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A COMPOSITE EXTRAGRADIENT-LIKE ALGORITHM FOR INVERSE-STRONGLY MONOTONE MAPPINGS AND STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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This paper is dedicated to Professor A. T. Lau on the occasion of his 70th Birthday.

Abstract. In this paper, we introduce a new composite extragradient-like algorithm for finding a common element of the solution set of variational inequality problem for an inverse-strongly monotone mapping and the fixed point set of a strictly pseudocontractive mapping in a Hilbert space. Under suitable control conditions, we prove the strong convergence of the sequence generated by the proposed algorithm to a common element of the solution set and the fixed point set, which is a solution of a certain variational inequality. As a direct consequence, we obtain the unique minimum norm common point of the solution set and the fixed point set.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be self-mapping on C. We denote by $Fix(S)$ the set of fixed points of S.

Let A be a nonlinear mapping of C into H. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denote the set of solutions of the variational inequality problem (1.1) by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature; see [2, 3, 13, 14, 22] and the references therein.

We recall that a mapping A of C into H is called *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x,\ y \in C;$$

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see [14]. For such a case, A is called α-inverse-strongly monotone. A mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C.$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|, \forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Recently, many authors have been devoting the studies on the problems of finding fixed points for k-stricly pseudocontractive mappings, see, for example, [1, 6, 9, 10], and the references therein.

Recently, in order to study the variational inequality problem coupled with the fixed point problem, many authors have introduced some iterative algorithms for finding a common element of the solution set of the variational inequality problem for a monotone mapping and the fixed point set of a nonexpansive mapping; see [5, 7, 8, 16, 19] and the references therein. In particular, In 2003, Takahashi and Toyoda [19] introduced Mann's type iterative algorithm for finding a common element of the solution set of the variational inequality problem for an inverse-strongly monotone mapping and the fixed point set of a nonexpansive mapping, and obtained the weak convergence of the proposed algorithm. Further, motivated by the idea of Korpelevich's extragradient method [12], Nadezhkina and Takahashi [16] proposed an iterative algorithm for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of the variational inequality problem for a monotone, Lipschitz continuous mapping. They proved a weak convergence theorem for two sequences generated by the proposed algorithm. Here, so-called extragradient method was first introduced by Korpelevich [12]. In 2005, Iiduka and Takahashi [7] provided Halpern's type iterative algorithm for finding a common element of the solution set of the variational inequality problem for an inverse-strongly monotone mapping and the fixed point set of a nonexpansive mapping, and showed the strong convergence of the proposed algorithm. In 2007, Chern et al. [5] extended the result of Iiduka and Takahashi [7] to the viscosity approximation method. In 2010, Jung [8] introduced a new composite extragradient-like algorithm by the viscosity approximation method, and established the strong convergence of the proposed algorithm to a common element of the solution set of the variational inequality problem for an inverse-strongly monotone mapping and the fixed point set of a nonexpansive mapping.

On the other hand, in 2001, Yamada [22] introduced the hybrid steepest descent method for the nonexpansive mapping to solve a variational inequality related to a Lipschitzian and strongly monotone mapping. Since then, by using ideas of Marino and Xu [15], Tien [20] and Ceng *et al.* [4] provided the general iterative algorithms for finding a fixed point of the nonexpansive mapping, which is a solution of a certain variational inequality related to a Lipschitzian and strongly monotone mapping. Cho *et al.* [6] and Jung [9, 10] gave the general iterative algorithms for finding a fixed point of the k-strictly pseudocontractive mapping, which is a solution of a certain variational inequality.

In this paper, motivated by the above-mentioned results, we introduce a new composite extragradient-like algorithm based on Yamada's the hybrid steepest descent method [22] for finding a common element of the solution set $VI(C, A)$ of the variational inequality problem for an inverse-strongly monotone mapping A and the fixed point set $Fix(T)$ of an k-strictly pseudocontractive mapping T for some $0 \leq k < 1$. Using appropriate control conditions, we prove that the sequence generated by the proposed algorithm converges strongly to a common point of $VI(C, A) \cap Fix(T)$, which is a solution of a certain variational inequality. As a direct consequence, we obtain the unique minimum norm point of $VI(C, A) \cap Fix(T)$. Our results improve and complement the corresponding results of Chen *et al.* [5], Iiduka and Takahashi [7] and Jung [8], and some recent results in the literature.

