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# UNIQUENESS OF POSITIVE SOLUTIONS OF BREZIS-NIRENBERG PROBLEMS ON $\mathbb{H}^n$

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ABSTRACT. We study the uniqueness of positive solutions of

$$\Delta_{\mathbb{H}^n}\varphi + \lambda\varphi + \varphi^p = 0$$

on the $n$-dimensional hyperbolic space $\mathbb{H}^n$, where $n \geq 2$, $\lambda \leq (n-1)^2/4$, and $p$ is subcritical or critical. In particular, in the case $n = 2$, we improve Mancini and Sandeep's uniqueness result.

## 1. Introduction

In this paper, we study the uniqueness of a positive solution of

$$\Delta_{\mathbb{H}}\varphi + \lambda\varphi + \varphi^p = 0 \quad \text{on } \mathbb{H}^n, \tag{1.1}$$

where $\mathbb{H} = \mathbb{H}^n$ is the $n$-dimensional hyperbolic space,

$$n \in \mathbb{N},\ n \geq 2,\ \lambda \leq \frac{(n-1)^2}{4}, \quad \begin{cases} 1 < p \leq \dfrac{n+2}{n-2} & \text{in the case } n \geq 3, \\ 1 < p < \infty & \text{in the case } n = 2, \end{cases} \tag{1.2}$$

and $\Delta_{\mathbb{H}}$ is the Laplace-Beltrami operator on $\mathbb{H}$. Uniqueness of positive solutions for the equations like (1.1) has been studied by many researchers; see [2–8, 10–13] and the references therein. One of such a equation is the scaler field equation

$$\Delta\varphi - \varphi + \varphi^p = 0 \quad \text{in } \mathbb{R}^n, \qquad \varphi(x) \to 0 \quad \text{as } |x| \to \infty,$$

and its uniqueness of a positive solution up to translation was established by Kwong [3]. Recently, the authors [10, 11] introduced a generalized Pohožaev identity and gave uniqueness results which are applicable to various equations including the scalar field equation.

For the hyperbolic space $\mathbb{H}$, it is well known [1] that

$$\inf_{\varphi \in H^1(\mathbb{H})\setminus\{0\}} \frac{\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\varphi|^2\, dV_{\mathbb{H}}}{\int_{\mathbb{H}} |\varphi|^2\, dV_{\mathbb{H}}} = \frac{(n-1)^2}{4},$$

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where $\nabla_{\mathbb{H}}$ is the gradient and $dV_{\mathbb{H}}$ is the volume element with respect to $\mathbb{H}$. It implies that

$$\|\varphi\|_{\mathbb{H}} = \left( \int_{\mathbb{H}} |\nabla_{\mathbb{H}}\varphi|^2 \, dV_{\mathbb{H}} \right)^{\frac{1}{2}}, \quad \varphi \in H^1(\mathbb{H})$$

is an equivalent norm on $H^1(\mathbb{H})$. We define

$$\|\varphi\|_{\mathcal{H}} = \left( \int_{\mathbb{H}} \left( |\nabla_{\mathbb{H}}\varphi|^2 - \frac{(n-1)^2}{4}\varphi^2 \right) dV_{\mathbb{H}} \right)^{\frac{1}{2}}$$

for $\varphi \in C_0^\infty(\mathbb{H})$, and we denote by $\mathcal{H}$ the completion of $C_0^\infty(\mathbb{H})$ with respect to this norm.

Now, we state our results. In the case $n = 2$, we improve [5, Theorem 1.3] and [5, Proposition 4.4]. More precisely, in the case $n = 2$, we do not need to assume $\lambda \le 2(p+1)/(p+3)^2$ in order to show the uniqueness of a positive solution of (1.1).

