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ON THE DYNAMICS OF LOVE: A MODEL INCLUDING SYNERGISM

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Abstract. In their paper [10], entitled "Dynamics of Love and Happiness: A Mathematical Analysis", D. Satsangi and Arun K. Sinha proposed a dynamical model which describes love affairs between two individuals. Similar to Rinaldi's model, it includes the phenomena of oblivion, return (both described by linear functions) and instinct (supposed constant), but introduces a nonlinear component for each member of the couple, with the meaning of synergism and emotional interaction. The concept is interesting and appears to be more realistic than those models known in the literature; however, unfortunately the stability analysis is wrong. In the present paper we follow their modelling approach to discuss the dynamics of the feelings between two people by proposing a further modification to the original model, and provide corrections to the stability analysis.

1. Introduction

The suggestions of love and the pursuit of happiness represent the dominant goal of life for many people, in present and ancient time. This tendency tourns out also in litterature, where they constitute the leading subjects from time out of mind, and countless poems have been inspired by stories of plagued and passionate loves.

Instead, the power of mathematics has rarely been applied to the dynamics of romance; at least until 1988, when Steven H. Strogatz (unconsciously) introduces a new trend ([12], [13]): the study of dibatted romantic relationships via dynamic systems.

In the original intentions, the Harvard mathematician Strogatz wants to describe a model that "relates mathematics to a topic that is already on the minds of many college students: the time evolution of a love affair between two people". Hence he bases the romantic affair on the story of two well known Shakespereanes characters, Romeo and Juliet. Nevertheless, in the considered case, it is not their families that keep them apart, but Romeo's fickleness.

Indeed Romeo recits

"My love for Juliet decreases in proportion to her love for me!",

while Juliet says

"My love for Romeo grows in proportion to his love for me!".

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Then the more Juliet loves Romeo, the more he wants to walk away, while if she shows herself detached, he is attracted to her. Conversely, Juliet warms up if he loves her and grows cold when he hates her.

The governing equations ([12]), based on simple harmonic oscillator system, are given by

$$\begin{cases} \frac{dR(t)}{dt} = -aJ(t) \\ \frac{dJ(t)}{dt} = bR(t). \end{cases} \tag{1.1}$$

where the parameters a, b are positive, while R(t) is Romeo's love for Juliet at time t and $J(t)$ is Juliet's love for Romeo at the same time.

In so far, a question can arise: what is meant by love in this new context? Positive values of variable states R and J mean friendship or love, negative values signify hate and disdain, while zero means indifference.

Hence love and hate are not two mutually exclusive feelings, they can coexist in a same couple and their combined effect produce a neverending cycle of love and hate. In order to point out this state of chaos, Clarence Peterson, in his account [6] on Strogatz article [12], adopts the emblematic title

"As usual, boy+girl=confusion"

In his book [13] Strogatz has a short section, in which he contemplated not only others but also own feelings. Then system (1.1) becomes more general and it is given by

$$\begin{cases} \frac{dR}{dt} = aR + bJ \\ \frac{dJ}{dt} = cR + dJ \end{cases} \tag{1.2}$$

where the parameters a, b, c, d can have variable sign.

Consequently four romantic styles can be determined for Romeo and Juliet respectively. From the perspective of Romeo, there exist the personalities [13]:

(1) Eager beaver ($a > 0$ and $b > 0$);
(2) Cautious (or secure, or synergic) lover ($a < 0$ and $b > 0$);
(3) Narcissistic nerd ($a > 0$ and $b < 0$);
(4) Hermit ($a < 0$ and $b < 0$).

The same styles can be exhibited for Juliet, discussing the parameters c and d.

In 2004 Sprott, from University of Wisconsin, considers the case in which the couple becomes a love triangle since Romeo has a mistresse, Guinevre [11]. He supposes that Romeo adopts the same romantic styles towards his lovers and (as often happens!) they do not know about each other.

The resulting model is

$$\begin{cases} \frac{dR_J}{dt} = aR_J + b(J - G) \\ \frac{dJ}{dt} = cR_J + dJ \\ \frac{dR_G}{dt} = aR_G + b(G - J) \\ \frac{dJ}{dt} = eR_G + fG \end{cases} \tag{1.3}$$

with the parameters a, b, c, d, e, f with variable signs.

From the first and third equation of system (1.3), it can be noted that Juliet's feelings for Romeo influence his feelings for Guinevre in a way that is exactly opposite to Guinevre's way to affect Romeo's feelings toward Juliet.

Let's look back to system (1.2). It has been generally acknowledged that the model, even if suggestive, is unrealistic. Indeed it does not explain why two individuals, who are initially completely indifferent, begin to develope a love affair.