2. Preliminaries and lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \quad \forall x,\ y \in H. \tag{2.1}$$

Moreover, $P_C(x)$ is characterized by the properties:

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$$

and

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H,\ y \in C.$$

In the context of the variational inequality problem for a nonlinear mapping A, this implies that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \forall \lambda > 0. \tag{2.2}$$

In a Hilbert space H, there holds the following identity:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x,\ y \in H. \tag{2.3}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Ax - Ay\rangle \geq \eta\|x - y\|^2, \quad \forall x,\ y \in C.$$

In such a case, we say A is η-strongly monotone. If A is η-strongly monotone and κ-*Lipschitz continuous*, that is, $\|Ax - Ay\| \leq \kappa\|x - y\|$ for all $x,\ y \in C$, then A is

$\frac{\eta}{\kappa^2}$-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x,\ y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2.\end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

The following result for the existence of solutions of the variational inequality problem for inverse-strongly monotone mappings was given in Takahashi and Toyoda [19].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and let A be an α-inverse-strongly monotone mapping of C into H. Then, $VI(C,A)$ is nonempty.*

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x,\ y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at v, that is, $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \text{for all } u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$: see [17,18].

We need the following lemmas for the proof of our main results.

Lemma 2.2. *In a real Hilbert space H, there holds the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.3 (Xu [21])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \beta_n + \gamma_n, \quad \forall n \geq 1,$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$;
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$;
(iii) $\gamma_n \geq 0$ $(n \geq 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([23])**.** *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudocontractive mapping on C. Then the following hold:*

(i) *The fixed point set $Fix(T)$ is closed convex, so that the projection $P_{Fix(T)}$ is well defined.*
(ii) *$Fix(P_C T) = Fix(T)$.*

(iii) *If we define a mapping* $S : C \to H$ *by* $Sx = \lambda x + (1-\lambda)Tx$ *for all* $x \in C$*. then, as* $\lambda \in [k, 1)$*,* S *is a nonexpansive mapping such that* $Fix(T) = Fix(S)$*.*

The following lemmas can be easily proven, and therefore, we omit the proofs (see [22]).

Lemma 2.5. *Let* H *be a real Hilbert space. Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*, and let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping with constants* $\kappa,\ \eta > 0$*. Then for* $0 \leq \gamma l < \mu\eta$*,*

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu F - \gamma V$ *is strongly monotone with constant* $\mu\eta - \gamma l$*.*

Lemma 2.6. *Let* H *be a real Hilbert space* H*. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping with constants* $\kappa,\ \eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t < \rho \leq 1$*. Then* $S := \rho I - t\mu F : H \to H$ *is a contractive mapping with constant* $\rho - t\tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$*.*

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace of H;
- $A : C \to H$ is an α-inverse-strongly monotone mapping;
- $VI(C, A)$ is the set of solutions of the variational inequality problem (1.1) for A;
- $F : C \to C$ is a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa,\ \eta > 0$;
- $V : C \to C$ is a l-LIpschitzian mapping with constant $l \geq 0$
- Constants $\mu > 0$ and $\gamma \geq 0$ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $T : C \to C$ is a k-strictly pseudocontractive mapping for some $0 \leq k < 1$;
- $Fix(T)$ is the set of fixed points of T;
- $T_n : C \to C$ is a mapping defined by $T_n x = k_n x + (1 - k_n)Tx$ for $0 \leq k \leq k_n \leq r < 1$ and $\lim_{n\to\infty} k_n = r$;
- P_C is a metric projection of H onto C;
- $VI(C, A) \cap Fix(T) \neq \emptyset$.

Now, we propose a new composite extragradient-like algorithm based on Yamada's the hybrid steepest descent method [22] for finding a common point of the solution set of the variational inequality problem for an inverse-strongly monotone mapping A and the fixed point set of a strictly pseudocontractive mapping T.

Algorithm 3.1. For an arbitrarily chosen $x_1 = x \in C$, let the iterative sequences $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T_n P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1)$.