**Theorem 1.1.** *Assume* (1.2)*. In the case $\lambda < (n-1)^2/4$, problem*

$$\varphi \in H^1(\mathbb{H}), \qquad \Delta_{\mathbb{H}}\varphi + \lambda\varphi + \varphi^p = 0 \tag{1.3}$$

*has at most one positive solution up to hyperbolic isometries, and in the case $\lambda = (n-1)^2/4$, problem*

$$\varphi \in \mathcal{H}, \qquad \Delta_{\mathbb{H}}\varphi + \lambda\varphi + \varphi^p = 0 \tag{1.4}$$

*has at most one positive solution up to hyperbolic isometries.*

**Remark 1.2.** Mancini-Sandeep showed that if $n \ge 4$, $p = (n+2)/(n-2)$ and $\lambda \le n(n-2)/4$, or $n = 3$ and $p = 5$, then problem (2.6) does not have a positive solution; see [5, Theorems 1.6 and 1.7].

For each $y \in \mathbb{H}$ and $r > 0$, we define

$$B_{\mathbb{H}}(y, r) = \{x \in \mathbb{H} : d_{\mathbb{H}}(x, y) < r\},$$

where $d_{\mathbb{H}}(x, y)$ is the distance of $x, y \in \mathbb{H}$.

**Theorem 1.3.** *Assume* (1.2)*. Let $x_0 \in \mathbb{H}$ and $R > 0$. Then problem*

$$\begin{cases} \Delta_{\mathbb{H}}\varphi + \lambda\varphi + \varphi^p = 0 & \text{in } B_{\mathbb{H}}(x_0, R), \\ \varphi = 0 & \text{on } \partial B_{\mathbb{H}}(x_0, R) \end{cases} \tag{1.5}$$

*has at most one positive solution belonging to $C^2(B_{\mathbb{H}}(x_0, R)) \cap C(\overline{B_{\mathbb{H}}(x_0, R)})$.*

In order to show Theorem 1.3, we can apply the results in [10, 11]; see Section 3. On the other hand, in order to show Theorem 1.1, we can not apply them directly. Indeed, when we consider problem (2.3) below, which is a transformed problem (1.1), the coefficient functions $g, h$ defined by (2.2) diverge as $r \to 1 - 0$ except $h$ in the case $n \ge 3$ and $p = (n+2)/(n-2)$. So we need to treat the problem carefully. Applying the essential part of the proofs in [10, 11], we give the proof of Theorem 1.1 in the next section.

## 2. Proof of Theorem 1.1

Using the Poincare disk model, we consider that

$$\frac{4|dx|^2}{(1-|x|^2)^2}$$

is the Riemannian metric tensor on $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$, $\Delta_{\mathbb{H}}$ is given by

$$\left(\frac{1-|x|^2}{2}\right)^2 \left(\Delta + \frac{2(n-2)}{1-|x|^2} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right)$$

and (1.1) is represented as

$$\left(\frac{1-|x|^2}{2}\right)^2 \Delta\varphi + \frac{(n-2)(1-|x|^2)}{2} x \cdot \nabla\varphi + \lambda\varphi + \varphi^p = 0 \quad \text{in } B(0,1). \tag{2.1}$$

Setting $u(x) = (1-|x|^2)^{(2-n)/2}\varphi(x)$, the equation above is transformed into

$$\Delta u - g(|x|)u + h(|x|)u^p = 0 \qquad \text{in } B(0,1),$$

where

$$g(r) = \frac{n(n-2) - 4\lambda}{(1-r^2)^2} \quad \text{and} \quad h(r) = 4(1-r^2)^{\frac{(n-2)p-(n+2)}{2}}. \tag{2.2}$$

