Linear and Nonlinear Analysis
Volume 1, Number 2, 2015, 305–315

Yokohama Publishers
ISSN 2188-8167

THE SPLIT COMMON FIXED POINT PROBLEM AND THE HYBRID METHOD IN BANACH SPACES

MAYUMI HOJO AND WATARU TAKAHASHI

Abstract. In this paper, we consider the split common fixed point problem in Banach spaces. Using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [5] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1},\ 1 \le i \le m$, and $B_j : H_2 \to 2^{H_2},\ 1 \le j \le n$, respectively, and bounded linear operators $T_j : H_1 \to H_2,\ 1 \le j \le n$, the *split common null point problem* [5] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0 \right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem in Hilbert spaces; see, for instance, [1, 5, 7, 10, 11, 22].

2010 *Mathematics Subject Classification.* 47H05, 47H09.

Key words and phrases. Split common fixed point problem, maximal monotone operator, fixed point, metric projection, metric resolvent, hybrid method, duality mapping.

The second author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

Recently, by using the ideas of [12, 13, 15], Takahashi [20] obtained the following result for the split common null point problem in Banach spaces; see also [19].

Theorem 1.1 ([20]). *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - \mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n), \\ C_n = \{z \in A^{-1}0 : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in A^{-1}0 : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\mu_n\} \subset (0, \infty)$ satisfies that for some $a, b \in \mathbb{R}$,

$$0 < a \leq \mu_n \leq b < \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

In this paper, we consider the split common fixed point problem in Banach spaces. Using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem and the split common null point problem in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [18] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [16].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [8, 14].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [16] and [17]. We know the following result:

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.2 ([16]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent*

(1) $z = P_C x_1$;
(2) $\langle z-y, J(x_1 - z) \rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x-y, u^*-v^* \rangle \geq 0$ for all

$x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [3]; see also [17, Theorem 3.5.4].

Theorem 2.3 ([3]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [17].

Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : E \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [21] if, for any $x \in E$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1 - \eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples. We know examples of η-demimetric mappings from [21].

(1) Let H be a Hilbert space and let k be a real number with $0 \leq k < 1$. Let U be a strict pseud-contraction [4] of H into itself such that $F(U) \neq \emptyset$. Then U is k-demimetric.

(2) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric.

(3) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric.

Furthermore, we know an important result for demimetric mappings in a smooth, strictly convex and reflexive Banach space.

Lemma 2.4 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

3. Main result

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. In this section, using the demimetric operators, we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Banach spaces.

Theorem 3.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - \lambda_n A^* J_F(Ax_n - UAx_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \text{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < (1-\eta)$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. Since

$$\begin{aligned} &\|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \Longleftrightarrow &\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z \rangle \leq 0, \end{aligned}$$

it follows that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that D_n is closed and convex. Then $C_n \cap D_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show that $F(T) \cap A^{-1}F(U) \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in F(T) \cap A^{-1}F(U)$. Then $z = Tz$ and $Az = UAz$. Since T is nonexpansive, we have that for $z \in F(T) \cap A^{-1}F(U)$,

$$\begin{aligned} \|z_n - z\|^2 &= \|T\Big(x_n - \lambda_n A^* J_F(Ax_n - UAx_n)\Big) - Tz\|^2 \\ &\leq \|x_n - \lambda_n A^* J_F(Ax_n - UAx_n) - z\|^2 \\ &= \|x_n - z - \lambda_n A^* J_F(Ax_n - UAx_n)\|^2 \\ &= \|x_n - z\|^2 - 2\langle x_n - z, \lambda_n A^* J_F(Ax_n - UAx_n)\rangle \\ &\quad + \|\lambda_n A^* J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n \langle Ax_n - Az, J_F(Ax_n - UAx_n)\rangle \\ &\quad + \lambda_n^2 \|A\|^2 \|J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|x_n - z\|^2 - \lambda_n(1-\eta)\|Ax_n - UAx_n\|^2 \end{aligned} \tag{3.1}$$

$$\begin{aligned}&+ \lambda_n^2\|A\|^2\|Ax_n - UAx_n\|^2\\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - (1-\eta))\|Ax_n - UAx_n\|^2\\
&\le \|x_n - z\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n x_n + (1-\alpha_n)z_n - z\|\\
&\le \alpha_n\|x_n - z\| + (1-\alpha_n)\|z_n - z\|\\
&\le \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\|\\
&\le \|x_n - z\|.\end{aligned}$$