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$ and $\{k_n\}$ satisfy the conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0, 1)$;
(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;
(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |k_{n+1} - k_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point $q \in VI(C, A) \cap Fix(T)$, which is the unique solution of the following variational inequality:

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in VI(C, A) \cap Fix(T). \tag{3.2}$$

Proof. First, let $Q = P_{\Omega}$, where $\Omega := VI(C, A) \cap Fix(T)$. By Lemma 2.6, it is easy to show that $Q(I - \mu F + \gamma V) : C \to C$ is a contractive mapping with constant $1 - (\tau - \gamma l)$. Thus, by Banach Contraction Principle, there exists a unique element $q \in C$ such that $q = P_{\Omega}(I - \mu F + \gamma V)q$. Equivalently, q is a solution of the variational inequality (3.2). Also, we can show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \leq \gamma l < \tau$ and $\mu\eta \geq \tau \iff \kappa \geq \eta$, it follows from Lemma 2.5 that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \geq (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \leq \gamma l < \tau \leq \mu\eta$. Hence the variational inequality (3.2) has only one solution. Below we use $q \in VI(C, A) \cap Fix(T)$ to denote the unique solution of the variational inequality (3.2).

From now, by the condition (i), without loss of generality, we assume that $2\alpha_n(\tau - \gamma l) < 1$ and $\alpha_n < 1 - \beta_n - \alpha_n$ for $n \geq 1$.

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, Let $z_n = P_C(x_n - \lambda_n A x_n)$ and $w_n = P_C(y_n - \lambda_n A y_n)$ for every $n \geq 1$. Let $p \in VI(C, A) \cap Fix(T)$ $(= VI(C, A) \cap Fix(T_n)$ by Lemma 2.4). Since $I - \lambda_n A$ is nonexpansive and $p = P_C(p - \lambda_n A p)$ from (2.2), we have

$$\begin{aligned} \|z_n - p\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(p - \lambda_n A p)\| \\ &\leq \|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\| \leq \|x_n - p\|. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\|w_n - p\| \leq \|y_n - p\|. \tag{3.4}$$

Now, let $p \in VI(C, A) \cap Fix(T)$. Then, from (3.3), (3.4), and Lemma 2.6, we obtain

$$\begin{aligned} &\|y_n - p\| \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \alpha_n \mu F)T_n z_n - (I - \alpha_n \mu F)p\| \\ &\leq (1 - \tau\alpha_n)\|z_n - p\| + \alpha_n \gamma \|V x_n - V p\| + \alpha_n \|\gamma V p - \mu F p\| \\ &\leq (1 - \tau\alpha_n)\|z_n - p\| + \alpha_n \gamma l \|x_n - p\| + \alpha_n \|\gamma V p - \mu F p\| \\ &= (1 - (\tau - \gamma l)\alpha_n)\|x_n - p\| + (\tau - \gamma l)\alpha_n \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\beta_n)(y_n-p)+\beta_n(T_nw_n-p)\| \\
&\le (1-\beta_n)\|y_n-p\|+\beta_n\|w_n-p\| \\
&\le (1-\beta_n)\|y_n-p\|+\beta_n\|y_n-p\| \\
&= \|y_n-p\| \\
&\le \max\left\{\|x_n-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma l}\right\}.
\end{aligned} \tag{3.6}
$$

By induction, it follows from (3.6) that

$$
\|x_n-p\| \le \max\left\{\|x_1-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma l}\right\}, \quad \forall n\ge 1.
$$

Therefore $\{x_n\}$ is bounded. So $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{Vx_n\}$, $\{Fx_n\}$, $\{Fy_n\}$, $FT_nz_n\}$, are bounded. Moreover, since $\|T_nz_n-p\|\le\|x_n-p\|$ and $\|T_nw_n-p\|\le\|y_n-p\|$, $\{T_nz_n\}$ and $\{T_nw_n\}$ are also bounded. And by the condition (i), we have

$$
\|y_n-T_nz_n\| = \ \alpha_n\|\gamma Vx_n-\mu FT_nz_n\|\to 0 \ \ (\text{as } n\to\infty). \tag{3.7}
$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Indeed, since $I-\lambda_nA$ and P_C are nonexpansive and $z_n=P_C(x_n-\lambda_nAx_n)$, we have

$$
\begin{aligned}
\|z_n-z_{n-1}\| &\le \|(x_n-\lambda_nAx_n)-(x_{n-1}-\lambda_{n-1}Ax_{n-1})\| \\
&\le \|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.
\end{aligned} \tag{3.8}
$$