We know from [5, Theorem 2.1] that if $\varphi$ is a positive solution of (1.1), then there is $x_0 \in \mathbb{H}$ such that $\varphi$ is constant on any hyperbolic spheres centered at $x_0$. So we consider the uniqueness of a positive solution $u$ which satisfies

$$u_r(0) = 0, \qquad u_{rr} + \frac{n-1}{r} u_r - g(r)u + h(r)u^p = 0, \quad 0 < r < 1 \tag{2.3}$$

and

$$\begin{cases} (1-r^2)^{\frac{n-2}{2}} u(r) \in H^1(\mathbb{H}) & \text{in the case } \lambda < (n-1)^2/4, \\ (1-r^2)^{\frac{n-2}{2}} u(r) \in \mathcal{H} & \text{in the case } \lambda = (n-1)^2/4. \end{cases} \tag{2.4}$$

We define the functions $a, b, c, G$ by

$$\begin{aligned}
a(r) &= 16^{-\frac{1}{p+3}} r^{\frac{2(n-1)(p+1)}{p+3}} \left(1-r^2\right)^{\frac{n+2-(n-2)p}{p+3}}, \\
b(r) &= -\frac{1}{2} a_r(r) + \frac{n-1}{r} a(r) \\
&= \frac{1}{16^{\frac{1}{p+3}}(p+3)} r^{\frac{2(n-1)(p+1)}{p+3} - 1} \left(1-r^2\right)^{\frac{n+2-(n-2)p}{p+3} - 1} \\
&\qquad \cdot \left(2(n-1) - r^2((n-2)p + n - 4)\right), \\
c(r) &= -b_r(r) + \frac{n-1}{r} b(r) \\
&= -\frac{(n+2)-(n-2)p}{16^{\frac{1}{p+3}}(p+3)^2} r^{\frac{2(n-1)(p+1)}{p+3} - 2} \left(1-r^2\right)^{\frac{n+2-(n-2)p}{p+3} - 2} \\
&\qquad \cdot \left(r^4((n-2)p + n - 4) + r^2(n(p-1)+4) - 2n + 2\right), \\
G(r) &= b(r)g(r) + \frac{1}{2} c_r(r) - \frac{1}{2}\big(a(r)g(r)\big)_r(r)
\end{aligned}$$

$$= \frac{2^{\frac{p-1}{p+3}}(n-1)(1+r^2)r^{\frac{2(n-1)(p+1)}{p+3}-3}}{(p+3)^3(1-r^2)^{\frac{(n+1)p-n+7}{p+3}}}(\alpha r^4+\beta r^2+\alpha)$$
$$= \frac{2^{\frac{p-1}{p+3}}(n-1)(1+r^2)r^{\frac{2(n-1)(p+1)}{p+3}-3}}{(p+3)^3(1-r^2)^{\frac{(n+1)p-n+7}{p+3}}}\Big(\alpha(1-r^2)^2+(2\alpha+\beta)r^2\Big),$$

where

$$\alpha = ((n+2)-(n-2)p)((n-2)p+n-4),$$
$$\beta = 2\big(\lambda(p-1)(p+3)^2 - n^2(p^2-1) - 2n(3p+1) + 2(p+1)(p+5)\big).$$

We also define

$$D(r) = b(r)^2 - a(r)(c(r)-a(r)g(r)).$$

For each positive solution $u$ of (2.3), we define

$$J(r;u) = \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(u)u(r) + \frac{1}{2}(c(r)-a(r)g(r))u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1} \quad \text{on } (0,1).$$

Then we can show the following Propositions 2.1 and 2.2. For their proofs, see [10, 11].

**Proposition 2.1.** *For each positive solution $u$ of* (2.3)*, there holds*

$$\frac{d}{dr}J(r;u) = G(r)u(r)^2 \quad \text{on } (0,1).$$

**Proposition 2.2.** *For each pair of distinct positive solutions $u, v$ of* (2.3) *such that $u(0) < v(0)$ and $J(r;u) \ge 0$ on $(0,1)$, there holds*

$$\frac{d}{dr}\left(\frac{v(r)}{u(r)}\right) < 0 \quad \text{on } (0,1).$$

Setting

$$\varphi(t) = (1-r^2)^{\frac{n-2}{2}}u(r) \quad \text{for } r \in [0,1) \text{ with } r = \tanh\frac{t}{2}, \tag{2.5}$$

we can see that problem (2.3) is transformed into

$$\varphi_t(0) = 0, \qquad \varphi_{tt}(t) + \frac{n-1}{\tanh t}\varphi_t(t) + \lambda\varphi(t) + \varphi(t)^p = 0, \quad t > 0. \tag{2.6}$$