With the aim to improve Strogatz's model, in 1998 ([7]) Sergio Rinaldi introduces the linear system

$$\begin{cases} \dot{x}_1(t) = -\alpha_1 x_1(t) + \beta_1 x_2(t) + \gamma_1 A_2 \\ \dot{x}_2(t) = -\alpha_2 x_2(t) + \beta_2 x_1(t) + \gamma_2 A_1 \end{cases} \tag{1.4}$$

with $\alpha_i, \beta_i, \gamma_i, A_i > 0$ for $i = 1, 2$.

It takes into account three aspects of a love affair: the *oblivion* $-\alpha_i x_i(t)$, that gives rise to a loss of interest in partner i; the *return* $\beta_i x_j(t)$, which measures the pleasure of i to being loved; the *instinct* $\gamma_i A_j$, expressing the reaction to the appeal of individual j.

The same processes are considered by the author in the model which describes the tempestuous love between Petrarca and his platonic dame Laura ([8]), given by

$$\begin{cases} \frac{dL(t)}{dt} = -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P \\ \frac{dP(t)}{dt} = -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)} \\ \frac{dZ(t)}{dt} = -\alpha_3 Z(t) + \beta_3 P(t) \end{cases} \tag{1.5}$$

where $P(t)$ and $L(t)$ are measures of Petrarca and Laura's emotions respectively, while $Z(t)$ is a new variable state, meaning poetic inspiration. On closer view, in the second equation of (1.5) the instinct function of the poet

$$\beta_2 \frac{A_L}{1 + \delta Z(t)}$$

depends not only upon Laura's appeal component A_L, but also upon his poetic inspiration $Z(t)$. Especially it increases when A_L boosts and decreases when the poetic inspiration intensifies. This would stress the fact that moral tension weakens the most basic instincts ([8]).

Compared to the previous model, the main innovative factor of system (1.4) is the appeal component A_i, that turns out to be the "driving force that creates order in the community"; where, the concept of order (or stability) is realized when the partner of the most attractive woman is the most attractive man [7].

The same results hold if the couple is composed by secure individuals (Rinaldi e Gragnani, [9]).

However Rinaldi's model is still minimal, because

- The influences of the external world are not contemplated, the world is kept frozen;
- The behavioral parameters α_i, β_i, γ_i and the appeals A_i are assumed to be constant, hence the model can be used only to do predictions for short periods of time;

- The mechanisms of synergism and adaptation are considered negligible, i.e. oblivion and return functions depend only upon one state variable.

Following the suggestion of Rinaldi, but including the emotional interaction component, in 2012 Satsangi e Sinha propose the dynamical model

$$\begin{cases} \frac{dx_1}{dt} = -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 \\ \frac{dx_2}{dt} = -\alpha_2 x_2 + \beta_2 x_1 + A_1 + d_2 x_1 x_2. \end{cases} \tag{1.6}$$

in which x_1, x_2 are a measure of love of first and second individual for their respective partners, and $\alpha_1, \beta_1, A_1, d_1,\ \alpha_2, \beta_2, A_2, d_2$ are positive constants with the following meaning: α_1, α_2 are oblivion parameters, β_1, β_2 are reactiveness coefficient, A_1, A_2 are individual appeals, d_1, d_2 mean synergism and emotional interaction. From model (1.6), it can be noted that the emotions are variable and, in addition, individual emotion cannot increase infinitely respect to the other.

Hence, compared to Rinaldi's model, the only new factor introduced here is the quadratic term $d_i x_i x_j$, with the meaning of synergism process. It is indicative of learning and adaptation process, deriving from the knowledge of the partner and the experience of relation, i.e. the learning effect after living together.

In our study we follow the same modelling approach but with a further modification to model (1.6). In that way a new dynamical system is obtained and it is presented in Section 2. Hence we go to determine its critical points. The aim of our reserch (Section 3) is to find suitable conditions in order to get asymptotic stability of the determined stationary points. This fact, from the perspective of love affair, allows us to establish if, at equilibrium, a romantic relation is characterized by a constant (or almost constant) behavior of the feelings or if it registers a brupt change of the emotions, if the initial emotional state (when they first meet) is slightly perturbed towards the equilibrium state. Here we assume that, when they meet for the first time, they are completely indifferent.

Finally, in Section 4 we consider the particular case of two romantic clones ("peas in a pod"), that is verified when the individuals have the same romantic styles. Also in this case we want to determine suitable conditions in order to obtain asymptotic stability.

The conclusion is that, under suitable hypothesis on the behavioral parameters, some solutions at equilibrium are asymptotically stable, while some others are ultimately unstable.