Similarly, we get

$$
\|w_n-w_{n-1}\|\le\|y_n-y_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ay_{n-1}\|. \tag{3.9}
$$

We also note that

$$
\begin{aligned}
\|T_nz_n-T_{n-1}z_{n-1}\| &\le \|T_nz_n-T_nz_{n-1}\|+\|T_nz_{n-1}-T_{n-1}z_{n-1}\| \\
&\le \|z_n-z_{n-1}\|+|k_n-k_{n-1}|\|z_{n-1}-Tz_{n-1}\|
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
&\|T_nw_n-T_{n-1}w_{n-1}\| \\
&\le \|T_nw_n-T_nw_{n-1}\|+\|T_nw_{n-1}-T_{n-1}w_{n-1}\| \\
&\le \|w_n-w_{n-1}\|+|k_n-k_{n-1}|\|w_{n-1}-Tw_{n-1}\|.
\end{aligned} \tag{3.11}
$$

Now, simple calculations show that

$$
\begin{aligned}
&y_n-y_{n-1} \\
&= \alpha_n\gamma Vx_n+(I-\alpha_n\mu F)T_nz_n-\alpha_{n-1}\gamma Vx_{n-1}-(I-\alpha_{n-1}\mu F)T_{n-1}z_{n-1} \\
&= (\alpha_n-\alpha_{n-1})(\gamma Vx_{n-1}-\mu FT_{n-1}z_{n-1})+\alpha_n\gamma(Vx_n-Vx_{n-1}) \\
&\quad +(I-\alpha_n\mu F)T_nz_n-(I-\alpha_n\mu F)T_{n-1}z_{n-1}.
\end{aligned}
$$

By (3.8), (3,10), and Lemma 2.6, we obtain

$$
\begin{aligned}
&\|y_n - y_{n-1}\| \\
\le&|\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|FT_{n-1}z_{n-1}\|) \\
&+ \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\tau\alpha_n)\|T_nz_n - T_{n-1}z_{n-1}\| \\
\le&|\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|FT_{n-1}z_{n-1}\|) \\
&+ \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\tau\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + |k_n - k_{n-1}|\|z_{n-1} - Tz_{n-1}\|.
\end{aligned} \tag{3.12}
$$

Also, observe that

$$
\begin{aligned}
x_{n+1} - x_n = (1-\beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(T_{n-1}w_{n-1} - y_{n-1}) \\
+ \beta_n(T_nw_n - T_{n-1}w_{n-1}).
\end{aligned} \tag{3.13}
$$

By (3.9), (3.11), (3.12) and (3.13), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
\le&\ (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|T_{n-1}x_{n-1}\| + \|y_{n-1}\|) \\
&+ \beta_n\|w_n - w_{n-1}\| + |k_n - k_{n-1}|\|w_{n-1} - Tw_{n-1}\| \\
\le&\ (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + \beta_n|\lambda_n - \lambda_{n-1}|\|Ay_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) \\
&+ |k_n - k_{n-1}|\|w_{n-1} - Tw_{n-1}\| \\
\le&\ \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ay_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) \\
&+ |k_n - k_{n-1}|\|w_{n-1} - Tw_{n-1}\| \\
\le&\ (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|FT_{n-1}z_{n-1}\|) \\
&+ |\lambda_n - \lambda_{n-1}|(\|Ay_{n-1}\| + \|Ax_{n-1}\|) \\
&+ |\beta_n - \beta_{n-1}|(\|T_{n-1}w_{n-1}\| + \|y_{n-1}\|) \\
&+ |k_n - k_{n-1}|(\|z_{n-1} - Tz_{n-1}\| + \|w_{n-1} - Tw_{n-1}\|) \\
\le&\ (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| + M_1|\alpha_n - \alpha_{n-1}| \\
&+ M_2|\lambda_n - \lambda_{n-1}| + M_3|\beta_n - \beta_{n-1}| + M_4|k_n - k_{n-1}|,
\end{aligned} \tag{3.14}
$$