(Of course, (2.6) can be obtained from (2.1) by $r = \tanh t/2$.) By [5, Lemmas 3.4 and 3.6], we also know that each of a positive solution $\varphi \in H^1(\mathbb{H})$ of (2.6) with $\lambda < (n-1)^2/4$ or a positive solution $\varphi \in \mathcal{H}$ of (2.6) with $\lambda = (n-1)^2/4$ satisfies

$$\lim_{t\to\infty}\frac{\log\varphi(t)^2}{t} = \lim_{t\to\infty}\frac{\log\varphi_t(t)^2}{t} = -\left(n-1+\sqrt{(n-1)^2+4\lambda}\right) \tag{2.7}$$

and

$$\lim_{t\to\infty}\frac{\varphi_t(t)}{\varphi(t)} = -\frac{n-1+\sqrt{(n-1)^2+4\lambda}}{2},$$

and that each positive solution $\psi \in \mathcal{H}$ of (2.6) with $\lambda = (n-1)^2/4$ has $A > 0$ satisfying

$$\lim_{t\to\infty} e^{\frac{n-1}{2}t}\psi(t) = A \quad \text{and} \quad \lim_{t\to\infty} e^{\frac{n-1}{2}t}\psi_t(t) = -\frac{n-1}{2}A. \tag{2.8}$$

**Proposition 2.3.** *Assume* (1.2). *Let $u$ be a positive solution of* (2.3) *with* (2.4). *Then $J(r;u) \geq 0$ for each $r \in (0,1)$.*

In order to show Proposition 2.3, we prepare the following lemma. We note that $n \geq 2$ and $p > 1$ implies

$$\frac{2(n-1)^2(p+1)}{(p+3)^2} < \frac{(n-1)^2}{4}.$$

**Lemma 2.4.** *Assume the assumptions in Proposition 2.3 and let $u$ be as in Proposition 2.3. Assume also*

$$\lambda < \frac{2(n-1)^2(p+1)}{(p+3)^2}. \tag{2.9}$$

*Then $J(r;u) \to 0$ as $r \to 1-0$.*

*Proof.* Let $\varphi$ be the function defined by (2.5). From (2.9), we can choose $\varepsilon > 0$ such that

$$\sqrt{(n-1)^2 - 4\lambda} > 1 - \frac{n+2-(n-2)p}{p+3} + 2\varepsilon.$$

From (2.7), (2.8) and (2.5), we have

$$\varphi(t) = O\left(e^{(-\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}+\varepsilon)t}\right) \quad \text{and} \quad \varphi_t(t) = O\left(e^{(-\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}+\varepsilon)t}\right)$$

as $t \to \infty$. So we have

$$u(r) = O\left((1-r)^{\frac{1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon}\right) \quad \text{as } r \to 1-0 \tag{2.10}$$

and

$$\begin{aligned} u_r(r) &= \frac{2-n}{2}(1-r^2)^{-\frac{n}{2}}(-2r)\varphi(t) + (1-r^2)^{\frac{2-n}{2}}\varphi_t(t)\frac{dt}{dr} \\ &= (1-r^2)^{-\frac{n}{2}}((n-2)r\varphi(t) + 2\varphi_t(t)) \\ &= O\left((1-r)^{\frac{-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon}\right) \qquad \text{as } r \to 1-0. \end{aligned} \tag{2.11}$$

Let

$$\gamma = \frac{n+2-(n-2)p}{p+3} - 1 + \sqrt{(n-1)^2-4\lambda} - 2\varepsilon.$$