2. The model

The proposed model is a nonlinear system of two first order differential equations, given by

$$\begin{cases} \dot{x}_1 = -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 \\ \dot{x}_2 = -\alpha_2 x_2 + \beta_2 x_1 + A_1 - d_2 x_1 x_2 \end{cases}, \tag{2.1}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, d_1, d_2, A_1, A_2$ are positive constants.

If we put

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$A = \begin{bmatrix} -\alpha_1 & \beta_1 \\ \beta_2 & -\alpha_2 \end{bmatrix}$$

and

$$f(\bar{x}) = \begin{bmatrix} f_1(\bar{x}) \\ f_2(\bar{x}) \end{bmatrix} = \begin{bmatrix} A_2 + d_1 x_1 x_2 \\ A_1 - d_2 x_1 x_2 \end{bmatrix},$$

it can be rewritten in the standard form

$$\dot{\bar{x}} = A\bar{x} + f(\bar{x}).$$

Such a system is an autonomous system, because it does not depend on time t.

In this context the state variables x_1 and x_2 are measures of first and second individual emotions respectively and, for $i = 1, 2$, positive values of $x_i(t)$ mean friendship or love, negative values signify hate and zero value means complete indifference.

Especially, for $i, j = 1, 2$ and $j \neq i$, we have that

(1) $-\alpha_i x_i(t)$ denotes the forgetting process, that gives rise to a loss of interest in individual i. So that, in absence of the person j, the feelings of individual i decay esponentially, according to $x_i(t) = x_i(0)e^{-\alpha_i t}$;
(2) $\beta_i x_j(t)$ is the reaction function, describing the pleasure of individual i to be loved by partner j;
(3) $d_i x_i(t) x_j(t)$ is the synergism function, which describe the emotional interaction process of the couple (i.e. the adaptation process after living together);
(4) A_i is a measure of the attractiveness of individual i.

From first equation of (2.1) we can observe that synergism component is a source of interest for the emotions of individual 1, similarly to the terms due to return and attractiveness processes; while for individual 2 the emotional interaction function contribute to the decay of love feelings, as it can be noted by equation 2 of (2.1). This is the basic difference factor that we have introduced in the original model. (1.6).

Moreover the model is still minimal. Indeed love is a complete mixture of feelings and it cannot be easily described by single state variable for each member of the couple and the personalities are assumed to be constant in time.

In the next section we will find stationary points of the previous model.

3. Critical points and stability analysis

In order to determine the critical points we give the following:

Proposition 3.1. *Let us assume that*

(1) $d_2 A_2 > d_1 A_1$,
(2) $\frac{d_2}{d_1} > \frac{\alpha_2 \beta_2 A_2}{\alpha_1 \beta_1 A_1}$,
(3) $\frac{d_2}{d_1} \neq \frac{\alpha_2}{\beta_1}$,
(4) $\frac{d_2}{d_1} \neq \frac{\beta_2}{\alpha_1}$,
(5) $-\alpha_2 \beta_2 + d_2 A_1 = 0$,

then system (2.1) *has three couples of critical points.*

Proof. First of all we recall that, if the general nonlinear autonomous system is

$$\dot{x}(t) = f(x(t)) \tag{3.1}$$

than x_e is a critical point for the system if and only if

$$f(x_e) = 0. \tag{3.2}$$

Hence, for system (2.1), we need to solve the system

$$\begin{cases} -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 = 0 \\ -\alpha_2 x_2 + \beta_2 x_1 + A_1 - d_2 x_1 x_2 = 0. \end{cases} \tag{3.3}$$

Supposed $x_1 \neq \frac{-\alpha_2}{d_2}$, we can obtain x_2 from the second equation of (3.3), thus system (3.3) becomes:

$$\begin{cases} -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 = 0, \\ x_2 = \frac{\beta_2 x_1 + A_1}{\alpha_2 + d_2 x_1}. \end{cases} \tag{3.4}$$

Otherwise if $x_1 = \frac{-\alpha_2}{d_2}$, for condition (5), the first equation of (3.3) is identically satisfied and system (3.3) reduces to

$$\begin{cases} x_1 = -\frac{\alpha_2}{d_2}, \\ -\alpha_2 x_2 + \beta_2 x_1 + A_1 - d_2 x_1 x_2 = 0. \end{cases} \tag{3.5}$$