where $M_1 = \sup\{\gamma\|Vx_n\| + \mu\|FT_nz_n\| : n \ge 1\}$, $M_2 = \sup\{\|Ay_n\| + \|Ax_n\| : n \ge 1\}$, $M_3 = \sup\{\|Sw_n\| + \|y_n\| : n \ge 1\}$, and $M_4 = \sup\{\|z_{n-1} - Tz_{n-1}\| + \|w_{n-1} - Tw_{n-1}\| : n \ge 1\}$. From the condition (i) and (iv), it is easy to see that

$$
\lim_{n\to\infty}(\tau-\gamma l)\alpha_n = 0, \quad \sum_{n=1}^{\infty}(\tau-\gamma l)\alpha_n = \infty,
$$

and

$$
\sum_{n=2}^{\infty}(M_1|\alpha_n - \alpha_{n-1}| + M_2|\lambda_n - \lambda_{n-1}| + M_3|\beta_n - \beta_{n-1}| + M_4|k_n - k_{n-1}|) < \infty.
$$

Applying Lemma 2.3 to (3.14), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Moreover, by (3.8) and (3.12), we also have

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_n z_n\| = 0$. Indeed,

$$\begin{aligned} \|x_{n+1} - y_n\| &= \beta_n \|T_n w_n - y_n\| \\ &\leq \beta_n(\|T_n w_n - T_n z_n\| + \|T_n z_n - y_n\|) \\ &\leq a(\|w_n - z_n\| + \|T_n z_n - y_n\|) \\ &\leq a(\|y_n - x_n\| + \|T_n z_n - y_n\|) \\ &\leq a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|T_n z_n - y_n\|) \end{aligned}$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|T_n z_n - y_n\|).$$

Obviously, by (3.7) and Step 2, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.15}$$

By (3.7) and (3.15), we also have

$$\|x_n - T_n z_n\| \leq \|x_n - y_n\| + \|y_n - T_n z_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 4. We show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$ and $\lim_{n\to\infty} \|y_n - z_n\| = 0$. To this end, let $p \in VI(C, A) \cap Fix(T)$. Since $z_n = P_C(x_n - \lambda_n A x_n)$ and $p = P_C(p - \lambda_n p)$, from Lemma 2.6, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \alpha_n \mu F) T_n z_n - (I - \alpha_n \mu F) p\|^2 \\ &\leq (\alpha_n \|\gamma V x_n - \mu F p\| + \|(I - \alpha_n \mu F) T_n z_n - (I - \alpha_n \mu F) T_n p\|)^2 \\ &\leq \alpha_n \|\gamma V x_n - \mu F p\|^2 + (1 - \tau\alpha_n)\|z_n - p\|^2 \\ &\quad + 2\alpha_n(1 - \tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\| \\ &\leq \alpha_n \|\gamma V(x_n) - \mu F p\|^2 + (1 - \tau\alpha_n)[\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A x_n - A p\|^2] \\ &\quad + 2\alpha_n(1 - \tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\| \\ &\leq \alpha_n \|\gamma V x_n - \mu F p\|^2 + \|x_n - p\|^2 + (1 - \tau\alpha_n)c(d - 2\alpha)\|A x_n - A p\|^2 \\ &\quad + 2\alpha_n \|\gamma V x_n - \mu F p\|\|z_n - p\|. \end{aligned}$$

So we obtain

$$\begin{aligned} &- (1 - \tau\alpha_n)c(d - 2\alpha)\|A x_n - A p\|^2 \\ &\leq \alpha_n \|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\ &\quad + 2\alpha_n \|\gamma V x_n - \mu F p\|\|z_n - p\| \\ &\leq \alpha_n \|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\ &\quad + 2\alpha_n \|\gamma V x_n - \mu F p\|\|z_n - p\|. \end{aligned}$$

