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UNIFORM NON-SQUARENESS FOR A-DIRECT SUMS OF BANACH SPACES WITH A STRICTLY MONOTONE NORM

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Abstract. We shall characterize the uniform non-squareness for a ψ-direct sum of finitely many Banach spaces with a strictly monotone norm, where the class $\Psi_N^{(1)}$ of convex functions on a certain convex set, which yields partial ℓ_1-norms on $\mathbb{C}^N$, plays an essential role. Also we shall show that a more general A-direct sum, a fortiori a Z-direct sum, is isometrically isomorphic to a ψ-direct sum. As a consequence we shall present a sequence of results on the uniform non-squareness in the general direct sum setting, which will cover some previously known results.

1. Introduction

Recently direct sums of Banach spaces have been often treated in the context of geometric properties of Banach spaces as well as the fixed point property (e.g. [1, 4–7, 9–13, 15, 18]). A Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ and a ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ are direct sums equipped with the norms induced from a Z-norm $\|\cdot\|_Z$, and a ψ-norm $\|\cdot\|_\psi$ or equivalently an absolute normalized norm on $\mathbb{R}^N$, respcetively (see Sections 2 and 5 for precise descriptions).

In Kato-Saito-Tamura [10] the following was shown: *A ψ-direct sum $X \oplus_\psi Y$ is uniformly non-square if and only if X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$, where ψ_1 and ψ_∞ are the convex functions on the unit interval corresponding to the ℓ_1- and ℓ_∞-norms, respectively.* They posed a question for the finitely many Banach spaces case, which turned out to be quite complicated. Dowling and Saejung [6, Theorem 13] gave a partial answer in the Z-direct sum setting and hence for ψ-direct sums, where the Z-norm $\|\cdot\|_Z$ is assumed to be strictly monotone (without this assumption for the case $N = 3$). On the other hand, Betiuk-Pilarska and Prus [1] showed that a Z-direct sum is uniformly non-square if and only if all the underlying Banach spaces and the Z-norm $\|\cdot\|_Z$ on $\mathbb{R}^N$ are uniformly non-square, where it remains still unknown when $\|\cdot\|_Z$ is uniformly non-square (cf. [14]).

In this paper we shall present a sequence of results on the uniform non-squareness for more general direct sums of Banach spaces, which we refer to as A-direct sums, with a strictly monotone norm: An A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ is the direct

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sum equipped with the norm induced from an *arbitrary* norm $\|\cdot\|_A$ on $\mathbb{R}^N$ (see Section 5). To do this, as it is enough to do so (Section 5), we shall first discuss a ψ-direct sum which enables us to use a powerful tool, the convex function ψ. In particular the class $\Psi_N^{(1)}$ of convex functions introduced in [12], which yields ℓ_1-like norms on $\mathbb{C}^N$, will play an essential role. As another point to do this the notions Properties T_1^N and T_∞^N in Dowling and Saejung [6] will be described in terms of the class $\Psi_N^{(1)}$.

In Section 2 definitions and preliminary results concerning ψ-direct sums will be mentioned. In Section 3 the uniform non-squareness for ψ-direct sums will be discussed. We shall first show that, under the condition that $\|\cdot\|_\psi$ is strictly monotone, $(X_1\oplus\cdots\oplus X_N)_\psi$ is uniformly non-square if and only if X_j's are uniformly non-square and $\psi\notin\Psi_N^{(1)}$ (Theorem 3.5). In the case the dual norm $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$ is strictly monotone, the same is true with $\psi^*\notin\Psi_N^{(1)}$ in place of $\psi\notin\Psi_N^{(1)}$, where ψ^* is the dual function of ψ (Theorem 3.7). From these results the following will be derived: Assume that $\|\cdot\|_\psi$ or $\|\cdot\|_\psi^*$ is strictly monotone. Then, $(X_1\oplus\cdots\oplus X_N)_\psi$ is uniformly non-square if and only if X_j's are uniformly non-square and $\psi,\psi^*\notin\Psi_N^{(1)}$ (Theorem 3.11).