Since, from hypothesis, conditions (1), (2), (3), (4), (5) hold, system (3.4) has the following two couples of solutions:

$$\begin{aligned} x_{1_1} = \frac{1}{2(\beta_2 d_1 - \alpha_1 d_2)} \big\{ & \alpha_1\alpha_2 - \beta_1\beta_2 - d_1 A_1 - d_2 A_2 \\ & + [\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2 A_1^2 + d_2^2 A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ & -2\alpha_1\alpha_2 d_1 A_1 + 2\alpha_1\alpha_2 d_2 A_2 - 2\beta_1\beta_2 d_1 A_1 + 2\beta_1\beta_2 d_2 A_2 \\ & +2 d_1 d_2 A_1 A_2 + 4\alpha_1\beta_1 d_2 A_1 - 4\alpha_2\beta_2 d_{1_{A_2}}]^{\frac{1}{2}} \big\} \end{aligned}$$

$$x_{1_2} = \frac{\beta_2 x_1 + A_1}{\alpha_2 + d_2 x_1}$$

and

$$\begin{aligned} x_{2_1} = \frac{1}{2(\beta_2 d_1 - \alpha_1 d_2)} \big\{ & \alpha_1\alpha_2 - \beta_1\beta_2 - d_1 A_1 - d_2 A_2 \\ & - [\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2 A_1^2 + d_2^2 A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ & -2\alpha_1\alpha_2 d_1 A_1 + 2\alpha_1\alpha_2 d_2 A_2 - 2\beta_1\beta_2 d_1 A_1 + 2\beta_1\beta_2 d_2 A_2 \\ & +2 d_1 d_2 A_1 A_2 + 4\alpha_1\beta_1 d_2 A_1 - 4\alpha_2\beta_2 d_1 A_2]^{\frac{1}{2}} \big\}, \end{aligned}$$

$$x_{2_2} = \frac{\beta_2 x_1 + A_1}{\alpha_2 + d_2 x_1}$$

,

while system (3.5) admits the unique solution

$$x_{3_1} = -\frac{\alpha_2}{d_2},$$
$$x_{3_2} = \frac{\alpha_1\alpha_2 + d_2A_2}{\alpha_2 d_1 - \beta_1 d_2}.$$

In order to semplify the notation we put

$$h = \alpha_1\alpha_2 - \beta_1\beta_2 - d_1A_1 - d_2A_2,$$
$$\begin{aligned} l = [&\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + d_1^2A_1^2 + d_2^2A_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 \\ &-2\alpha_1\alpha_2 d_1A_1 + 2\alpha_1\alpha_2 d_2A_2 - 2\beta_1\beta_2 d_1A_1 + 2\beta_1\beta_2 d_2A_2 \\ &+2d_1d_2A_1A_2 + 4\alpha_1\beta_1 d_2A_1 - 4\alpha_2\beta_2 d_1A_2]^{\frac{1}{2}}, \end{aligned}$$
$$n = 2(\beta_2 d_1 - \alpha_1 d_2),$$

hence the equilibrium points can be rewritten as

$$\left(\frac{h+l}{n}, \frac{\beta_2(h+l) + A_1 n}{\alpha_2 n + d_2(h+l)}\right), \tag{3.6}$$

$$\left(\frac{h-l}{n}, \frac{\beta_2(h-l) + A_1 n}{\alpha_2 n + d_2(h-l)}\right), \tag{3.7}$$

$$\left(-\frac{\alpha_2}{d_2}, \frac{\alpha_1\alpha_2 + d_2A_2}{2(\alpha_2 d_1 - \beta_1 d_2)}\right). \tag{3.8}$$

Let us introduce

$$k = \alpha_1\alpha_2 - \beta_1\beta_2 + d_1A_1 + d_2A_2,$$
$$m = 2(\alpha_2 d_1 - \beta_1 d_2).$$

Since it can be proved that

$$\frac{\beta_2(h+l) + A_1 n}{\alpha_2 n + d_2(h+l)} = \frac{k+l}{m}$$

and

$$\frac{\beta_2(h-l) + A_1 n}{\alpha_2 n + d_2(h-l)} = \frac{k-l}{m},$$

definitively the critical points can be rapresented as

$$\left(\frac{h+l}{n}, \frac{k+l}{m}\right),$$
$$\left(\frac{h-l}{n}, \frac{k-l}{m}\right).$$
$$\left(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2 + d_2A_2}{m}\right).$$

□

The next purpose is to study asymptotic stability of the determined points.

First of all, we observe that for this class of points the following definitions hold [5]:

Definition 3.2. The equilibrium state x_e of system (3.2) is stable if for every given $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if

$$\|x(0) - x_e\| < \delta$$

then

$$\|x(t) - x_e\| < \epsilon$$

for all $t \geq 0$.

Otherwise it is unstable.