Then we have that $F(T) \cap A^{-1}F(U) \subset C_n$ for all $n \in \mathbb{N}$. We show that $F(T) \cap A^{-1}F(U) \subset D_n$ for all $n \in \mathbb{N}$. It is obvious that $F(T) \cap A^{-1}F(U) \subset D_1$. Suppose that $F(T) \cap A^{-1}F(U) \subset D_k$ for some $k \in \mathbb{N}$. Then $F(T) \cap A^{-1}F(U) \subset C_k \cap D_k$. From $x_{k+1} = P_{C_k \cap D_k} x_1$, we have that

$$\langle x_{k+1} - z, x_1 - x_{k+1}\rangle \ge 0, \quad \forall z \in C_k \cap D_k$$

and hence

$$\langle x_{k+1} - z, x_1 - x_{k+1}\rangle \ge 0, \quad \forall z \in F(T) \cap A^{-1}F(U).$$

Then, $F(T) \cap A^{-1}F(U) \subset D_{k+1}$. By mathematical induction, we have that $F(T) \cap A^{-1}F(U) \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that $F(T) \cap A^{-1}F(U)) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(T) \cap A^{-1}F(U)$ is a nonempty, closed and convex subset of H, there exists $z_0 \in F(T) \cap A^{-1}F(U)$ such that $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \le \|x_1 - y\|$$

for all $y \in C_n \cap D_n$. Since $z_0 \in F(T) \cap A^{-1}F(U) \subset C_n \cap D_n$, we have that

$$\|x_1 - x_{n+1}\| \le \|x_1 - z_0\|. \tag{3.2}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of D_n, we have that $x_n = P_{D_n} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$ we have $x_{n+1} \in D_n$. Thus

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in D_n$ we have that

$$\langle x_n - x_{n+1}, x_1 - x_n\rangle \ge 0.$$

This implies from (2.3) that

$$0 \le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

We have from $x_{n+1} \in C_n$ and the definition of C_n that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ we have that $\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0$. Using this, we have that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

We have from (3.1) that for any $z \in F(T) \cap A^{-1}F(U)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\qquad + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - (1-\eta))\|Ax_n - UAx_n\|^2 \\
&\le \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - (1-\eta))\|Ax_n - UAx_n\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
(1-\alpha_n)\lambda_n(1-\eta-\lambda_n \|A\|^2)\|Ax_n - UAx_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, $0 \le \alpha_n \le a < 1$ and $0 < b \le \lambda_n\|A\|^2 \le c < (1-\eta)$, we have that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\|^2 = 0. \tag{3.5}$$

We also have that $\|y_n - x_n\| = \|\alpha_n x_n + (1-\alpha_n)z_n - x_n\| = (1-\alpha_n)\|z_n - x_n\|$. From $\|y_n - x_n\| \to 0$ and $0 \le \alpha_n \le a < 1$, we have that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Since $\lim_{n\to\infty} \|Ax_n - UAx_n\|^2 = 0$ and U is demiclosed, we have that $Aw \in F(U)$. We also have that

$$\begin{aligned}
\|x_n - Tx_n\| &= \|x_n - z_n + z_n - Tx_n\| \\
&= \|x_n - z_n + T\Big(x_n - \lambda_n A^* J_F(Ax_n - UAx_n)\Big) - Tx_n\| \\
&\le \|x_n - z_n\| + \|x_n - \lambda_n A^* J_F(Ax_n - UAx_n) - x_n\| \\
&= \|x_n - z_n\| + \lambda_n\|A^* J_F(Ax_n - UAx_n)\| \to 0.
\end{aligned}$$

Since $x_{n_i} \rightharpoonup w$ and a nonexpansive mapping T is demiclosed [18], we have $w = Tw$. This implies that $w \in F(T) \cap A^{-1}F(U)$.

From $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$ and $w \in F(T) \cap A^{-1}F(U)$, we have from (3.2) that

$$\begin{aligned}
\|x_1 - z_0\| \le \|x_1 - w\| &\le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_0\|.
\end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|.$$

Since H satisfies the Kadec-Klee property, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_0.$$

Therefore, we have $x_n \to w = z_0$. This completes the proof. □

4. Applications

In this section, using Theorem 3.1, we get well-known and new strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [9]; see also [23].