In Section 4 we shall discuss the relation between the class $\Psi_N^{(1)}$ and the notions of Properties T_1^N and T_∞^N. We shall show that Properties T_1^N and T_∞^N are respectively equivalent to $\psi\notin\Psi_N^{(1)}$ and $\psi^*\notin\Psi_N^{(1)}$ (Theorem 4.3).

In the final Section 5 we shall discuss Z-direct sums and A-direct sums. A ψ-direct sum is a Z-direct sum and a Z-direct sum is an A-direct sum. In Theorem 5.2 we shall show that these three kinds of notions of direct sum are equivalent, more precisely, for any norm $\|\cdot\|_A$ on $\mathbb{R}^N$ there exists $\psi\in\Psi_N$ such that the A-direct sum $(X_1\oplus\cdots\oplus X_N)_A$ is isometrically isomorphic to the ψ-direct sum $(X_1\oplus\cdots\oplus X_N)_\psi$.

Combining the results in Section 3 with Theorems 4.3 and 5.2 we shall obtain the following. Assume that $\|\cdot\|_A$ (resp., $\|\cdot\|_A^*$) is strictly monotone. Then, $(X_1\oplus\cdots\oplus X_N)_A$ is uniformly non-square if and only if X_j's are uniformly non-square and $\|\cdot\|_A$ has Property T_1^N (resp., T_∞^N) (Theorems 5.3, 5.4). In particular, if $\|\cdot\|_A$ or $\|\cdot\|_A^*$ is strictly monotone, $(X_1\oplus\cdots\oplus X_N)_A$ is uniformly non-square if and only if all X_j are uniformly non-square and $\|\cdot\|_A$ has Properties T_1^N and T_∞^N (Theorem 5.5), which covers the main result of Dowling and Saejung [6, Theorem 13] for Z-direct sums.

2. Definitions and preliminary results

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *absolute* if $\|(z_1,\cdots,z_N)\| = \|(|z_1|,\cdots,|z_N|)\|$ for all $(z_1,\cdots,z_N)\in\mathbb{C}^N$ and *normalized* if $\|(1,0,\cdots,0)\| = \cdots = \|(0,\cdots,0,1)\| = 1$. A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *monotone* provided that, if $|z_j|\le|w_j|$ for $1\le j\le N$, $\|(z_1,\ldots,z_N)\|\le\|(w_1,\ldots,w_N)\|$. A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *strictly monotone* provided that it is monotone and, if $|z_j|<|w_j|$ for some $1\le j\le N$, $\|(z_1,\ldots,z_N)\|<\|(w_1,\ldots,w_N)\|$. The following is known.

Lemma 2.1 (Bhatia [2], see also [13]). *A norm $\|\cdot\|$ on $\mathbb{C}^N$ is absolute if and only if it is monotone.*

Let us recall that for every absolute normalized norm on $\mathbb{C}^N$ there corresponds a convex function on a certain convex set in $\mathbb{R}^{N-1}$ ([3, 16]). For any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^N$ let

$$\psi(s) = \left\|\left(1-\sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1}\right)\right\| \text{ for } s=(s_1,\cdots,s_{N-1}) \in \Delta_N, \tag{2.1}$$

where

$$\Delta_N = \left\{ s=(s_1,\cdots,s_{N-1}) \in \mathbb{R}^{N-1} : \sum_{i=1}^{N-1} s_i \le 1,\ s_i \ge 0 \right\}$$

Then ψ is convex (continuous) on Δ_N and satisfies the following:

(A_0) $\psi(0,\cdots,0) = \psi(1,0,\cdots,0) = \cdots = \psi(0,\cdots,0,1) = 1$,

(A_1) $\psi(s_1,\dots,s_{N-1}) \ge \left(\sum_{i=1}^{N-1} s_i\right)\psi\left(\frac{s_1}{\sum_{i=1}^{N-1} s_i},\cdots,\frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\right)$ if $0 < \sum_{i=1}^{N-1} s_i \le 1$,

(A_2) $\psi(s_1,\cdots,s_{N-1}) \ge (1-s_1)\psi\left(0,\frac{s_2}{1-s_1},\cdots,\frac{s_{N-1}}{1-s_1}\right)$ if $0 \le s_1 < 1$,

....

