \in \Lambda$, the equality of this lemma holds. □

Theorem 3.5. *Let $\bar{x} \in S$. Then (C) implies (B).*

Proof. Assume (C). There exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0\rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. That is, for each $i \in I(\bar{x})$,

$$g_i^\circ(\bar{x}, y_0) = \max_{\xi\in\partial^\circ g_i(\bar{x})} \langle \xi_i, y_0\rangle < 0.$$

Since $g_i^\circ(\bar{x}, \cdot)$ is a real-valued convex function on $\mathbb{R}^n$ and $g_i^\circ(\bar{x}, y_0) < 0$, by using Theorem 2.1,

$$\operatorname{int}\{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) \leq 0\} = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) < 0\}.$$

Also, it is clear that $\partial^\circ g_i(\bar{x})^- = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) \leq 0\}$. Thus,

$$\operatorname{int}\partial^\circ g_i(\bar{x})^- = \{y \in \mathbb{R}^n \mid g_i^\circ(\bar{x}, y) < 0\} \ni y_0. \tag{3.1}$$

Consequently, we have $\bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) \neq \emptyset$. By using Lemma 3.4, we have

$$\operatorname{cl} \bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) = \bigcap_{i\in I(\bar{x})} \operatorname{cl}(\partial^\circ g_i(\bar{x})^-) = \bigcap_{i\in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-). \tag{3.2}$$

Next, we show

$$\bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-) \subseteq T_S(\bar{x}). \tag{3.3}$$

Let $y \in \bigcap_{i\in I(\bar{x})} \operatorname{int}(\partial^\circ g_i(\bar{x})^-)$. For each $i \in I(\bar{x})$, from (3.1) and the regularity of g_i at $\bar{x}$, we have $g_i'(\bar{x}, y) < 0$. Then, there exists $t_i > 0$ such that $g_i(\bar{x} + ty) < 0$ for each $t \in (0, t_i]$. Moreover, for each $i \in I \setminus I(\bar{x})$, from the continuity of g_i and $g_i(\bar{x}) < 0$, there exists $t_i > 0$ such that $g_i(\bar{x} + ty) < 0$ for each $t \in (0, t_i]$. Put $t_0 = \min\{t_i \mid i \in I\}$, for each $t \in (0, t_0)$

$$\text{for each } i \in I, g_i(\bar{x} + ty) < 0. \tag{3.4}$$

Then $\bar{x} + ty \in S$ for each $t \in (0, t_0]$. For each $k \in \mathbb{N}$, put $x_k = \bar{x} + \frac{t_0}{k} y$ and $\alpha_k = \frac{k}{t_0}$. Then $\{\alpha_k(x_k - \bar{x})\} \subseteq \operatorname{cone}(S - \bar{x})$ and $\alpha_k(x_k - \bar{x}) \to y$, that is, $y \in T_S(\bar{x})$. Thus (3.3) holds. By using (3.2) and (3.3), we have

$$\bigcap_{i\in I(\bar{x})} (\partial^\circ g_i(\bar{x})^-) \subseteq T_S(\bar{x}).$$

The converse inclusion $T_S(\bar{x}) \subseteq \bigcap_{i\in I(\bar{x})}(\partial^\circ g_i(\bar{x})^-)$ holds from Lemma 3.1.

Finally, we prove that $\operatorname{cone}\operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is closed, that is,

$$\operatorname{cl}\operatorname{cone}\operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}) \subseteq \operatorname{cone}\operatorname{co} \bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x}).$$

