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APPROXIMATIONS WITH WEAK CONTRACTIONS IN HADAMARD MANIFOLDS

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Abstract. This article is devoted to constructing the Mann iterative method of Moudafi type, a convex combination of the Mann and viscosity iterations, with a weak contraction in a Hadamard manifold. This type of iterative methods include the class of Halpern type of iterations. We first deal with strong convergence theorems for Mann iterations of Halpern type and then generalize them to those of Moudafi type with an effective approach.

1. Introduction

A wide variety of problems can be solved by finding a fixed point of a particular operator, and algorithms for finding such points play a prominent role in a number of applications. In this paper we show how to construct an iterative method for fixed point approximation problems with a weak contraction in a Hadamard manifold, which is complete simply connected Riemannian manifold with nonpositive sectional curvature. The spaces of nonpositive curvature such as hyperbolic spaces play a significant role in many areas: Lie group theory, combinatorial and geometric group theory, dynamical system, geometric topology, Kleinian group theory and Teichmüller theory.

Let (X, d) be a metric space. Suppose that $T : X \to X$ is a nonexpansive mapping, i.e., $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in X$. We shall denote $\mathfrak{F}(T)$ the fixed point set of T. Kim and Xu [12] proposed the following modified Halpern iteration in a uniformly smooth Banach space X. Let C be a closed convex subset of X, $T : C \to C$ a nonexpansive mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ twe sequences in $[0, 1]$. Given $u, x_1 \in C$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n u + (1 - \beta_n)[\alpha_n x_n + (1 - \alpha_n)Tx_n], \quad n \in \mathbb{N}. \tag{1.1}$$

The strong convergence of $\{x_n\}$ is established [12, Theorem 1] under certain conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, in addition to assuming

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty. \tag{1.2}$$

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The iterative scheme (1.1) is Halpern algorithm [9] when $\alpha_n \equiv 0$ and it is Mann algorithm [15] when $\beta_n \equiv 0$; they are often used to approximate a fixed point of a nonexpansive mapping. Hu [10] presented some convergence theorems of the iteration (1.1) in Banach spaces which have a uniformly Gâteaux differentiable norm, where $\{\beta_n\}$ satisfies (1.2) and $\{\alpha_n\}$ satisfies $0 < a \leq \alpha_n \leq b < 1$ or $\lim_{n\to\infty} \alpha_n = 0$. It is worth emphasizing that Wang [21] found a counterexample to illustrate the failure of the strong convergence of $\{x_n\}$ defined by (1.1) when $\lim_{n\to\infty} \alpha_n = 1$. Cuntavepanit and Panyanak [6] extended Kim-Xu's result to CAT(0) spaces (see Section 2 for the definition). It is known that the fixed point set of a nonexpansive mapping defined on a CAT(0) space is closed and convex.

An important class of CAT(0) spaces is given by Hadamard manifolds. A CAT(0) space is not a manifold, in general; it can be a tree, for example. In [14] Li *et al.* present analogs of Halpern and Mann algorithms for approximating fixed points of nonexpansive mappings in Hadamard manifolds.

A mapping $f : X \to X$ is a contraction if there exists $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y), \quad \forall x, y \in X;$$

it is a *φ-weak contraction*, due to Alber and Guerre-Delabriere [2], if

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1.3}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$. It is seen that a φ-weak contraction is a k-contraction by taking $\varphi(t) = (1 - k)t$ in (1.3). The well-known Banach contraction principle [4] guarantees the existence and uniqueness of fixed points of contractions on a complete metric space. The validity of Banach contraction principle for weak contractions in complete metric spaces was proved by Rhoades [17]. We state this result as follows.

Theorem 1.1 (Rhoades [17])**.** *Let (X, d) be a complete metric space and f a weak contraction on X. Then f has a unique fixed point.*

Moudafi [16] introduced the following viscosity approximation with a contraction f and generalized Halpern's theorems in another direction: let $x_1 \in X$ and define $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Recently, Huang [11] obtains some convergence theorems by Moudafi viscosity approximations with a weak contraction for a sequence of nonexpansive mappings in a CAT(0) space.