Then we can see $\gamma > 0$ and

$$\begin{cases} a(r)u_r(r)^2 = O((1-r)^\gamma), \\ b(r)u_r(r)u(r) = O((1-r)^\gamma), \\ c(r)u(r)^2 = O((1-r)^\gamma), \\ a(r)g(r)u(r)^2 = O((1-r)^\gamma) \end{cases} \quad \text{as } r \to 1-0.$$

Moreover, we have

$$a(r)h(r)u(r)^{p+1} = O\left((1-r)^{\gamma+(p-1)(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon)}\right) \quad \text{as } r \to 1-0.$$

Hence, we can infer $J(r;u) \to 0$ as $r \to 1-0$. □

*Proof of Proposition 2.3.* First, we consider the case $n \geq 3$ and

$$\lambda \geq \frac{2(n-1)^2(p+1)}{(p+3)^2}. \tag{2.12}$$

We have $\alpha \geq 0$. From (2.12) and

$$2\alpha + \beta = 2(p-1)\left(\lambda(p+3)^2 - 2(n-1)^2(p+1)\right),$$

we also have $2\alpha + \beta \geq 0$. So we have $G(r) \geq 0$ on $(0,1)$. From $n \geq 3$, we can see $J(r;u) \to 0$ as $r \to +0$. Hence, by Proposition 2.1, we obtain $J(r;u) \geq 0$ on $(0,1)$. Next, we consider the case $n = 2$ and (2.12). From $n = 2$, we have

$$D(r) = -\left(\frac{2r}{1-r^2}\right)^{\frac{2(p-1)}{p+3}} \frac{\left((1-r^2)^2 + \left(\lambda(p+3)^2 - 2(p+1)\right)r^2\right)}{(p+3)^2}. \tag{2.13}$$

From (2.12), we can find $D(r) < 0$ on $(0,1)$, which yields

$$\begin{aligned} J(r;u) &> \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}(c(r) - a(r)g(r))u(r)^2 \\ &= \frac{1}{2}a(r)u(r)^2\left(\left(\frac{u_r(r)}{u(r)} + \frac{b(r)}{a(r)}\right)^2 - \frac{D(r)}{a(r)^2}\right) > 0 \end{aligned}$$

for each $r \in (0,1)$. Hence, in the case (2.12), we have shown our assertion.

Next, we consider the case $n \geq 3$ and (2.9). From $\alpha \geq 0$ and $2\alpha + \beta < 0$, there exists $r_0 \in [0,1)$ such that $G(r) > 0$ on $(0, r_0)$ and $G(r) < 0$ on $(r_0, 1)$. We can easily see $J(r;u) \to 0$ as $r \to +0$. Using Lemma 2.4 and Proposition 2.1, we can infer the assertion. Next, we consider the case $n = 2$ and (2.9). We have $\alpha = -8$ and $2\alpha + \beta < 0$, which yields $G(r) < 0$ on $(0,1)$. Using Lemma 2.4 and Proposition 2.1 again, we can infer the assertion. □

**Remark 2.5.** Although we can not completely cover [5, Theorems 1.6 and 1.7] stated in Remark 1.2, we can easily see that problem (1.3) does not have a positive solution in the case $n \geq 3$, $p = (n+2)/(n-2)$ and $\lambda < n(n-2)/4$. Indeed, if there is a positive solution $\varphi$ in such a case, by Lemma 2.4, we can see the corresponding function $u$ defined by (2.5) satisfies $J(r;u) \to 0$ as $r \to +0$ and $r \to 1-0$. From $\alpha = 0$ and $2\alpha + \beta < 0$, we have $G(r) < 0$ on $(0,1)$. So we can obtain a contradiction by Proposition 2.1.