(A_N) $\psi(s_1,\cdots,s_{N-1}) \ge (1-s_{N-1})\psi\left(\frac{s_1}{1-s_{N-1}},\cdots,\frac{s_{N-2}}{1-s_{N-1}},0\right)$ if $0 \le s_{N-1} < 1$.

In fact the condition (A_0) is equivalent to that the norm $\|\cdot\|$ is normalized. The conditions $(A_j), 1 \le j \le N$, are equivalent to the monotonicity in the jth entry of the norm respectively, which is equivalent to that the norm is absolute by Lemma 2.1.

Conversely, let Ψ_N denote the class of all convex functions ψ on Δ_N satisfying (A_0)–(A_N). Then, for any $\psi \in \Psi_N$ we can construct an absolute normalized norm $\|\cdot\|_\psi$ on $\mathbb{C}^N$ by the formula

$$\|(z_1,\cdots,z_N)\|_\psi = \begin{cases} \left(\sum_{j=1}^N |z_j|\right)\psi\left(\frac{|z_2|}{\sum_{j=1}^N |z_j|},\cdots,\frac{|z_N|}{\sum_{j=1}^N |z_j|}\right) & \\ \qquad \text{if } (z_1,\cdots,z_N) \ne (0,\cdots,0), & \\ 0 \qquad \text{if } (z_1,\cdots,z_N) = (0,\cdots,0), & \end{cases} \tag{2.2}$$

where $\|\cdot\|_\psi$ satisfies (2.1) ([16]; see [3] for the case $N=2$). Thus every absolute normalized norm $\|\cdot\|$ corresponds to a unique convex function $\psi \in \Psi_N$ with the equation (2.1). We refer the norm $\|\cdot\|_\psi$ to as *ψ-norm*. The ℓ_p-norms

$$\|(z_1,\cdots,z_N)\|_p = \begin{cases} \{|z_1|^p + \cdots + |z_N|^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|z_1|,\cdots,|z_N|\} & \text{if } p = \infty \end{cases}$$

are basic examples of such norms and the corresponding functions ψ_p are given by

$$\psi_p(s_1,\cdots,s_{N-1}) = \begin{cases} \left\{\left(1-\sum_{i=1}^{N-1} s_i\right)^p + s_1^p + \cdots + s_{N-1}^p\right\}^{1/p} & \\ \qquad \text{if } 1 \le p < \infty, & \\ \max\{1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-1}\} \qquad \text{if } p = \infty. & \end{cases}$$

In particular, the function $\psi_1(t) = 1$ corresponds to the ℓ_1-norm. For all $\psi \in \Psi_N$ we have $\|\cdot\|_\infty \le \|\cdot\|_\psi \le \|\cdot\|_1$ ([16, Lemma 4.1]).

Let $X_1, ..., X_N$ be Banach spaces. The ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$, $\psi \in \Psi_N$, is their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_\psi := \|(\|x_1\|, \cdots, \|x_N\|)\|_\psi \text{ for } (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

([9, 18]). The dual function ψ^* of ψ is defined by

$$\psi^*(s_1, \ldots, s_{N-1}) = \sup_{(t_1,\ldots,t_{N-1})\in\Delta_N} \frac{(1-\sum_{i=1}^{N-1} s_i)(1-\sum_{i=1}^{N-1} t_i) + \sum_{i=1}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})},$$

$(s_1, \ldots, s_{N-1}) \in \Delta_N$ ([15], see also [12]):