The iterative schemes we consider here are modified Mann iterations of Halpern type and Moudafi type, respectively, in a Hadamard manifold. The natural question then arises: under what conditions on the control coefficients may the convergence of the iterative methods of Halpern type be proved? In addition, it is of great importance to know whether these convergence theorems can be extended to those of Moudafi type without imposing other extra conditions on control coefficients. It is customary to derive the results for the case of Moudafi type straightforward; nevertheless, it is very difficult to prove. Accordingly, the plan is to pass from Halpern type cases to Moudafi type cases with an effective approach. To accomplish this, preliminary facts about complete Riemannian manifolds are needed. Section 2 presents some basic concepts and fundamental theorems in Riemannian Geometry. In Section 3, the approximation of Mann iteration of Halpern type with a weak contraction on Hadamard manifolds is first discussed in two cases; see Theorems 3.1 and 3.2. Then we adapt the technique as shown in [11, Proposition 3.5] so that it applies to the more general case; see Theorems 3.3 and 3.4.

2. Preliminaries

Let M be a differentiable manifold with finite dimension n, T_xM the tangent space of M at x (T_xM is a linear space and has the same dimension of M) and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. Because we restrict ourselves to real manifolds, T_xM is isomorphic to $\mathbb{R}^n$. When M is endowed with a Riemannian metric g and the corresponding norm denoted by $\|\cdot\|$, M is a Riemannian manifold. The inner product of two vectors $u, v \in T_xM$ is written as $\langle u, v\rangle_x = g_x(u, v)$, where g_x is the metric at a point x. The *norm* of a vector $v \in T_xM$ is set by $\|v\|_x = \sqrt{\langle v, v\rangle_x}$ and the *angle* $\angle_x(u, v)$ between $u, v \in T_xM$ $(u, v \neq 0)$ by $\cos\angle_x(u, v) = \langle u, v\rangle_x/\|u\|\|v\|$. If there is no confusion, we denote $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_x$, $\|\cdot\| = \|\cdot\|_x$ and $\angle(u, v) = \angle_x(u, v)$. The metrics can be used to define the length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining $\gamma(a) = x$ to $\gamma(b) = y$ through

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}dt.$$

Minimizing this length functional over the set of all such curves, we obtain a Riemannian distance $d(x, y)$ which induces the original topology on M; see [7, Proposition 2.6, p.146].

Let ∇ be a Levi-Civita connection associated to $(M, \langle\cdot,\cdot\rangle)$. This connection defines a unique covariant derivative D/dt, where for each vector field V, along a smooth curve $\gamma : [a, b] \to M$, another vector field DV/dt is obtained, called the covariant derivative of V along γ. A curve $\gamma : [a, b] \to M$ is called a *geodesic* if $D\gamma'/dt = 0$ and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, then γ is said to be *normalized*. A geodesic joining x and y in M is said to be *minimizing* if its length is equal to $d(x, y)$. A Riemannian manifold is complete if its geodesics are defined for any value of $t \in \mathbb{R}$. Hopf-Rinow Theorem [7, Theorem 2.8, p.146] asserts that if M is complete, then any pair of points in M can be joined by a minimal geodesic (the exponential map is an essential tool for this understanding), and every bounded closed subset of M is compact.

Let M be a complete Riemannian manifold and $x \in M$. The exponential map $\exp_x : T_xM \to M$ is defined as $\exp_x v = \gamma_v(1, x)$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v. Then, for any value of t, we have $\exp_x tv = \gamma_v(t, x)$.

The following well-known result, as an application of Hopf-Rinow Theorem, can be found in [7, Theorem 3.1, p.149] and [18, Theorem 4.1, p.221].