Definition 3.3. The equilibrium state x_e of system (3.2) is asymptotically stable if it is stable and exists $\delta > 0$ such that if

$$\|x(0) - x_e\| < \delta$$

then

$$\lim_{t\to\infty} \|x(t) - x_e\| = 0.$$

In order to testing the previous stability properties for (3.6), (3.7) and (3.8), we recall a well known criterion, proposed for the first time by Lyapunov, often cited in litterature as *Lyapunov's indirect method.* This method is based on the linearization of the nonlinear system (3.2) in a neighborhood of the considered equilibrium state x_e, that is

$$\dot{\delta x(t)} = J(x_e)\delta x(t) \tag{3.9}$$

where $\delta x(t) = x(t) - x_e$ is a measure of the distance between the perturbed state and the equilibrium state at time t, and

$$J(x_e) = \left|\begin{matrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{matrix}\right|_{x=x_e}$$

is the Jacobian matrix associated to f at $x = x_e$. System (3.9) is called *linearized* system and its dynamic matrix is the Jacobian matrix of f at $x = x_e$.

At system (3.9) can be applied the analysis techniques which hold for linear systems, and the stability results obtained implies the stability of the original non linear system (3.2) in a way that is expressed in the following [4]:

Theorem 3.4. *(Lyapunov's first criterion)*

- *If all eigenvalues of the matrix $J(x_e)$ have negative real parts, then the equilibrium state x_e is asymptotically stable for the original nonlinear system;*
- *If the matrix $J(x_e)$ has one or more eigenvalues with positive real part, the the equilibrium state x_e is unstable.*

Remark 3.5. If $J(x_e)$ has at least an eigenvalue λ on the imaginary axis ($Re(\lambda) = 0$) and all others are in the left half of the complex plane, then one cannot conclude any type of stability at $x = x_e$ for the nonlinear original system.

Therefore stability properties of equilibrium points in a nonlinear system can be analyzed by locating zeros of the characteristic polynomial associated to the considered problem.

Let us proceed to apply the Lyapunov indirect method to our case.

The Jacobian matrix associated to (2.1) in $x_e = (x_{e_1}, x_{e_2})$ turns out to be

$$J(x_e) = \begin{vmatrix} -\alpha_1 + d_1 x_{e_2} & \beta_1 + d_1 x e_1 \\ \beta_2 - d_2 x_{e_2} & -\alpha_2 - d_2 x_{e_1} \end{vmatrix}$$

and its corresponding characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.10}$$

where

$$a_1 = \alpha_1 + \alpha_2 + d_2 x_{e_1} - d_1 x_{e_2}, \tag{3.11}$$

$$a_2 = \alpha_1\alpha_2 - \beta_1\beta_2 - \frac{n}{2}x_{e_1} - \frac{m}{2}x_{e_2}. \tag{3.12}$$

The following result holds:

Proposition 3.6. *Let us assume that* (1), (2), (3), (4), (5) *hold. Then*

(1) *The critical point* $(\frac{h+l}{n}, \frac{k+l}{m})$ *is unstable for system* (2.1)*;*

(2) *If the conditions*

$$\alpha_2 < \beta_1, \tag{3.13}$$

$$d_1 < d_2, \tag{3.14}$$

$$d_1 A_1 > 4\alpha_1\alpha_2 \tag{3.15}$$

$$d_1 A_1 > 4\beta_1\beta_2 \tag{3.16}$$

$$d_1 A_1 > 4 d_2 A_2 \tag{3.17}$$

hold, then the critical point $\left(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2 + d_2 A_2}{m}\right)$ *is aymptotically stable for system* (2.1)*;*

(3) *Supposed that* A_1 *and* A_2 *are both negligible compared to the other parameters, then the critical point* $(\frac{h-l}{n}, \frac{k-l}{m})$ *is asymptotically stable for system* (2.1)*.*

Proof. (1) In the present case, the characteristic polynomial associated to the problem at $x_e = (\frac{h+l}{n}, \frac{k+l}{m})$ is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.18}$$

with coefficients a_1 and a_2 given by

$$a_1 = \alpha_1 + \alpha_2 + d_2\frac{h+l}{n} - d_1\frac{k+l}{m},$$

$$a_2 = -l.$$

Since $a_2 < 0$, there exists at least a positive root of $P(\lambda)$. This fact implies, by Lyapunov's first criterion, that $(\frac{h+l}{n}, \frac{k+l}{m})$ is an unstable equilibrium state for system (2.1).