Since $\alpha_n \to 0$ from the condition (i) and $\|x_n - y_n\| \to 0$ from Step 3, we have $\|Ax_n - Ap\| \to 0$ $(n \to \infty)$. Moreover, from (2.1) and (2.3), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\le \langle x_n - \lambda_n Ax_n - (p - \lambda_n Ap), z_n - p\rangle \\
&= \frac{1}{2}[\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\|^2 + \|z_n - p\|^2 \\
&\qquad - \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap) - (z_n - p)\|^2] \\
&\le \frac{1}{2}[\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\
&\qquad + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2].
\end{aligned}$$

and so

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2.$$

Thus

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\|\gamma Vx_n - \mu Fp\|^2 + (1 - \tau\alpha_n)\|z_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \tau\alpha_n)\|\gamma Vx_n - \mu Fp\|\|z_n - p\| \\
&\le \alpha_n\|\gamma Vx_n - \mu Fp\|^2 + \|x_n - p\|^2 - (1 - \tau\alpha_n)\|x_n - z_n\|^2 \\
&\quad + 2(1 - \tau\alpha_n)\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - (1 - \tau\alpha_n)\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n\|\gamma Vx_n - \mu Fp\|\|z_n - p\|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(1 - \tau\alpha_n)\|x_n - z_n\|^2 \\
&\le \alpha_n\|\gamma Vx_n - \mu Fp\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\
&\quad + 2(1 - \tau\alpha_n)\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - (1 - \tau\alpha_n)\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n\|\gamma Vx_n - \mu Fp\|\|z_n - p\| \\
&\le \alpha_n\|\gamma Vx_n - \mu Fp\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\
&\quad + 2(1 - \tau\alpha_n)\lambda_n\langle x_n - z_n, Ax_n - Ap\rangle - (1 - \tau\alpha_n)\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n\|\gamma Vx_n - \mu Fp\|\|z_n - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_n - y_n\| \to 0$ and $\|Ax_n - Ap\| \to 0$, we get $\|x_n - z_n\| \to 0$. Also by (3.15)

$$\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \to 0 \ (n \to \infty). \tag{3.16}$$

Step 5. We show that $\lim_{n\to\infty}\|T_n z_n - z_n\| = 0$. In fact, since

$$\begin{aligned}
\|T_n z_n - z_n\| &\le \|T_n z_n - y_n\| + \|y_n - z_n\| \\
&= \alpha_n\|\gamma Vx_n - \mu FT_n z_n\| + \|y_n - z_n\|,
\end{aligned}$$

from (3.7) and (3.16), we have $\lim_{n\to\infty}\|T_n z_n - z_n\| = 0$.

Step 6. We show that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F))q, y_n - q\rangle = \limsup_{n\to\infty}\langle(\gamma V - \mu F)q, y_n - q\rangle \le 0,$$

where q is the unique solution of the variational inequality (3.2).

First we prove that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu F)q, T_n z_n - q\rangle \le 0.$$

Since $\{z_n\}$ is bounded, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu F)q, T_n z_n - q\rangle = \lim_{i\to\infty}\langle(\gamma V-\mu F)q, T_{n_i} z_{n_i} - q\rangle.$$

Without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z \in C$.

Now we will show that $z \in VI(C,A) \cap Fix(T)$. First, let us that $z \in VI(C,A)$. Let

$$Qv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset & v \notin C. \end{cases}$$

Then Q is maximal monotone. Let $(v,w) \in G(Q)$. Since $w - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, w - Av\rangle \ge 0.$$

On the other hand, from $z_n = P_C(x_n - \lambda_n A x_n)$, we have $\langle v - z_n, z_n - (x_n - \lambda_n A x_n)\rangle \ge 0$ and hence

$$\left\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + Ax_n\right\rangle \ge 0.$$

Therefore we have

$$\begin{aligned}\langle v - z_{n_i}, w\rangle &\ge \langle v - z_{n_i}, Av\rangle \\ &\ge \langle v - z_{n_i}, Av\rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i}\right\rangle \\ &= \left\langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\ge \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

Since $\|z_n - x_n\| \to 0$ in Step 4 and A is α-inverse-strongly monotone, we have $\langle v - z, w\rangle \ge 0$ as $i \to \infty$. Since Q is maximal monotone, we have $z \in Q^{-1}0$ and hence $z \in VI(C,A)$.