Theorem 2.1. *Let M be a Hadamard manifold. Then M is diffeomorphic to the Euclidian space $\mathbb{R}^n$, where $n = \dim M$; more precisely, at any point $x \in M$, the exponential map mapping $\exp_x : T_xM \to M$ is a diffeomorphism. Moreover, for any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y, which is, in fact, a minimal geodesic.*

Proposition 2.2 ([5, II.2.2] and [18])**.** *If M is a Hadamard manifold, then the distance function $d : M \times M \to \mathbb{R}$ is convex with respect to the product Riemannian metric, that is, for any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$ we have*

$$d(\gamma_1(t), \gamma_2(t)) \le (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)), \quad \forall t \in [0,1].$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \to \mathbb{R}$ is a convex function.

A *geodesic triangle* in a Riemannian manifold M is a set formed by three minimizing normalized geodesics $\gamma_i : [0, \ell_i] \to M$ (the *sides*) such that $\gamma_i(\ell_i) = \gamma_{i+1}(0)$, indices taken modulo 3. The points $p_i = \gamma_{i+2}(0)$ are called the *vertices* of the triangle and $\alpha_i = \angle(-\gamma'_{i+1}(\ell_{i+1}), \gamma'_{i+2}(0))$ the *corresponding angles*. Let us denote by $\triangle(p_1, p_2, p_3)$ this geodesic triangle.

One of the most important characterizations of Hadamard manifolds is described in the following proposition; see [7, Lemma 3.1, p.259] and [18, Proposition 4.5, p.223].

Proposition 2.3. *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in a Hadamard manifold and α_i the corresponding angles at the vertices p_i, $i = 1, 2, 3$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \le \pi$,

(ii) $d(p_1, p_2)^2 + d(p_1, p_3)^2 - 2\langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle \le d(p_2, p_3)^2$.

The following relation between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$ can be referred to [5, I.2.14].

Lemma 2.4. *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in a Hadamard manifold. Then there exists a triangle $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3) \subset \mathbb{R}^2$ for $\triangle(p_1, p_2, p_3)$ such that $d(p_i, p_{i+1}) = \|\bar{p}_i - \bar{p}_{i+1}\|$, indices taken modulo 3; it is unique up to an isometry of $\mathbb{R}^2$.*

The triangle $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ in Lemma 2.4 is said to be a *comparison triangle* for $\triangle(p_1, p_2, p_3)$. The geodesic side from x to y will be denoted $[x, y]$. A point $\bar{x} \in [\bar{p}_1, \bar{p}_2]$ is called a *comparison point* for $x \in [p_1, p_2]$ if $\|\bar{x} - \bar{p}_1\| = d(x, p_1)$. The interior angle of $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ at $\bar{p}_1$ is called the *comparison angle* between $\bar{p}_2$ and $\bar{p}_3$ at $\bar{p}_1$ and is denoted $\angle_{\bar{p}_1}(\bar{p}_2, \bar{p}_3)$. With all notation as in the statement of Proposition 2.3, according to the law of cosines, Condition (ii) is valid if and only if

$$\langle \bar{p}_2 - \bar{p}_1, \bar{p}_3 - \bar{p}_1 \rangle_{\mathbb{R}^2} \le \langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle \tag{2.1}$$

or,

$$\alpha_1 \le \angle_{\bar{p}_1}(\bar{p}_2, \bar{p}_3),$$

or, equivalently, $\triangle(p_1, p_2, p_3)$ satisfies the CAT(0) inequality (see [5, II.1.7(5) and II.1.9(2)]), that is, if, given a comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$ for $\triangle(p_1, p_2, p_3)$, for all $x, y \in \triangle(p_1, p_2, p_3)$,

$$d(x, y) \le \|\bar{x} - \bar{y}\|,$$

where $\bar{x}, \bar{y} \in \overline{\triangle}$ are the respective comparison points of x, y.

A geodesic space X is a CAT(0) space if all geodesic triangles in X satisfy the CAT(0) inequality. It turns out that a complete Riemannian manifold is CAT(0) if and only if it is a Hadamard manifold; see also [5, II.1.5, II.1A.8, and II.4.1(2)].