We may assume that $I(\bar{x}) \neq \emptyset$. Let $y \in \operatorname{cl}\operatorname{cone}\operatorname{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. There exists $\{y_k\} \subseteq \operatorname{cone}\operatorname{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ such that $y_k \to y$. For each $k \in N$, there exist $\lambda^k = (\lambda_i^k)_{i\in I(\bar{x})} \in \mathbb{R}_+^{I(\bar{x})}$ and $x^k = (x_i^k)_{i\in I(\bar{x})} \in \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ such that $y_k = \sum_{i\in I(\bar{x})} \lambda_i^k x_i^k$. From (C), there exists $y_0 \in \mathbb{R}^n$ such that $g_i^\circ(\bar{x}, y_0) < 0$ for each $i \in I(\bar{x})$. Put $r = \max_{i\in I(\bar{x})} g_i^\circ(\bar{x}, y_0)$. For each $i \in I(\bar{x})$, $\langle x_i^k, y_0\rangle \le r < 0$. Thus, $\langle y_k, y_0\rangle \le r \sum_{i\in I(\bar{x})} \lambda_i^k$. Since $\langle y_k, y_0\rangle \to \langle y, y_0\rangle$,

$$\langle y, y_0\rangle - 1 < \langle y_k, y_0\rangle \le r \sum_{i\in I(\bar{x})} \lambda_i^k$$

hold for sufficiently large k, that is,

$$\|\lambda^k\| \le \sum_{i\in I(\bar{x})} \lambda_i^k \le \frac{\langle y, y_0\rangle - 1}{r} (=: K).$$

Therefore, $\{(\lambda^k, x^k)\} \subseteq \operatorname{cl} B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. From the compactness of $\operatorname{cl} B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$, there exist $(\lambda, x) = (\lambda_i, x_i)_{i\in I(\bar{x})} \in \operatorname{cl} B(0,K) \times \prod_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$ and a subsequence $\{(\lambda^{k_j}, x^{k_j})\}$ of $\{(\lambda^k, x^k)\}$ such that $(\lambda^{k_j}, x^{k_j}) \to (\lambda, x)$. Moreover, we have $\lambda_i \ge 0$, $x_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, and $y = \sum_{i\in I(\bar{x})} \lambda_i x_i$. Thus, $y \in \operatorname{cone}\operatorname{co}\bigcup_{i\in I(\bar{x})} \partial^\circ g_i(\bar{x})$. This completes the proof. □

Remark 3.6. (1) The converse of Theorem 3.5 is not true in general, see Example 3.7.

(2) From (3.4), (C) implies the Slater condition. However, the converse is not true in general, see Example 3.8.

(3) In Example 3.8, the Slater condition does not imply (A). Therefore, the Slater condition is not a constraint qualification for the optimality conditions in convex optimization problem (P).

Example 3.7. Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$g(x) = |x|.$$

Then $S = \{0\}$, $T_S(0) = \{0\}$ and $\partial^\circ g(0) = [-1, 1]$. So that, $\partial^\circ g(0)^- = \{0\}$ and $\partial^\circ g(0)$ is closed. Thus (B) holds. On the other hand, for each $y \in \mathbb{R}$, $\frac{y}{|y|+1} \in \partial^\circ g(0)$ and $\frac{y}{|y|+1} y \ge 0$, and then (C) does not hold.

Example 3.8. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$g(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if} \quad x_1 \ge 0, x_2 \ge 0, \\ \|(x_1, x_2)\| + x_2 & \text{if} \quad x_1 \ge 0, x_2 < 0, \\ \|(x_1, x_2)\| + x_1 & \text{if} \quad x_1 < 0, x_2 \ge 0, \\ -x_1 x_2 & \text{if} \quad x_1 < 0, x_2 < 0. \end{cases}$$

Then $S = -\mathbb{R}_+^2$, S is convex, g is regular at $(0,0)$ and the Slater condition holds. On the other hand, $N_S(0,0) = \mathbb{R}_+^2$ and $\operatorname{cone}\partial^\circ g(0,0) = \{(0,0)\} \cup \operatorname{int}\mathbb{R}_+^2$. Hence, (A) does not hold. Thus (C) does not hold.

Next we consider the relationship of (C), (D), (E) and (F). From Theorem 2.2, condition (D), given by Dutta and Lalitha, is a sufficient constraint qualification for the optimality conditions in convex optimization problem (P). Conditions (E) and (F) are motivated by (C) and (D), respectively.