(2) The considered equilibrium state is $x_e = \big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2+d_2A_2}{m}\big)$, therefore the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{3.19}$$

with coefficients

$$a_1 = \alpha_1 - 2d_1\frac{\alpha_1\alpha_2 + d_2A_2}{m},$$

$$a_2 = -\beta_1\beta_2 + \frac{n}{2}\frac{\alpha_2}{d_2} - d_2A_2.$$

From hypothesis (3.13), (3.14), (3.15), (3.16) and (3.17) we get that

$$a_1 = -d_2\frac{\alpha_1\beta_1 + A_2d_1}{\alpha_2d_1 - \beta_1d_2} > 0$$

and

$$a_2 = -\alpha_1\alpha_2 - \beta_1\beta_2 + d_1A_1 - d_2A_2 > 0.$$

Hence the characteristic polynomial $P(\lambda)$ has both the roots with negative real part. From Lyapunov's first criterion, we conclude that the point $\big(-\frac{\alpha_2}{d_2}, 2\frac{\alpha_1\alpha_2+d_2A_2}{m}\big)$ is asymptotically stable for system (2.1).

(3) As in the above point, we consider the characteristic polynomial associated to system (2.1) at $x_e = (\frac{h-l}{n}, \frac{k-l}{m})$, that is given by

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 \tag{3.20}$$

and it has coefficients

$$a_1 = \alpha_1 + \alpha_2 - d_1\frac{k-l}{m} + d_2\frac{h-l}{n},$$

$$a_2 = l.$$

We readily note that a_2 is positive.
According to the hypothesis for which A_1 and A_2 must be negligible compered to the other behavioral parameters, we can assume that

$$A_1 = \epsilon_1$$

and

$$A_2 = \epsilon_2,$$

with ϵ_1, ϵ_2 both negligible compared to the other parameters, so that

$$l \simeq |\alpha_1\alpha_2 - \beta_1\beta_2|,$$

$$k \simeq \alpha_1\alpha_2 - \beta_1\beta_2,$$

$$h \simeq \alpha_1\alpha_2 - \beta_1\beta_2.$$

Hence we obtain that

$$a_1 \simeq \alpha_1 + \alpha_2 > 0.$$

As in the previous case, we can conclude that the critical point $x_e = (\frac{h-l}{n}, \frac{k-l}{m})$ is asymptotically stable for system (2.1).

□

Remark 3.7. As we observed earlier, while in system

$$\begin{cases} \frac{dx_1}{dt} = -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 \\ \frac{dx_2}{dt} = -\alpha_2 x_2 + \beta_2 x_1 + A_1 + d_2 x_1 x_2, \end{cases}$$

introduced by Satsangi and Sinha, the synergism component is a source of interest for both individuals 1 and 2, in our model

$$\begin{cases} \frac{dx_1}{dt} = -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 \\ \frac{dx_2}{dt} = -\alpha_2 x_2 + \beta_2 x_1 + A_1 - d_2 x_1 x_2 \end{cases}$$

the same component is source of interest for the first individual, but contribute to the decay of the emotions in the second individual. This is the basic, but not unique, difference factor between the two models.

Indeed, article [10] contains some errors in the determination of solutions at equilibrium and in the study of stability properties of the determined critical points. Concerning the first point, it is not guaranted the existence of solutions at equilibrium, at least without doing further hypothesis. In addition, the same solutions at equilibrium, determined in the article, are not correct.

While, as regards the second point, it can be observed a wrong application of Routh-Hurwitz stability criterion. This last criterion allows us to decide if the roots, of the characteristic polynomial associated to the considered problem, all lye in the left half complex plain. Especially, in case $n = 2$, corresponding to the polynomial

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2,$$

the coefficients a_1 and a_2 must both be positive in order to satisfy Routh Hurwitz criterion ([1]). Instead, in article [10], Satsangi and Sinha deduce asymptotic stability property for system (1.6) in its critical points, starting from the fact that a_1 and a_2 have different sign, under suitable hypothesis. According to the above statement, these conclusions are not correct.

Remark 3.8. In case (2) of the above Proposition, asymptotic stability is achieved if the ratio of appeals $\frac{A_1}{A_2}$ is greater than the the quatruple of reciprocal of ratio of synergism coefficients (3.17), and the geometric mean of $d_1 A_1$ is greater than the double of geometric mean of forgetting parameters (3.15) and reactiveness coefficients (3.16). Moreover the second individual must be less reticent then the reactiveness of the first one and more synergic compared to individual 1, in order to get asymptotic stability.

Remark 3.9. We recall that, in Rinaldi's model, the introduction of components A_1 and A_2 explains why two individuals initially indifferent begin to develop a love affair, while, in system (2.1), they must be negligible compared to other components in order to get asymptotic stability for the equilibrium state $x_e = (\frac{h-l}{n}, \frac{k-l}{m})$ in case (3) of the previous Proposition. This fact does not seems too strange if one considers the presence of the forth component d_i of emotional interaction. This last constitues a measure of the synergism, adaptation and learning effect of the couple, i.e., after living together. Hence the obtained result seems to suggest that the equilibrium of the couple depends further upon the experience of the couple rather than the mutual attractiveness, that plays a crucial role in the early stage, although A_1 and A_2 must be never null.