A set C is *convex* in a Hadamard manifold M if it includes the geodesic joining any pair of its points, that is, for any $x, y \in C$, if $\gamma : [a, b] \to M$ is a geodesic joining x to y, then $\gamma(t) \in C$ for all $t \in [a, b]$. In particular, if C is a closed convex subset of M, then there exists a unique point $P_C x \in C$ such that $d(x, P_C x) = \inf\{d(x, y) : y \in C\}$. Therefore P_C defines a mapping of M onto C, called the metric projection of M onto C (see [5, II.2.4]), which is a nonexpansive retraction from M onto C; see [8, Proposition 3.5].

Lemma 2.5 ([20])**.** *Let C be a closed convex subset of a Hadamard manifold M and $x \in M$. Then*

$$\left\langle \exp_{P_C x}^{-1} x, \exp_{P_C x}^{-1} y \right\rangle \leq 0, \quad \textit{for all } y \in C.$$

The following result is an analog of the Banach space version of Suzuki's Lemma [19, Lemma 2.2] in Hadamard manifolds.

Lemma 2.6. *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Hadamard manifold and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose that*

$$z_{n+1} = \exp_{w_n} (1 - \alpha_n) \exp_{w_n}^{-1} z_n, \quad n \in \mathbb{N},$$

and

$$\limsup_{n\to\infty} [d(w_{n+1}, w_n) - d(z_{n+1}, z_n)] \leq 0.$$

Then $\lim_{n\to\infty} d(w_n, z_n) = 0$.

Proof. Just apply Lemma 2.2 in [19] for Banach spaces to the Euclidean spaces. Fix any $n \in \mathbb{N}$. Consider a geodesic triangle $\triangle(z_n, w_n, w_{n+1})$ and its comparison triangle $\triangle(\bar{z}_n, \bar{w}_n, \bar{w}_{n+1})$ in $\mathbb{R}^2$ so that $d(w_n, w_{n+1}) = \|\bar{w}_n - \bar{w}_{n+1}\|$, $d(z_n, z_{n+1}) = \|\bar{z}_n - \bar{z}_{n+1}\|$ and

$$\bar{z}_{n+1} = \alpha_n \bar{w}_n + (1 - \alpha_n)\bar{z}_n.$$

Lemma 2.2 in [19] assures that $\lim_{n\to\infty} d(w_n, z_n) = \lim_{n\to\infty} \|\bar{w}_n - \bar{z}_n\| = 0$. □

The following two technical lemmas are crucial to the study of our problem.

Lemma 2.7 ([3, Lemma 2.3])**.** *Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\zeta_n\}$ be three sequences of nonnegative numbers and $\{\xi_n\}$ a sequence of real numbers such that $\{\alpha_n\} \subset [0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \zeta_n < \infty$ and $\limsup_{n\to\infty} \xi_n \leq 0$. Suppose that*

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \alpha_n \xi_n + \zeta_n, \quad n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} \lambda_n = 0$.

Lemma 2.8 ([1])**.** *Let $\{\lambda_n\}$ and $\{\eta_n\}$ be two sequences of nonnegative numbers and $\{\alpha_n\}$ a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \eta_n/\alpha_n = 0$. Suppose that*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \phi(\lambda_n) + \eta_n, \quad n \in \mathbb{N},$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$. Then $\lim_{n\to\infty} \lambda_n = 0$.

3. Modified Halpern Iteration

This section contains four convergence theorems in a Hadamard manifold. We first consider an iterative scheme of Halpern type with a weak contraction and carry out our discussions in two cases. Recall that for $x, y \in \mathbb{R}^2$ (in fact, any Hilbert space),

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall t \in [0,1].$$

Theorem 3.1. *Let C be a closed convex subset of a Hadamard manifold M, $T : C \to C$ a nonexpansive mapping with $\mathfrak{F}(T) \neq \emptyset$, and $\{\alpha_n\}$, $\{\beta_n\}$ two sequences in $(0,1)$ satisfying*

$$\begin{aligned}
&\text{(C1)} \quad \lim_{n\to\infty} \alpha_n = 0,\\
&\text{(C2)} \quad \sum_{n=1}^{\infty} \alpha_n = \infty,\\
&\text{(D1)} \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.
\end{aligned}$$

Let $u, x_1 \in C$ and define a sequence $\{x_n\}$ by

$$\begin{aligned}
y_n &= \exp_u (1-\alpha_n) \exp_u^{-1} Tx_n,\\
x_{n+1} &= \exp_{x_n} (1-\beta_n) \exp_{x_n}^{-1} y_n.
\end{aligned} \tag{3.1}$$

Then $\{x_n\}$ converges strongly to the point $q = P_{\mathfrak{F}(T)} u$.