Theorem 2.2 ([12, Theorem 4.1, 4.2]; cf. [15]). *Let $X_1, \cdots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Then, $\psi^* \in \Psi_N$ and*

$$(X_1 \oplus \cdots \oplus X_N)^*_\psi = (X_1^* \oplus \cdots \oplus X_N^*)_{\psi^*}$$

A Banach space X is called *uniformly non-square* if there exists a constant $\varepsilon > 0$ such that

$$\min\{\|x+y\|, \|x-y\|\} \le 2(1-\varepsilon) \text{ for all } x, y \in X \text{ with } \|x\| = \|y\| = 1.$$

(If $\min\{\|x+y\|, \|x-y\|\} < 2$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$, X is called *non-square.*) Throughout the paper let $\mathbb{R}^N_+ = \{(a_1, \ldots, a_N) \in \mathbb{R}^N : \ a_j \ge 0, 1 \le j \le N\}$.

3. Uniform non-squareness for ψ-direct sums

Lemma 3.1 ([16], [2, p.36, Lemma 3]). *Let $\psi \in \Psi_N$.*

(i) *The ψ-norm $\|\cdot\|_\psi$ is monotone.*
(ii) *If $|z_j| < |w_j|$ for all $1 \le j \le N$, $\|(z_1, \ldots, z_N)\|_\psi < \|(w_1, \ldots, w_N)\|_\psi$.*

Lemma 3.2 ([10]). *Let $\{x_n\}$ and $\{y_n\}$ be nonzero bounded sequences in a Banach space X with $\lim_{n\to\infty} \|x_n\| > 0$ and $\lim_{n\to\infty} \|y_n\| > 0$. Then the following are equivalent.*

(i) $\displaystyle\lim_{n\to\infty} \|x_n + y_n\| = \lim_{n\to\infty} \{\|x_n\| + \|y_n\|\}$.
(ii) $\displaystyle\lim_{n\to\infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2.$

Definition 3.3 (cf. [12]). Let $\psi \in \Psi_N$. We say $\psi \in \Psi_N^{(1)}$ if there exist $(a_1, \ldots, a_N) \in \mathbb{R}^N_+$ and a nonempty proper subset T of $\{1, \ldots, N\}$ for which

$$\begin{aligned} \|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\ &\quad + \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi, \end{aligned} \tag{3.1}$$

where $(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)$ and $(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)$ are nonzero, χ_T denotes the characteristic function of T. A ψ-norm with $\psi \in \Psi_N^{(1)}$ is referred to as a *partial ℓ_1-norm.*

Theorem 3.4 (cf. [12, Theorem 5.8]). *Let $\psi \in \Psi_N$. Then the following are equivalent.*

(i) $\psi \in \Psi_N^{(1)}$.

(ii) *There exist* $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ *and a nonempty proper subset* T *of* $\{1, \dots, N\}$ *for which the formula* (3.1) *holds true with*

$$\|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\|_\psi = \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|_\psi = 1.$$

(iii) *There exists a nonempty subset* S *of* $\{1, \dots, N-1\}$ *and an element* $(s_1, \dots, s_{N-1}) \in \Delta_N$ *with* $0 < \sum_{i=1}^{N-1} \chi_S(i)s_i < 1$ *such that*

$$\begin{aligned} \psi(s_1, \dots, s_{N-1}) &= M\psi\left(\frac{\chi_S(1)s_1}{M}, \dots, \frac{\chi_S(N-1)s_{N-1}}{M}\right) \\ &\quad +(1-M)\psi\left(\frac{\chi_{S^c}(1)s_1}{1-M}, \dots, \frac{\chi_{S^c}(N-1)s_{N-1}}{1-M}\right). \end{aligned}$$

We refer the reader to [12,13] for a sequence of results on the class $\Psi_N^{(1)}$. Now we shall have the following characterization of uniform non-squareness for a ψ-direct sum with a strictly monotone norm.