Moreover, we give the following example

Example 3.10. *If we consider the couple of values*

$$(\alpha_1, \beta_1, d_1; \alpha_2, \beta_2, d_2) = \Big(\frac{1}{2}, \frac{1}{4}, \frac{1}{5}; \frac{1}{3}, \frac{1}{5}, \frac{1}{5}\Big),$$

with $\alpha_i, \beta_i, d_i \in (0,1)$ *for* $i = 1, 2$, *we can choose*

$$\epsilon_1 = \frac{1}{70} < \min\{\alpha_1, \beta_1, d_1\alpha_2, \beta_2, d_2\} \cdot 10^{-1}$$

and

$$\epsilon_2 = \frac{1}{60} < \min\{\alpha_1, \beta_1, d_1\alpha_2, \beta_2, d_2\} \cdot 10^{-1},$$

in order to get asymptotic stability.

4. Peas in a pod

Following a suggestion of Sprott ([11]), we consider the case in which the individuals involved in the romantic relationship have the same behavioral styles ("romantic clones"), e.g.

$$a_1 = a_2, \tag{4.1}$$

$$b_1 = b_2, \tag{4.2}$$

$$d_1 = d_2, \tag{4.3}$$

$$A_1 = A_2, \tag{4.4}$$

with $\alpha_1, \beta_1, d_1, A_1 > 0$, then the starting model becomes

$$\begin{cases} \dot{x}_1 = -\alpha_1 x_1 + \beta_1 x_2 + A_1 + d_1 x_1 x_2 \\ \dot{x}_2 = -\alpha_1 x_2 + \beta_1 x_1 + A_1 - d_1 x_1 x_2 \end{cases}. \tag{4.5}$$

In order to determine the critical points give the following:

Proposition 4.1. *Let us suppose that*

(6) $\alpha_1 \neq \beta_1$,

(7) $-\alpha_1\beta_1 + d_1 A_1 = 0$,

then system (4.5) *has three couples of critical points.*

Proof. For the perspective to find critical points we need to solve the system

$$\begin{cases} -\alpha_1 x_1 + \beta_1 x_2 + A_1 + d_1 x_1 x_2 = 0 \\ -\alpha_1 x_2 + \beta_1 x_1 + A_1 - d_1 x_1 x_2 = 0 \end{cases}. \tag{4.6}$$

If we assume that $x_1 \neq \frac{-\alpha_1}{d_1}$, we can get x_2 from second equation of (4.6), thus system (4.6) becomes

$$\begin{cases} -\alpha_1 x_1 + \beta_1 x_2 + A_2 + d_1 x_1 x_2 = 0 \\ x_2 = \frac{\beta_1 x_1 + A_1}{\alpha_1 + d_1 x_1} \end{cases}. \tag{4.7}$$

Otherwise if $x_1 = \frac{-\alpha_1}{d_1}$, since condition (7) hold, the first equation of (4.6) is identically satisfied and system (4.6) reduces to

$$\begin{cases} x_1 = -\frac{\alpha_1}{d_1}, \\ -\alpha_1 x_2 + \beta_1 x_1 + A_1 - d_1 x_1 x_2 = 0. \end{cases} \tag{4.8}$$

Let us introduce h, k, l, m, n, which are definied as in the previous section, considering the conditions (4.1), (4.2), (4.3), (4.4) on the behavioral parameters:

$$\begin{aligned} h &= \alpha_1^2 - \beta_1^2 - 2d_1 A_1, \\ k &= \alpha_1^2 - \beta_1^2 + 2d_1 A_1, \\ l &= \sqrt{(\alpha_1^2 - \beta_1^2)^2 + 4d_1^2 A_1^2}, \\ m &= 2(\alpha_1 - \beta_1)d_1, \\ n &= 2(\beta_1 - \alpha_1)d_1. \end{aligned}$$

Since conditions (6) and (7) hold, system (4.7) has solutions

$$\left(\frac{h+l}{n}, \frac{k+l}{m}\right),$$

$$\left(\frac{h-l}{n}, \frac{k-l}{m}\right),$$

while system (4.8) admit solution

$$\left(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2 + d_1 A_1}{m}\right).$$

□

Analougsly to the preceding section, we will study stability properties of the obtained critical points.