Proof. This proof consists primarily of the verification of a collection of four claims.

Claim 1: $\{x_n\}$ is bounded. To prove this, choose any $p \in \mathfrak{F}(T)$ and fix $n \in \mathbb{N}$. Let γ_n^1 and γ_n^2 be two respective geodesics joining from u to Tx_n, and from x_n to y_n. Then

$$\begin{aligned}
d(y_n, p) &= d(\gamma_n^1(1-\alpha_n), p)\\
&\le \alpha_n d(u,p) + (1-\alpha_n) d(Tx_n, p)\\
&\le \alpha_n d(u,p) + (1-\alpha_n) d(x_n, p)
\end{aligned}$$

and

$$\begin{aligned}
d(x_{n+1}, p) &= d(\gamma_n^2(1-\beta_n), p)\\
&\le \beta_n d(x_n,p) + (1-\beta_n) d(y_n, p)\\
&\le \alpha_n(1-\beta_n) d(u,p) + [1-\alpha_n(1-\beta_n)] d(x_n, p).
\end{aligned}$$

We use induction on n to the two inequalities above and get

$$\begin{aligned}
d(y_n, p) &\le \max\{d(u,p), d(x_1,p)\}, \quad \text{for all } n \in \mathbb{N},\\
d(x_{n+1}, p) &\le \max\{d(u,p), d(x_1,p)\}, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Hence $\{x_n\}$ and $\{y_n\}$ are bounded, and so is $\{Tx_n\}$.

Claim 2: $\lim_{n\to\infty} d(x_n, y_n) = 0$. Recall that

$$x_{n+1} = \exp_{y_n} \beta_n \exp_{y_n}^{-1} x_n.$$

First we have

$$\begin{aligned} d(y_{n+1}, y_n) &= d(\gamma^1_{n+1}(1-\alpha_{n+1}), \gamma^1_n(1-\alpha_n)) \\ &\le d(\gamma^1_{n+1}(1-\alpha_{n+1}), \gamma^1_n(1-\alpha_{n+1})) + d(\gamma^1_n(1-\alpha_{n+1}), \gamma^1_n(1-\alpha_n)) \\ &\le (1-\alpha_{n+1})d(Tx_{n+1}, Tx_n) + |\alpha_{n+1}-\alpha_n| d(u, Tx_n) \\ &\le (1-\alpha_{n+1})d(x_{n+1}, x_n) + |\alpha_{n+1}-\alpha_n| d(u, Tx_n) \qquad (3.2) \end{aligned}$$

which implies that

$$d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \le |\alpha_{n+1}-\alpha_n| d(u, Tx_n) - \alpha_{n+1} d(x_{n+1}, x_n).$$

This shows that

$$\limsup_{n\to\infty} [d(y_{n+1}, y_n) - d(x_{n+1}, x_n)] \le 0.$$

Therefore by Condition (D1), Lemma 2.6 asserts that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Claim 3: Observe that

$$\limsup_{n\to\infty} \left\langle \exp_q^{-1} u, \exp_q^{-1} Tx_n \right\rangle \le 0. \qquad (3.3)$$

First, from (3.1) and Condition (C1), we obtain

$$d(y_n, Tx_n) = \alpha_n d(u, Tx_n) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$d(x_n, Tx_n) \le d(x_n, y_n) + d(y_n, Tx_n) \to 0 \quad \text{as } n \to \infty.$$

Next, since the sequence $\left\{\left\langle \exp_q^{-1} u, \exp_q^{-1} Tx_n \right\rangle\right\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \left\langle \exp_q^{-1} u, \exp_q^{-1} Tx_n \right\rangle = \lim_{k\to\infty} \left\langle \exp_q^{-1} u, \exp_q^{-1} Tx_{n_k} \right\rangle.$$