Next, we show that $z \in Fix(T)$. To this end, define $S : C \to C$ by $Sx = rx + (1-r)Tx$, $\forall x \in C$, for $0 \le k \le k_n \le r < 1$ and $\lim_{n\to\infty} k_n = r$. Then S is nonexpansive with $Fix(S) = Fix(T)$ by Lemma 2.4 (iii). Notice that

$$\begin{aligned}\|Sz_{n_i} - z_{n_i}\| &\le \|Sz_{n_i} - T_{n_i} z_{n_i}\| + \|T_{n_i} z_{n_i} - z_{n_i}\| \\ &= (r - k_{n_i})\|z_{n_i} - Tz_{n_i}\| + \|T_{n_i} z_{n_i} - z_{n_i}\| \\ &= \frac{r - k_{n_i}}{1 - k_{n_i}}\|z_{n_i} - T_{n_i} z_{n_i}\| + \|T_{n_i} z_{n_i} - z_{n_i}\| \\ &= \frac{1 + r - 2k_{n_i}}{1 - k_{n_i}}\|T_{n_i} z_{n_i} - z_{n_i}\|.\end{aligned}$$

By Step 5 and $k_{n_i} \to r$, we have $\|Sz_{n_i} - z_{n_i}\| \to 0$. Assume that $z \notin Fix(T)(= Fix(S))$. Since $z_{n_i} \rightharpoonup z$ and $Sz \neq z$, by the Opial condition, we obtain

$$\begin{aligned}\liminf_{i\to\infty} \|z_{n_i} - z\| &< \liminf_{i\to\infty} \|z_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty}(\|z_{n_i} - Sz_{n_i}\| + \|Sz_{n_i} - Sz\|) \\ &= \liminf_{i\to\infty} \|Sz_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty} \|z_{n_i} - z\|,\end{aligned}$$

which is a contradiction. So, we get $z \in Fix(S)$. By Lemma 2.4 (iii), $z \in Fix(T)$. Therefore, $z \in Fix(T) \cap VI(C, A)$.

Now, from Step 5, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, T_n z_n - q\rangle &= \lim_{i\to\infty}\langle(\gamma V - \mu F)q, T_{n_i} z_{n_i} - q\rangle \\ &= \lim_{i\to\infty}\langle(\gamma V - \mu F)q, z_{n_i} - q\rangle \\ &= \langle(\gamma V - \mu F)q, z - q\rangle \leq 0.\end{aligned} \tag{3.17}$$

By (3.7) and (3.17) , we conclude that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, y_n - q\rangle \\ \leq\ &\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, y_n - T_n z_n\rangle + \limsup_{n\to\infty}\langle u + (\gamma V - \mu F)q, T_n z_n - q\rangle \\ \leq\ &\limsup_{n\to\infty}\|(\gamma V - \mu F)q\|\|y_n - T_n z_n\| + \limsup_{n\to\infty}\langle(\gamma V - \mu F)q, T_n z_n - q\rangle \\ \leq\ &0.\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$, where q is a solution of the variational inequality (3.2). Indeed from (3.1), Lemma 2.2, and Lemma 2.6, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F q) + (I - \alpha_n\mu F)T_n z_n - (I - \alpha_n\mu F)q\|^2 \\ &\leq \|(I - \alpha_n\mu F)T_n z_n - (I - \alpha_n\mu F)q\|^2 + 2\alpha_n\langle\gamma V x_n - \mu F q, y_n - q\rangle \\ &\leq (1 - \tau\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\gamma\langle V x_n - V q, y_n - q\rangle \\ &\quad + 2\alpha_n\langle\gamma V q - \mu F q, y_n - q\rangle \\ &\leq (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|\|y_n - q\| \\ &\quad + 2\alpha_n\langle(\gamma V - \mu F)q, y_n - q\rangle \\ &\leq (1 - \tau\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) \\ &\quad + 2\alpha_n\langle(\gamma V - \mu F)q, y_n - q\rangle \\ &= (1 - 2(\tau - \gamma l)\alpha_n)\|x_n - q\|^2 \\ &\quad + \alpha_n^2\tau^2\|x_n - q\|^2 + 2\alpha_n\gamma l\|x_n - q\|\|y_n - x_n\| \\ &\quad + 2\alpha_n\langle(\gamma V - \mu F)q, y_n - q\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2+\alpha_n^2\tau^2M_5^2+2\alpha_n\gamma l\|y_n-x_n\|M_5\\&\quad+2\alpha_n\langle(\gamma V-\mu F)q,y_n-q\rangle\\&=(1-\overline{\alpha_n})\|x_n-q\|^2+\overline{\beta_n},\end{aligned}$$

where $M_5=\sup\{\|x_n-q\|:n\ge 1\}$, $\overline{\alpha_n}=2(\tau-\gamma l)\alpha_n$ and

$$\overline{\beta_n}=\alpha_n[\alpha_n\tau^2M_5^2+2\gamma l\|y_n-x_n\|M_5+2\langle u+(\gamma V-\overline{F})q,y_n-q\rangle].$$