Now, we prove Theorem 1.1. Assume to the contrary that problem (2.3) has distinct positive solutions $u$ and $v$ such that

$$\begin{cases} (1-r^2)^{\frac{n-2}{2}}u(r), (1-r^2)^{\frac{n-2}{2}}v(r) \in H^1(\mathbb{H}) & \text{in the case } \lambda < (n-1)^2/4, \\ (1-r^2)^{\frac{n-2}{2}}u(r), (1-r^2)^{\frac{n-2}{2}}v(r) \in \mathcal{H} & \text{in the case } \lambda = (n-1)^2/4. \end{cases}$$

Let $\varphi$ and $\psi$ be the functions defined by (2.5) with $u$ and $v$, respectively. Without loss of generality, we may assume $u(0) < v(0)$. We define

$$\begin{aligned} X(r) &= \left(\frac{v(r)}{u(r)}\right)^2 J(r;u) - J(r;v) \\ &= \frac{1}{2}a(r)v(r)^2\left(\frac{u_r(r)^2}{u(r)^2} - \frac{v_r(r)^2}{v(r)^2}\right) + b(r)v(r)^2\left(\frac{u_r(r)}{u(r)} - \frac{v_r(r)}{v(r)}\right) \\ &\quad + \frac{1}{p+1}a(r)h(r)v(r)^2\left(u(r)^{p-1} - v(r)^{p-1}\right) \end{aligned}$$

for $r \in (0,1)$.

**Lemma 2.6.** *$X(r) \to 0$ as $r \to +0$ and $X(r) \to 0$ as $r \to 1-0$.*

*Proof.* It is easy to see $X(r) \to 0$ as $r \to +0$. We will show $X(r) \to 0$ as $r \to 1-0$. Let $\varepsilon > 0$ arbitrary such that

$$\frac{(n-1)(p^2-1)}{2(p+3)} > (p+5)\varepsilon.$$

From (2.7) and (2.8), we have

$$\lim_{t\to\infty}(\sinh^{n-1}t)(\varphi_t(t)\psi(t) - \varphi(t)\psi_t(t)) = 0,$$

and hence we obtain

$$\begin{aligned} &\left|(\sinh^{n-1}t)(\varphi_t(t)\psi(t) - \varphi(t)\psi_t(t))\right| \\ &\qquad = \left|\int_t^\infty \sinh^{n-1}(\tau)\varphi(\tau)\psi(\tau)(\varphi(\tau)^{p-1} - \psi(\tau)^{p-1})\,d\tau\right| \\ &\qquad \le C\int_t^\infty e^{(n-1)\tau}e^{(p+1)(-\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}+\varepsilon)\tau}\,d\tau \\ &\qquad = Ce^{((n-1)-(p+1)(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon))t}, \end{aligned}$$

where $C > 0$ is an appropriate constant. So we have

$$\varphi_t(t)\psi(t) - \varphi(t)\psi_t(t) = O\left(e^{-(p+1)(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon)t}\right) \quad \text{as } t \to \infty.$$

Since we have

$$\frac{1}{\varphi(t)} = O\left(e^{(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}+\varepsilon)t}\right) \quad \text{and} \quad \frac{1}{\psi(t)} = O\left(e^{(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}+\varepsilon)t}\right)$$

as $t \to \infty$, we obtain

$$\begin{aligned} \frac{\varphi_t(t)}{\varphi(t)} - \frac{\psi_t(t)}{\psi(t)} &= \frac{\varphi_t(t)\psi(t) - \varphi(t)\psi_t(t)}{\varphi(t)\psi(t)} \\ &= O\left(e^{-(p+1)(\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon)t-2(-\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon)t}\right) \\ &= O\left(e^{-(p-1)\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}t+(p+3)\varepsilon t}\right) \end{aligned}$$

$$= O\left((1-r)^{(p-1)\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-(p+3)\varepsilon}\right) \qquad \text{as } r \to 1-0.$$