By passing to a subsequence it may be assumed that $\{x_{n_k}\}$ converges to some $x_0 \in M$. We then derive that

$$d(x_0, Tx_0) \le d(x_0, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx_0) \to 0 \quad \text{as } n \to \infty.$$

That is, $x_0 \in \mathfrak{F}(T)$. Lemma 2.5 guarantees that

$$\lim_{k\to\infty} \left\langle \exp_q^{-1} u, \exp_q Tx_{n_k} \right\rangle = \left\langle \exp_q^{-1} u, \exp_q^{-1} x_0 \right\rangle \le 0,$$

which proves Claim 3.

Claim 4: $\lim_{n\to\infty} d(x_n, q) = 0$. To prove this, fix any $n \in \mathbb{N}$. For a geodesic triangle $\triangle(u, q, Tx_n) \subset M$, let $\triangle(\bar{u}, \bar{q}, \overline{Tx_n}) \subset \mathbb{R}^2$ be a comparison triangle for $\triangle(u, q, Tx_n)$. We then have

$$\bar{y}_n = \alpha_n \bar{u} + (1-\alpha_n)\overline{Tx_n}.$$

Proposition 2.3 and (2.1) show that

$$\langle \bar{u} - \bar{q}, \overline{Tx_n} - \bar{q} \rangle_{\mathbb{R}^2} \le \left\langle \exp_q^{-1} u, \exp_q^{-1} Tx_n \right\rangle$$

which implies that

$$\begin{aligned}
d(y_n,q)^2 &\le \|\bar{y}_n-\bar{q}\|^2\\
&= \alpha_n^2\|\bar{u}-\bar{q}\|^2+(1-\alpha_n)^2\|\overline{Tx_n}-\bar{q}\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle \bar{u}-\bar{q}, \overline{Tx_n}-\bar{q}\rangle_{\mathbb{R}^2}\\
&\le \alpha_n^2 d(u,q)^2+(1-\alpha_n)d(x_n,q)^2+2\alpha_n\left\langle \exp_q^{-1}u, \exp_q^{-1}Tx_n\right\rangle. \qquad (3.4)
\end{aligned}$$

Again, consider a geodesic triangle $\triangle(q,x_n,y_n)$ in M and let $\triangle(\bar{q},\bar{x}_n,\bar{y}_n)\subset\mathbb{R}^2$ be the corresponding comparison triangle. Then $d(x_n,q)=\|\bar{x}_n-\bar{q}\|$, $d(y_n,q)=\|\bar{y}_n-\bar{q}\|$ and

$$\bar{x}_{n+1}=\beta_n\bar{x}_n+(1-\beta_n)\bar{y}_n$$

and hence by (3.4) we get

$$\begin{aligned}
d(x_{n+1},q)^2 &\le \|\bar{x}_{n+1}-\bar{q}\|^2\\
&\le \beta_n\|\bar{x}_n-\bar{q}\|^2+(1-\beta_n)\|\bar{y}_n-\bar{q}\|^2\\
&= \beta_n d(x_n,q)^2+(1-\beta_n)d(y_n,q)^2\\
&\le [1-\alpha_n(1-\beta_n)]d(x_n,q)^2+\alpha_n^2(1-\beta_n)d(u,q)^2\\
&\quad + 2\alpha_n(1-\beta_n)\left\langle \exp_q^{-1}u,, \exp_q^{-1}Tx_n\right\rangle\\
&\le [1-\alpha_n(1-\beta_n)]\lambda_n+\alpha_n(1-\beta_n)\xi_n,
\end{aligned}$$

where $\lambda_n=d(x_n,q)^2$ and

$$\xi_n=\alpha_n d(u,q)^2+2\left\langle \exp_q^{-1}u,, \exp_q^{-1}Tx_n\right\rangle.$$

We then apply (3.3) and Lemma 2.7 to conclude that

$$\lim_{n\to\infty} d(x_n,q)=0,$$

as desired. This finishes the proof. □

Theorem 3.2. *Let C be a closed convex subset of a Hadamard manifold M, $T: C\to C$ a nonexpansive mapping with $\mathfrak{F}(T)\neq\emptyset$, and $\{\alpha_n\}$, $\{\beta_n\}$ two sequences in $(0,1)$ such that $\{\alpha_n\}$ satisfies* (C1), (C2) *and*

(C3) *either* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *or* $\lim_{n\to\infty}\alpha_n/\alpha_{n+1}=1$,

and $\{\beta_n\}$ satisfies

(D2) $\lim_{n\to\infty}\beta_n=0$,
(D3) $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$.