Lemma 4.1 ([9])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \le k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseud-contraction such that $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T\Big(x_n - \lambda_n A^*(Ax_n - UAx_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the conditions such that

$$0 \le \alpha_n \le a < 1, \quad \text{and} \quad 0 < b \le \lambda_n \|A\|^2 \le c < (1-k)$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. Since U be a k-strict pseud-contraction of H_2 into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is k-demimetric. Furthermore, from Lemma 4.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 4.3. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $T : H \to H$ be a nonexpansive mapping, let $A : H \to F$ be a bounded linear operator such that*

$A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = P_C\Big(x_n - \lambda_n A^* J_F(Ax_n - P_D A x_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \textit{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < 2$$

for some $a, b, c \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C \cap A^{-1}D} x_1$.

Proof. Since P_C is the metric projection of H onto C, P_C is nonexpansive. Furthermore, since P_D is the metric projection of F onto D, from (2) in Examples, P_D is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$, then $p = P_D p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$. It is clear that $P_D x_n \rightharpoonup p$ and $\|J_F(x_n - P_D x_n)\| = \|x_n - P_D x_n\| \to 0$. Since P_D is the metric projection of F onto D, we have that

$$\langle P_D x_n - P_D p, J_F(x_n - P_D x_n) - J_F(p - P_D p) \rangle \geq 0.$$

Therefore, $-\|p - P_D p\|^2 = \langle p - P_D p, -J_F(p - P_D p) \rangle \geq 0$ and hence $p = P_D p$. Therefore, we have the desired result from Theorem 3.1. $\square$

Theorem 4.4. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into H and F into F^*, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$, respectively. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_\lambda\Big(x_n - \lambda_n T^* J_F(Tx_n - Q_\mu T x_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \textit{and} \quad 0 < b \leq \lambda_n \|T\|^2 \leq c < 2$$

for some $a, b, c \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Proof. Since J_λ is the resolvent of A on H, J_λ is nonexpansive. Furthemore, since Q_μ is the metric resolvent of B on F, from (3) in Examples, Q_μ is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - Q_\mu x_n \to 0$, then $p = Q_\mu p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - Q_\mu x_n \to 0$. It is clear that $Q_\mu x_n \rightharpoonup p$ and $\|J_F(x_n - Q_\mu x_n)\| = \|x_n - Q_\mu x_n\| \to 0$. Since Q_μ is the metric resolvent of B, we have from [2] that

$$\langle Q_\mu x_n - Q_\mu p, J_F(x_n - Q_\mu x_n) - J_F(p - Q_\mu p)\rangle \geq 0.$$

Therefore, $-\|p - Q_\mu p\|^2 = \langle p - Q_\mu p, -J_F(p - Q_\mu p)\rangle \geq 0$ and hence $p = Q_\mu p$. Therefore, we have the desired result from Theorem 3.1. □

References

[1] S. M. Alsulami and W. Takahashi, *The split common null point problem for maximal monotone mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **15** (2014), 793–808.

[2] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[3] F. E. Browder, *Nonlinear maximal monotone operators in Banach spaces*, Math. Ann. **175** (1968), 89–113.

[4] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[5] C. Byrne Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.

[6] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[7] Y. Censor and A. Segal *The split common fixed-point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[8] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[9] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strich pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[10] E. Masad and S. Reich, *A note on the multiple-set split convex feasibility problem in Hilbert space*, J. Nonlinear Convex Anal. **8** (2007), 367–371.

[11] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems **26** (2010), 055007, 6 pp.

[12] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mapping and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[13] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **81** (2003), 439–445.

[14] S. Reich, *Book Review: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Bull. Amer. Math. Soc. **26** (1992), 367–370.

[15] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Programming Ser. A. **87** (2000), 189–202.

[16] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[17] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000.

[18] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[19] W. Takahashi, *The split feasibility problem in Banach spaces*, J. Nonlinear Convex Anal. **15** (2014), 1349–1355.

[20] W. Takahashi, *The split common null point problem in Banach spaces*, Arch. Math. **104** (2015), 357–365.
[21] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces*, J. Convex Anal., to appear.
[22] W. Takahashi, H.-K. Xu, and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. **23** (2015) 205–221.
[23] W. Takahashi, N.-C. Wong and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.

Manuscript received 15 April 2015
revised 10 May 2015

Mayumi Hojo
Shibaura Institute of Technology, Tokyo 135-8548, Japan
E-mail address: `mayumi-h@shibaura-it.ac.jp`

Wataru Takahashi
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan;

Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan;

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`