From (2.5) and (2.11), we have

$$\begin{aligned}\frac{u_r(r)}{u(r)} &= \frac{(1-r^2)^{-\frac{n}{2}}((n-2)r\varphi(t)+2\varphi_t(t))}{(1-r^2)^{\frac{2-n}{2}}\varphi(t)} \\ &= (1-r^2)^{-1}\left((n-2)r+2\frac{\varphi_t(t)}{\varphi(t)}\right).\end{aligned}$$

Noting (2.10), we obtain

$$\begin{aligned}&a(r)v(r)^2\left(\frac{u_r(r)^2}{u(r)^2}-\frac{v_r(r)^2}{v(r)^2}\right) \\ &= 4a(r)v(r)^2(1-r^2)^{-2}\left((n-2)r+\frac{\varphi_t(t)}{\varphi(t)}+\frac{\psi_t(t)}{\psi(t)}\right)\left(\frac{\varphi_t(t)}{\varphi(t)}-\frac{\psi_t(t)}{\psi(t)}\right) \\ &= O\left((1-r)^{\frac{n+2-(n-2)p}{p+3}+1+\sqrt{(n-1)^2-4\lambda}-2\varepsilon-2+(p-1)\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-(p+3)\varepsilon}\right) \\ &= O\left((1-r)^{\frac{(n-1)(p^2-1)}{2(p+3)}+\frac{(p+1)\sqrt{(n-1)^2-4\lambda}}{2}-(p+5)\varepsilon}\right) \quad \text{as } r \to 1-0,\end{aligned}$$

and

$$\begin{aligned}&b(r)v(r)^2\left(\frac{u_r(r)}{u(r)}-\frac{v_r(r)}{v(r)}\right) \\ &= O\left((1-r)^{\frac{n+2-(n-2)p}{p+3}-1+1+\sqrt{(n-1)^2-4\lambda}-2\varepsilon-1+(p-1)\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}-(p+3)\varepsilon}\right) \\ &= O\left((1-r)^{\frac{(n-1)(p^2-1)}{2(p+3)}+\frac{(p+1)\sqrt{(n-1)^2-4\lambda}}{2}-(p+5)\varepsilon}\right) \quad \text{as } r \to 1-0.\end{aligned}$$

Hence, as $r \to 1-0$, we have

$$a(r)v(r)^2\left(\frac{u_r(r)^2}{u(r)^2}-\frac{v_r(r)^2}{v(r)^2}\right) \to 0, \qquad b(r)v(r)^2\left(\frac{u_r(r)}{u(r)}-\frac{v_r(r)}{v(r)}\right) \to 0.$$

We also have

$$\begin{aligned}&a(r)h(r)v(r)^2\left(u(r)^{p-1}-v(r)^{p-1}\right) \\ &= O\left((1-r)^{\frac{n+2-(n-2)p}{p+3}+\frac{(n-2)p-(n+2)}{2}+(p+1)(\frac{1+\sqrt{(n-1)^2-4\lambda}}{2}-\varepsilon)}\right) \\ &= O\left((1-r)^{\frac{(n-1)(p^2-1)}{2(p+3)}+\frac{(p+1)\sqrt{(n-1)^2-4\lambda}}{2}-(p+1)\varepsilon}\right)\end{aligned}$$

as $r \to 1-0$. Therefore we can find $X(r) \to 0$ as $r \to 1-0$. □

*Proof of Theorem 1.1.* By Remark 1.2, we do not need to treat the case $n \geq 3$, $p = (n+2)/(n-2)$ and $\lambda \leq n(n-2)/4$. We will show

$$G(r) \not\equiv 0 \quad \text{on } (0,1). \tag{2.14}$$

First, we consider the case $n \geq 3$. Since we exclude the case $p = (n+2)/(n-2)$ and $\lambda \leq n(n-2)/4$, we have $\alpha > 0$ or $2\alpha+\beta > 0$, which yields (2.14). In the case $n = 2$, we have $\alpha = -8$, and hence we have (2.14). From (2.14) and Propositions 2.1 and 2.3, we have $J(r;u) \geq 0$ and $J(r;u) \not\equiv 0$ on $(0,1)$. We also have