Theorem 3.5. *Let* $X_1, \dots, X_N$ *be Banach spaces and let* $\psi \in \Psi_N^{(1)}$. *Assume that the* ψ*-norm* $\|\cdot\|_\psi$ *is strictly monotone. Then the following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is uniformly non-square.*

(ii) X_j*'s are uniformly non-square and* $\psi \notin \Psi_N^{(1)}$.

Proof. (i) $\Rightarrow$ (ii). Let $(X_1 \oplus \cdots \oplus X_N)_\psi$ be uniformly non-square. Then all X_j are uniformly non-square as they are embedded into $(X_1 \oplus \cdots \oplus X_N)_\psi$. Suppose that $\psi \in \Psi_N^{(1)}$. Then, by Theorem 3.4 there exist $(a_1, \dots, a_N) \in \mathbb{R}_+^N$ and a nonempty proper subset T of $\{1, \dots, N\}$ for which

$$(a_1, \dots, a_N)\|_\psi = \|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\|_\psi + \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|_\psi,$$

where

$$\|(\chi_T(1)a_1, \dots, \chi_T(N)a_N)\|_\psi = \|(\chi_{T^c}(1)a_1, \dots, \chi_{T^c}(N)a_N)\|_\psi = 1.$$

Without loss of generality we may assume that $T = \{1, \dots, r\}$ with some $1 \le r < N$, that is,

$$\|(a_1, \dots, a_N)\|_\psi = \|(a_1, \dots, a_r, 0, \dots, 0)\|_\psi + \|(0, \dots, 0, a_{r+1}, \dots, a_N)\|_\psi \tag{3.2}$$

and

$$\|(a_1, \dots, a_r, 0, \dots, 0)\|_\psi = \|(0, \dots, 0, a_{r+1}, \dots, a_N)\|_\psi = 1. \tag{3.3}$$

Take $x_j \in S_{X_j}, 1 \le j \le N$, and let

$$u = (a_1x_1, \dots, a_rx_r, 0, \dots, 0) \in (X_1 \oplus \cdots \oplus X_N)_\psi$$

and

$$v = (0, \dots, 0, a_{r+1}x_{r+1}, \dots, a_Nx_N) \in (X_1 \oplus \cdots \oplus X_N)_\psi.$$

Then by (3.2) and (3.3) we have

$$\|u\|_\psi = \|v\|_\psi = 1 \text{ and } \|u \pm v\|_\psi = \|(a_1, \dots, a_N)\|_\psi = 2,$$

whence $(X_1 \oplus \cdots \oplus X_N)_\psi$ is not uniformly non-square, a contradiction. Thus we have $\psi \notin \Psi_N^{(1)}$.

(ii) $\Rightarrow$ (i). Let all X_j be uniformly non-square and let $\psi \notin \Psi_N^{(1)}$. Suppose that $(X_1 \oplus \cdots \oplus X_N)_\psi$ is not uniformly non-square. Then for every n there exist $(x_1^{(n)}, \ldots, x_N^{(n)})$ and $(y_1^{(n)}, \ldots, y_N^{(n)})$ in the unit ball of $(X_1 \oplus \cdots \oplus X_N)_\psi$ such that

$$\|(\|x_1^{(n)} \pm y_1^{(n)}\|, \ldots, \|x_N^{(n)} \pm y_N^{(n)}\|)\|_\psi \geq 2 - \frac{1}{n}. \tag{3.4}$$

By choosing subsequences if necessary, we may assume that for all $1 \leq j \leq N$

$$\lim_{n\to\infty} \|x_j^{(n)} + y_j^{(n)}\| = \alpha_j, \ \lim_{n\to\infty} \|x_j^{(n)} - y_j^{(n)}\| = \beta_j,$$

and

$$\lim_{n\to\infty} \|x_j^{(n)}\| = \mu_j, \ \lim_{n\to\infty} \|y_j^{(n)}\| = \nu_j.$$