4.1. **Stability analysis.** Also in this context the principal tool of our analysis are the previously stated Lyapunov's first criterion and Routh theorem. Thus we must consider the characteristic poynomial associated to the system in each point and study the sign of its coefficients.

To this aim, first of all we introduce the Jacobian matrix associated to problem (4.5) at quilibrium state $x_e = (x_{e_1}, x_{e_2})$, given by

$$J(x_e) = \begin{vmatrix} -\alpha_1 + d_1 x_{e_2} & \beta_1 + d_1 x_{e_1}, \\ \beta_1 - d_1 x_{e_2} & -\alpha_1 - d_1 x_{e_1}. \end{vmatrix}$$

Thus, the corresponding characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1 \lambda + a_2,$$

with coefficients

$$a_1 = 2\alpha_1 - d_1 x_{e_2} + d_2 x_{e_1},$$

and

$$a_2 = \alpha_1^2 - \beta_1^2 - \frac{m}{2} x_{e_2} - \frac{n}{2} x_{e_1}.$$

In order to study stability properties we give the following:

Proposition 4.2. *Let us suppose that* (6) *and* (7) *hold, then*

(1) *The critical point* $(\frac{h+l}{n}, \frac{k+l}{m})$ *is unstable for system* (4.5)*;*
(2) *The critical point* $(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2+d_1A_1}{m})$ *is unstable for system* (4.5)*;*
(3) *Let us assume that*

$$\beta_1 = \frac{d_1 A_1}{\alpha_1}, \tag{4.9}$$

$$\alpha_1 > \beta_1, \tag{4.10}$$

then the critical point $(\frac{h-l}{n}, \frac{k-l}{m})$ *is asymptotically stable for system* (4.5).

Proof. (1) If the equilibrium state is $x_e = (\frac{h+l}{n}, \frac{k+l}{m})$, then the characteristic polynomial associated to the linearized system of problem (4.5) is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{4.11}$$

where

$$a_1 = 2\alpha_1 - d_1\frac{k+l}{m} + d_2\frac{h+l}{n}$$

and

$$a_2 = -l.$$

We readily note that $a_2 < 0$. Therefore there exists at least a positive root of $P(\lambda)$ and, for Lyapunov's first criterion, the point $(\frac{h+l}{n}, \frac{k+l}{m})$ is unstable for system (4.5).

(2) If the equilibrium state is $x_e = (-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2+d_1A_1}{m})$ then the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{4.12}$$

where

$$a_1 = -\frac{\alpha_1\beta_1 + d_1A_1}{\alpha_1 - \beta_1},$$

and

$$a_2 = -\alpha_1^2 - \beta_1^2 + \alpha_1\beta_1 - d_1A_1.$$

From relation (7) we get that

$$a_2 = -\alpha_1^2 - \beta_1^2 < 0.$$

Hence, as in the previous case, the characteristic polynomial (4.12) has at least a positive root and we conclude that the point $(-\frac{\alpha_1}{d_1}, 2\frac{\alpha_1^2+d_1A_1}{m})$ is unstable for system (4.5).

(3) The characteristic polynomial $P(\lambda)$ associated to the critical point $(\frac{h-l}{n}, \frac{k-l}{m})$ is given by

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 \tag{4.13}$$

where

$$a_1 = 2\alpha_1 - d_1\frac{k-l}{m} + d_2\frac{h-l}{n}$$

and

$$a_2 = l.$$

We readily see that a_2 is positive. Under hypothesis (4.9) and (4.10), we obtain that

$$a_2 = l = \alpha_1^2 + \beta_1^2 > 0$$

and

$$a_1 = 2\alpha_1 + \frac{2\beta_1^2}{\alpha_1 - \beta_1} > 0.$$

Hence the characteristic polynomial (4.13) has both the roots with negative real part. From Lyapunov's first criterion, we can conclude that $(\frac{h-l}{n}, \frac{k-l}{m})$ is asymptotically stable for system (4.5). □

Remark 4.3. In case (3) of Proposition (4.2), the assumptions for stability suggest that if the individuals are two romantic clones then the stability is guaranteed if the individuals have forgetting process coefficients greater then the reactiveness parameters, condition (4.10). In other words, they have the tendency to forget each other rather than to be reactive to partner's love, thus the feelings start to flag. This result leads us to believe that if individuals have the same behavior they can hardly generate instability in the relationship, instead of of what happens between individuals with different. Hence this case could confirm that only "the opposites attact". Instead the condition (4.9) would mean that the reactiveness increases if the forgetting process decreases et vice versa, while it increases if the appeal and synergism grow up. This last result is in agreement with the fact than β_1, d_1, A_1 are sources of interest for the couple instead of α_1.

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