Let $u,x_1\in C$ and define a sequence $\{x_n\}$ by (3.1). Then $\{x_n\}$ converges strongly to the point $q=P_{\mathfrak{F}(T)}u$.

Proof. With all notations as in the proof of Theorem 3.1, it follows that $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ are all bounded.

Observe that $\lim_{n\to\infty}d(x_{n+1},x_n)=0$. In fact, from (3.2) we get

$$d(y_n,y_{n-1})\le(1-\alpha_n)d(x_n,x_{n-1})+|\alpha_n-\alpha_{n-1}|d(u,Tx_{n-1})$$

and therefore

$$\begin{aligned}
d(x_{n+1}, x_n) &\le d(\gamma_n^2(1-\beta_n), \gamma_{n-1}^2(1-\beta_n)) + d(\gamma_{n-1}^2(1-\beta_n), \gamma_{n-1}^2(1-\beta_{n-1}))\\
&\le \beta_n d(x_n, x_{n-1}) + (1-\beta_n) d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}| d(x_{n-1}, y_{n-1})\\
&\le \beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n) d(x_n, x_{n-1})\\
&\quad + |\alpha_n - \alpha_{n-1}|(1-\beta_n) d(u, Tx_{n-1}) + |\beta_n - \beta_{n-1}| d(x_{n-1}, y_{n-1})\\
&\le [1 - \alpha_n(1-\beta_n)] d(x_n, x_{n-1}) + K(1-\beta_n)|\alpha_n - \alpha_{n-1}|\\
&\quad + K|\beta_n - \beta_{n-1}|\\
&= [1 - \alpha_n(1-\beta_n)] d(x_n, x_{n-1}) + K\alpha_n(1-\beta_n)\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right|\\
&\quad + K|\beta_n - \beta_{n-1}|,
\end{aligned}$$

where K is a constant such that $K > \sup\{d(u, Tx_n), d(x_n, y_n) : n \in \mathbb{N}\}$. According to Conditions (C2), (C3) and (D3), Lemma 2.7 assures that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

This yields that

$$\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, Tx_n)\\
&= d(x_n, x_{n+1}) + \beta_n d(x_n, y_n) + \alpha_n d(u, Tx_n) \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

by Conditions (C1) and (D2). Now, using an argument similar to that of Claims 3 and 4 in the preceding theorem, we have $\lim_{n\to\infty} x_n = q$, as required. □

We next study an iterative method of viscosity type with a weak contraction. The approaches are similar to that of [11, Proposition 3.5].

Let C be a closed convex subset of a Hadamard manifold M, T a nonexpansive mapping on C and f a φ-weak contraction on C. Then $T \circ f$ is a φ-weak contraction on C which follows from the following inequality:

$$d(T \circ fx, T \circ fy) \le d(f(x), f(y)) \le d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in C.$$

Theorem 3.3. *Let M, C, T, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 3.1. Suppose that f is a φ-weak contraction on C, where φ is strictly increasing. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned}
y_n &= \exp_{f(x_n)}(1-\alpha_n)\exp^{-1}_{f(x_n)} Tx_n,\\
x_{n+1} &= \exp_{x_n}(1-\beta_n)\exp^{-1}_{x_n} y_n.
\end{aligned} \tag{3.5}$$

Then $\{x_n\}$ converges strongly to the unique point $q \in C$ such that $q = P_{\mathfrak{F}(T)} f(q)$.