Then

$$\|(\mu_1, \ldots, \mu_N)\|_\psi = \lim_{n\to\infty} \|(\|x_1^{(n)}\|, \ldots, \|x_N^{(n)}\|)\|_\psi \leq 1 \tag{3.5}$$

and in the same way

$$\|(\nu_1, \ldots, \nu_N)\|_\psi \leq 1. \tag{3.6}$$

By (3.4) we have for any $n \in \mathbb{N}$

$$\begin{aligned} 2 - \frac{1}{n} &\leq \|(\|x_1^{(n)} \pm y_1^{(n)}\|, \ldots, \|x_N^{(n)} \pm y_N^{(n)}\|)\|_\psi \\ &\leq \|(\|x_1^{(n)}\| + \|y_1^{(n)}\|, \ldots, \|x_N^{(n)}\| + \|y_N^{(n)}\|)\|_\psi \\ &\leq \|(\|x_1^{(n)}\|, \ldots, \|x_N^{(n)}\|)\|_\psi + \|(\|y_1^{(n)}\|, \ldots, \|y_N^{(n)}\|)\|_\psi \\ &\leq 2. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} \|(\alpha_1, \ldots, \alpha_N)\|_\psi &= \|(\beta_1, \ldots, \beta_N)\|_\psi \\ &= \|(\mu_1 + \nu_1, \ldots, \mu_N + \nu_N)\|_\psi = 2. \end{aligned} \tag{3.7}$$

Since the norm $\|\cdot\|_\psi$ is strict monotone, we have

$$\alpha_j = \beta_j = \mu_j + \nu_j \text{ for all } 1 \leq j \leq N.$$

Now we shall show that

$$\min\{\mu_j, \nu_j\} = 0 \text{ and hence } \alpha_j = \beta_j = \max\{\mu_j, \nu_j\} \text{ for all } 1 \leq j \leq N. \tag{3.8}$$

Suppose that $\min\{\mu_{j_0}, \nu_{j_0}\} > 0$ with some $1 \leq j_0 \leq N$. Then

$$\lim_{n\to\infty} \|x_{j_0}^{(n)} \pm y_{j_0}^{(n)}\| = \mu_{j_0} + \nu_{j_0} = \lim_{n\to\infty} \|x_{j_0}^{(n)}\| + \lim_{n\to\infty} \|y_{j_0}^{(n)}\|,$$

from which it follows that

$$\lim_{n\to\infty} \left\| \frac{x_{j_0}^{(n)}}{\|x_{j_0}^{(n)}\|} \pm \frac{y_{j_0}^{(n)}}{\|y_{j_0}^{(n)}\|} \right\| = 2$$

by Lemma 3.2. Therefore X_{j_0} is not uniformly non-square, which is a contradiction. Thus we have (3.8).

Next let

$$T = \{j : \ \alpha_j = \mu_j > 0, \ 1 \leq j \leq N\}.$$

Then T is a nonempty proper subset of $\{1, \dots, N\}$. Indeed, if T is empty, we have

$$\alpha_j = 0 \text{ or } \alpha_j = \nu_j \text{ for all } 1 \le j \le N$$

and hence

$$2 = \|(\alpha_1, \dots, \alpha_N)\|_\psi \le \|(\nu_1, \dots, \nu_N)\|_\psi \le 1,$$

a contradiction. If $T = \{1, \dots, N\}$, we have

$$2 = \|(\alpha_1, \dots, \alpha_N)\|_\psi = \|(\mu_1, \dots, \mu_N)\|_\psi \le 1,$$

a contradiction. Thus, without loss of generality, we may assume that $T = \{1, \dots, r\}$ with some $1 \le r < N$. Then, by (3.7) we have

$$\begin{aligned} 2 &= \|(\alpha_1, \dots, \alpha_N)\|_\psi \\ &\le \|(\alpha_1, \dots, \alpha_r, 0, \dots, 0)\|_\psi + \|(0, \dots, 0, \alpha_{r+1}, \dots, \alpha_N)\|_\psi \\ &\le \|(\mu_1, \dots, \mu_r, 0, \dots, 0)\|_\psi + \|(0, \dots, 0, \nu_{r+1}, \dots, \nu_N)\|_\psi \le 2, \end{aligned}$$