From (i), $\|y_n-x_n\|\to 0$ in Step 3 and Step 6, it is easily seen that $\overline{\alpha_n}\to 0$, $\sum_{n=1}^{\infty}\overline{\alpha_n}=\infty$, and $\limsup_{n\to\infty}\frac{\overline{\beta_n}}{\overline{\alpha_n}}\le 0$. Hence, by Lemma 2.3, we conclude $x_n\to q$ as $n\to\infty$. This completes the proof. □

From Theorem 3.2, we deduce the following result.

Corollary 3.3. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_1=x\in C\\ y_n=(1-\alpha_n)T_nP_C(x_n-\lambda_nAx_n),\\ x_{n+1}=(1-\beta_n)y_n+\beta_nT_nP_C(y_n-\lambda_nAy_n),\quad\forall n\ge 1,\end{cases}$$

where $\{\lambda_n\}\subset[0,2\alpha]$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1)$. *Let $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, and $\{k_n\}$ satisfy the conditions* (i), (ii), (iii) *and* (iv) *in Theorem 3.2. Then $\{x_n\}$ converges strongly to a point $q\in VI(C,A)\cap Fix(T)$, which is the unique solution of the following minimum norm problem: find $x^*\in VI(C,A)\cap Fix(T)$ such that*

$$\|x^*\|=\min_{x\in VI(C,A)\cap Fix(T)}\|x\|. \tag{3.19}$$

Proof. Take $F=I$, $\mu=1$, $\tau=1$, $V=0$, and $l=0$ in Theorem 3.2. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q,q-p\rangle\le 0,\quad\forall p\in VI(C,A)\cap Fix(T).$$

This obviously implies that

$$\|q\|^2\le\langle q,p\rangle\le\|q\|\|p\|,\quad\forall p\in VI(C,A)\cap Fix(T).$$

It turns out that $\|q\|\le\|p\|$ for all $p\in VI(C,A)\cap Fix(T)$. Therefore q is minimum norm point of $VI(C,A)\cap Fix(T)$. □

Taking $\beta_n=0$ for $n\ge 1$ in Theorem 3.2 and Corollary 3.3, we derive the following results.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_1=x\in C\\ x_{n+1}=\alpha_n\gamma Vx_n+(I-\alpha_n\mu F)T_nP_C(x_n-\lambda_nAx_n),\qquad\forall n\ge 1,\end{cases}$$

where $\{\lambda_n\}\subset[0,2\alpha]$, and $\{\alpha_n\}$ is a sequence in $[0,1)$. *Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{k_n\}$ satisfy the conditions* (i), (iii) *and* (iv) *in Theorem 3.2. Then $\{x_n\}$ converges strongly to a point $q\in Fix(T)$, which is the unique solution of a variational inequality* (VI1).

Corollary 3.5. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = (1-\alpha_n)T_n P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$ *and* $\{\alpha_n\}$ *is a sequence in* $[0,1)$. *Let* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{k_n\}$ *satisfy the conditions* (i), (iii) *and* (iv) *in Theorem 3.2. Then* $\{x_n\}$ *converges strongly to a point* $q \in VI(C,A) \cap Fix(T)$, *which solves the minimum norm problem* (3.19).

Remark 3.6. 1) Theorem 3.2 and Corollary 3.4 improve, extend, and develop the corresponding results in [5, 7, 8] in following aspects:

(a) The nonexpansive mapping S in [5,7,8] is extended to the case of a k-strictly pseudocontractive mapping T.
(b) A κ-Lipschitzian and η-strongly monotone mapping F is used.
(c) The contractive mapping f with constant $\xi \in (0,1)$ in [5, 8] is extended to the case of a Lipschitzian mapping V with constant $l \geq 0$.

2) Corollary 3.5 is also a new result for finding the minimum norm point of $Fix(T) \cap VI(C,A)$.

3) In all our results, we can replace the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on the control parameter $\{\alpha_n\}$ by the condition $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ ([21]), or by the perturbed control condition $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ ([11]).

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