$$\begin{aligned}\frac{d}{dr}X(r) &= \left(\frac{d}{dr}\left(\frac{v(r)}{u(r)}\right)^2\right) J(r;u) + \left(\frac{v(r)}{u(r)}\right)^2 G(r)u(r)^2 - G(r)v(r)^2 \\ &= \left(\frac{d}{dr}\left(\frac{v(r)}{u(r)}\right)^2\right) J(r;u)\end{aligned}$$

for $r \in (0,1)$. Using Proposition 2.2, we have $d/drX(r) \leq 0$ and $d/drX(r) \not\equiv 0$ on $(0,1)$, which contradicts Lemma 2.6. Therefore, we have shown that each of problems (1.3) and (1.4) has at most one positive solution. $\square$

## 3. Proof of Theorem 1.3

We can see that problem (1.5) can be transformed into

$$\begin{cases} \Delta u - g(|x|)u + h(|x|)u^p = 0 & \text{in } B(0,r_0), \\ u = 0 & \text{on } \partial B(0,r_0) \end{cases} \tag{3.1}$$

with some $r_0 \in (0,1)$, where $g$, $h$ are the functions defined by (2.2) and $B(0,r_0) = \{x \in \mathbb{R}^n : |x| < r_0\}$. Putting $u(x) = v(x/r_0)$, problem (3.1) is transformed into

$$\begin{cases} \Delta v - r_0^2 g(r_0|x|)v + r_0^2 h(r_0|x|)v^p = 0 & \text{in } B(0,1) \\ v = 0 & \text{on } \partial B(0,1). \end{cases}$$

Setting $f : (0,1) \times (0,\infty) \to \mathbb{R}$ by $f(r,s) = -r_0^2 g(r_0 r)s + r_0^2 h(r_0 r)s^p$, we have

$$(1 - r_0^2 r^2)^{\frac{n+2}{2}} f(r, (1 - r_0^2 r^2)^{-\frac{n-2}{2}} s) = r_0^2(-(n(n-2) - 4\lambda)s + s^p),$$

which is independent of $r$. Applying [9, Theorem 1], we can see that each positive solution of problem (3.1) is radially symmetric. Hence, it is enough to show that problem

$$\begin{cases} u_{rr}(r) + \dfrac{n-1}{r} u_r(r) - g(r)u(r) + h(r)u(r)^p = 0, & 0 < r < r_0, \\ u_r(0) = 0, \quad u(r_0) = 0 \end{cases} \tag{3.2}$$

has at most one positive solution. We define $a, b, c, G, D$ as in the previous section. We can easily see $a(r) \to 0$ and $b(r) \to 0$ as $r \to +0$, and

$$\lim_{r \to +0} c(r) = \begin{cases} 0 & \text{in the case of } n \geq 3, \\ \infty & \text{in the case of } n = 2. \end{cases}$$

In the case $n \geq 3$, noting $\alpha \geq 0$, there is $r_1 \in [0, r_0]$ such that $G(r) \geq 0$ on $(0, r_1)$ and $G(r) \leq 0$ on $(r_1, r_0)$. In the case $n = 2$, we can see

$$\{r \in (0,r_0) : G(r) = 0,\ D(r) > 0\} = \emptyset. \tag{3.3}$$

In fact, in the case $2\alpha + \beta \geq 0$, which is equivalent to (2.12), from (2.13), we find $D(r) < 0$ on $(0, r_0)$. In the case $2\alpha + \beta < 0$, from $\alpha = -8$, we have $G(r) < 0$ on $(0, r_0)$. Thus we have (3.3). Hence, we can infer $J(r;u) \geq 0$ on $(0, r_0)$. Therefore, by [11, Theorem 1], we can see that problem (3.2) has at most one positive solution.

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