We show the relationship of (C), (D), (E) and (F) as follows:

Theorem 3.9. *Let $\bar{x} \in S$. Then (C), (D), (E) and (F) are equivalent.*

Proof. First, we prove (C) implies (D). Assume (C). There exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi_i, y_0 \rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. It is clear that $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$. In addition, the Slater condition holds from (2) of Remark 3.6. Thus (D) holds.

Next, we prove (D) implies (F). Assume (D). Then $0 \notin \partial^\circ g_i(\bar{x})$ for each $i \in I(\bar{x})$, and it is easy to show that $\operatorname{int} S$ is non-empty from the Slater condition and the continuity of all g_i. Thus (F) holds.

Next, we prove (F) implies (E). Assume that (E) dos not hold. Then, there exist $\lambda_i \in \mathbb{R}_+$ and $\xi_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that

$$\begin{cases} \sum_{i \in I(\bar{x})} \lambda_i = 1, \\ \sum_{i \in I(\bar{x})} \lambda_i \xi_i = 0. \end{cases}$$

From (F), we have $\xi_i \neq 0$ for each $i \in I(\bar{x})$. Also from (F), there exists $x_0 \in \mathbb{R}^n$ and $r > 0$ such that $B(x_0, r) \subseteq S$. For each $i \in I(\bar{x})$, since $x_0 + \frac{r}{2\|\xi_i\|}\xi_i \in B(x_0, r) \subseteq S$, then for each $i \in I(\bar{x})$, $\partial^\circ g_i(\bar{x}) \subseteq N_S(\bar{x})$ from Lemma 3.1, that is, $\xi_i \in N_S(\bar{x})$. So for each $i \in I(\bar{x})$,

$$\langle \xi_i, x_0 - \bar{x} \rangle + \frac{r}{2}\|\xi_i\| = \left\langle \xi_i, x_0 + \tfrac{r}{2\|\xi_i\|}\xi_i - \bar{x} \right\rangle \leq 0.$$

Therefore,

$$\frac{r}{2} \sum_{i \in I(\bar{x})} \lambda_i \|\xi_i\| = \left\langle \textstyle\sum_{i \in I(\bar{x})} \lambda_i \xi_i, x_0 - \bar{x} \right\rangle + \frac{r}{2} \sum_{i \in I(\bar{x})} \lambda_i \|\xi_i\| \leq 0.$$

From $\sum_{i \in I(\bar{x})} \lambda_i = 1$ and $\xi_i \neq 0$ for each $i \in I(\bar{x})$,

$$0 < \frac{r}{2} \sum_{i \in I(\bar{x})} \lambda_i \|\xi_i\|.$$

This is a contradiction.

Finally, we prove (E) implies (C). Assume (E). Since $\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ is non-empty, convex and closed and $0 \notin \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$ from (E), the point 0 can be strongly separated from $\operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$, that is there exists $y_0 \in \mathbb{R}^n$ such that $\langle \xi, y_0 \rangle < 0$ for each $\xi \in \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial^\circ g_i(\bar{x})$. Thus, $\langle \xi_i, y_0 \rangle < 0$ for each $i \in I(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$. Therefore (C) holds. This completes the proof. □

Finally, we consider the relationship of (E) and (G). When all g_i are continuously differentiable at $\bar{x}$, condition (G), that is

$$\{\nabla g_i(\bar{x})\}_{i \in I(\bar{x})} \text{ is linearly independent,}$$

is called the linearly independent constraint qualification, see [3, 10].