from which it follows that

$$\|(\alpha_1, \dots, \alpha_N)\|_\psi = \|(\alpha_1, \dots, \alpha_r, 0, \dots, 0)\|_\psi + \|(0, \dots, 0, \alpha_{r+1}, \dots, \alpha_N)\|_\psi$$

and

$$\|(\alpha_1, \dots, \alpha_r, 0, \dots, 0)\|_\psi = \|(0, \dots, 0, \alpha_{r+1}, \dots, \alpha_N)\|_\psi = 1.$$

Consequently we have $\psi \in \Psi_N^{(1)}$ by Theorem 3.4, which is a contradiction. This completes the proof. □

Remark 3.6. The assertion (i) $\Rightarrow$ (ii) in Theorem 3.5 is valid without the assumption on the strict monotonicity of $\|\cdot\|_\psi$.

Since $(X_1 \oplus \cdots \oplus X_N)_\psi^* = (X_1^* \oplus \cdots \oplus X_N^*)_{\psi^*}$ by Theorem 2.2 and X^* is uniformly non-square if and only if X is ([17]), we shall have the next theorem.

Theorem 3.7. *Let $X_1, \dots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Assume that the ψ^*-norm $\|\cdot\|_{\psi^*}$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square.*

(ii) *All X_j are uniformly non-square and $\psi^* \notin \Psi_N^{(1)}$.*

Remark 3.8. The assertion (i) $\Rightarrow$ (ii) in Theorem 3.7 is valid without the assumption on the strict monotonicity of $\|\cdot\|_{\psi^*}$.

According to Remarks 3.6 and 3.8 we have the following.

Proposition 3.9. *Let $X_1, \dots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Let $(X_1 \oplus \cdots \oplus X_N)_\psi$ be uniformly non-square. Then $\psi, \psi^* \notin \Psi_N^{(1)}$.*

Remark 3.10. Proposition 3.9 is valid under the assumption "non-squareness" in place of uniform non-squareness (cf. [6, Theorem 3]). In fact, in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 3.5 we proved that, if $(X_1 \oplus \cdots \oplus X_N)_\psi$ is "non-square", then $\psi \notin \Psi_N^{(1)}$.

From Theorems 3.5, 3.7 and Proposition 3.9 the next result follows.

Theorem 3.11. *Let $X_1, \dots, X_N$ be Banach spaces and let $\psi \in \Psi_N^{(1)}$. Assume that $\|\cdot\|_\psi$ or $\|\cdot\|_{\psi^*}$ is strictly monotone. Then the following are equivalent.*

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is uniformly non-square.*

(ii) *All* X_j *are uniformly non-square and* $\psi, \psi^* \notin \Psi_N^{(1)}$.

4. The class $\Psi_N^{(1)}$ and properties T_1^N and T_∞^N

A similar result to Theorem 3.11 was obtained in Dowling and Saejung [6] in terms of the notions Properties T_1^N and T_∞^N. We shall see that these results are equivalent.

Definition 4.1 ([6])**.** A norm $\|\cdot\|$ on $\mathbb{C}^N$ is said to have *Property* T_1^N if for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$ with

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a} + \boldsymbol{b}\| = 1$$

one has $\operatorname{supp} \boldsymbol{a} \cap \operatorname{supp} \boldsymbol{b} \neq \emptyset$, where $\operatorname{supp} \boldsymbol{a} = \{j : a_j \neq 0\}$. A norm $\|\cdot\|$ is said to have *Property* T_∞^N if for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$ with

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a} + \boldsymbol{b}\| = 1$$

one has $\operatorname{supp} \boldsymbol{a} \cap \operatorname{supp} \boldsymbol{b} \neq \emptyset$.