Proof. Since the metric projection $P_{\mathfrak{F}(T)}$ is nonexpansive, the existence and the uniqueness of q is guaranteed by Theorem 1.1. Define a sequence $\{z_n\}$ by

$$\begin{aligned}
w_n &= \exp_{f(q)}(1-\alpha_n)\exp^{-1}_{f(q)} Tz_n,\\
z_{n+1} &= \exp_{z_n}(1-\beta_n)\exp^{-1}_{z_n} w_n.
\end{aligned}$$

An appeal to Theorem 3.1 establishes the strong convergence of $\{z_n\}$ with limit $P_{\mathfrak{F}(T)}f(q) = q$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+1}, z_{n+1}) &\le \beta_n d(x_n, z_n) + (1-\beta_n) d(y_n, w_n) \\
&\le \beta_n d(x_n, z_n) + (1-\beta_n)[\alpha_n d(f(x_n), f(q)) + (1-\alpha_n) d(x_n, z_n)] \\
&\le \beta_n d(x_n, z_n) + \alpha_n(1-\beta_n)[d(f(x_n), f(z_n) + d(f(z_n), f(q))] \\
&\quad + (1-\alpha_n)(1-\beta_n) d(x_n, z_n) \\
&\le d(x_n, z_n) - \alpha_n(1-\beta_n)\varphi(d(x_n, z_n)) + \alpha_n(1-\beta_n) d(z_n, q).
\end{aligned}$$

Apply Lemma 2.8 by setting $\lambda_n = d(x_n, z_n)$ and $\eta_n = \alpha_n(1-\beta_n)d(z_n, q)$ to get

$$\lim_{n\to\infty} d(x_n, z_n) = 0.$$

Consequently, this shows that $\{x_n\}$ converges strongly to q. □

Using the same argument as that of Theorem 3.3 (so the proof is omitted), we obtain the following result as an extension of Theorem 3.2.

Theorem 3.4. *Let M, C, T, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem* 3.2. *Suppose that f is a φ-weak contraction on C, where φ is strictly increasing. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by* (3.5). *Then $\{x_n\}$ converges strongly to the unique point $q \in C$ such that $q = P_{\mathfrak{F}(T)}f(q)$.*

Remark 3.5. The sequence $\{x_n\}$ defined by (3.1) is

$$x_{n+1} = \exp_{x_n}(1-\beta_n)\exp_{x_n}^{-1}[\exp_u(1-\alpha_n)\exp_u^{-1} Tx_n].$$

In contrast to the analog of this iteration in a Hilbert space, it is given by two formulations:

$$\begin{aligned}
x_{n+1} &= \beta_n x_n + (1-\beta_n)[\alpha_n u + (1-\alpha_n)Tx_n] \\
&= \alpha_n(1-\beta_n)u \\
&\quad + [1-\alpha_n(1-\beta_n)]\left[\frac{\beta_n}{1-\alpha_n(1-\beta_n)}x_n + \frac{(1-\alpha_n)(1-\beta_n)}{1-\alpha_n(1-\beta_n)}Tx_n\right];
\end{aligned}$$

so we ask whether (3.1) in a Hadamard manifold M can be rewritten as follows:

$$\exp_u[1-\alpha_n(1-\beta_n)]\exp_u^{-1}\left[\exp_{x_n}\frac{(1-\alpha_n)(1-\beta_n)}{1-\alpha_n(1-\beta_n)}\exp_{x_n}^{-1} Tx_n\right].$$

The following observation makes clear the answer to this problem. Let $x, y \in M$. Then the unique minimal geodesic joining x and y is defined by $\gamma(t) = \exp_x t\exp_x^{-1} y$, $t \in [0,1]$. For any $z \in M$, is the curve $\tilde{\gamma}(t) = \exp_z[(1-t)\exp_z^{-1} x + t\exp_z^{-1} y]$, $t \in [0,1]$, still a minimal geodesic joining x and y? If this were true, then the formula above is a reformulation of (3.1). Unfortunately, $\tilde{\gamma}$ may not be a minimal geodesic joining x and y; see [13, Example 1.1 and Theorem 2.1].

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