Theorem 3.10. *Let $\bar{x} \in S$. Then (G) implies (E).*

Proof. Assume that (E) does not hold. Then, there exist $\lambda_i \in \mathbb{R}_+$ and $x_i \in \partial^\circ g_i(\bar{x})$, $i \in I(\bar{x})$, such that

$$\begin{cases} \sum_{i \in I(\bar{x})} \lambda_i = 1, \\ \sum_{i \in I(\bar{x})} \lambda_i x_i = 0. \end{cases}$$

Thus (G) does not hold. □

The converse of Theorem 3.10 is not true in general. See the following example:

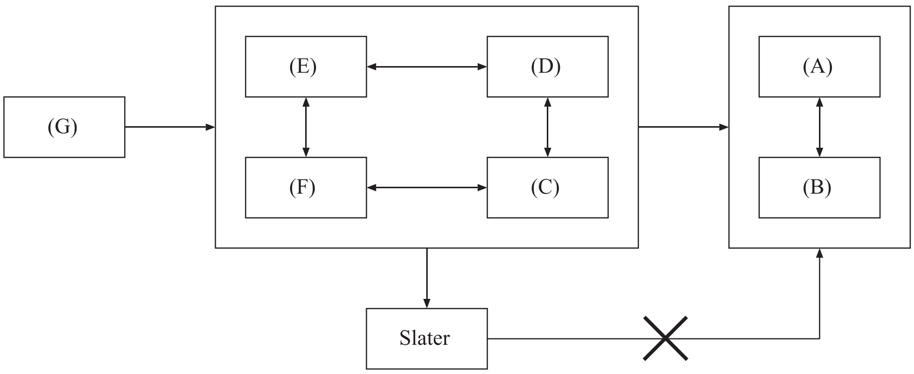
Example 3.11. Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be functions as follows:

$$g_1(x) = (x-1)(x+1), g_2(x) = \frac{1}{2}(x-1)(x+1).$$

Then $S = [-1, 1]$, $\operatorname{int} S \neq \emptyset$, $I(1) = \{1, 2\}$, $\partial^\circ g_1(1) = \{2\}$ and $\partial^\circ g_2(1) = \{1\}$. Thus (F) holds. On the other hand, it is clear that $\{2, 1\}$ is not linearly independent. Hence (G) does not hold.

4. Conclusion

In this paper, we have presented constraint qualifications for KKT optimality condition in a convex optimization problem under locally Lipschitz constraints which was discuss by Dutta and Lalitha in [2], and compared our results to previous ones. First, we introduced two necessary and sufficient constraint qualifications for KKT optimality condition. Moreover we proposed constraint qualifications, and discussed the relationship of these constraint qualifications. On the other hand, it was shown that the Slater condition was not a constraint qualification in this optimization. The following figure shows the relationship of the constraint qualifications, which were introduced in this paper, for optimality conditions:



References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
[2] J. Dutta and C. S. Lalitha, *Optimality conditions in convex optimization revisited*, Optim. Lett. **7** (2013), 221–229.
[3] R. G. Eustaquio, E. W. Karas and A. A. Ribeiro, *Constraint qualifications for nonlinear programming*, Federal University of Parana, Brazil, 2008.

[4] J. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms I*, Grundlehren der Mathematschen Wissenschaften **305**, Springer-verlag, New York, 1993.
[5] H. Komiya, *Elementary proof for Sions minimax theorem*, Kodai Math. J. **11** (1988), 5–7.
[6] J. B. Lasserre, *On representations of the feasible set in convex optimization*, Optim. Lett. **4** (2010), 1–5.
[7] C. Li, K. F. Ng and T. K. Pong, *Constraint qualifications for convex inequality system with applications in constrained optimization*, SIAM J. Optim. **19** (2008), 163–187.
[8] M. Sion, *On general minimax theorems*, Pac. J. Math. **8** (1958), 171–176.
[9] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Glob. Optim. **45** (2009), 551–563.
[10] Z. Wang, S.-C. Fang and W. X. Xing, *On constraint qualification, Design and inter-relations*, J. Ind. Manag. Optim. **9** (2013), 983–1001.

Manuscript received 14 October 2015
revised 16 February 2016

S. Yamamoto
Interdisciplinary Graduate School of Science and Engineering, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: `yamamoto@math.shimane-u.ac.jp`

D. Kuroiwa
Major in Interdisciplinary Science and Engineering, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: `kuroiwa@math.shimane-u.ac.jp`