These properties for an absolute norm are interpreted in words of partial ℓ_1-norms or the class $\Psi_N^{(1)}$. First we shall see the next result.

Proposition 4.2. *Let* $\psi \in \Psi_N$ *and let* T *be a nonempty proper subset of* $\{1, \ldots, N\}$. *Then the following are equivalent.*

(i) *There exists* $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ *such that*

$$\begin{aligned} \|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\ &= \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi = 1. \end{aligned}$$

(ii) *There exists* $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ *such that*

$$\begin{aligned} \|(a_1, \ldots, a_N)\|_{\psi^*} &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_{\psi^*} \\ &\quad + \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_{\psi^*}, \end{aligned}$$

where $\|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_{\psi^*} = \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_{\psi^*} = 1.$

Proof. (i) $\Rightarrow$ (ii). Assume that there exists $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ such that

$$\begin{aligned} \|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\ &= \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi = 1. \end{aligned}$$

Then there exist $(c_1, \ldots, c_N),\ (d_1, \ldots, d_N) \in \mathbb{R}_+^N$ such that

$$\begin{aligned} \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi &= \sum_{j=1}^N \chi_T(j)a_jc_j, \\ \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi &= \sum_{j=1}^N \chi_{T^c}(j)a_jd_j, \end{aligned}$$

and

$$\|(\chi_T(1)c_1, \ldots, \chi_T(N)c_N)\|_{\psi^*} = \|(\chi_{T^c}(1)d_1, \ldots, \chi_{T^c}(N)d_N)\|_{\psi^*} = 1.$$

Let $a_j' = \chi_T(j)c_j + \chi_{T^c}(j)d_j$ $(1 \le j \le N)$. Then, since

$$\begin{aligned}
\|(a_1', \ldots, a_N')\|_{\psi^*} &= \|(\chi_T(1)c_1 + \chi_{T^c}(1)d_1, \ldots, \chi_T(N)c_N + \chi_{T^c}(N)d_N)\|_{\psi^*} \\
&\ge \sum_{j=1}^N \chi_T(j)c_j a_j + \sum_{j=1}^N \chi_{T^c}(j)d_j a_j \\
&= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\
&\quad + \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi = 2,
\end{aligned}$$

we have $\|(a_1', \ldots, a_N')\|_\psi = 2$. Consequently we have

$$\|(a_1', \ldots, a_N')\|_{\psi^*} = \|(\chi_T(1)a_1', \ldots, \chi_T(N)a_N')\|_{\psi^*} + \|(\chi_{T^c}(1)a_1', \ldots, \chi_{T^c}(N)a_N')\|_{\psi^*},$$

where $\|(\chi_T(1)a_1', \ldots, \chi_T(N)a_N')\|_{\psi^*} = \|(\chi_{T^c}(1)a_1', \ldots, \chi_{T^c}(N)a_N')\|_{\psi^*} = 1$.

(ii) $\Rightarrow$ (i). Assume that there exists $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ satisfying the condition in (ii). Take $(a_1', \ldots, a_N') \in \mathbb{R}_+^N$ with $\|(a_1', \ldots, a_N')\|_\psi = 1$ so that

$$\|(a_1, \ldots, a_N)\|_{\psi^*} = \sum_{j=1}^N a_j a_j'.$$

Then, since $\|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_{\psi^*} = \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_{\psi^*} = 1$, we have

$$\begin{aligned}
2 &= \|(a_1, \ldots, a_N)\|_{\psi^*} \\
&= \sum_{j=1}^N \chi_T(j)a_j a_j' + \sum_{j=1}^N \chi_{T^c}(j)a_j a_j' \\
&= \sum_{j=1}^N \chi_T(j)a_j \chi_T a_j' + \sum_{j=1}^N \chi_{T^c}(j)a_j \chi_{T^c} a_j' \\
&\le \|(\chi_T(1)a_1', \ldots, \chi_T(N)a_N')\|_\psi + \|(\chi_{T^c}(1)a_1', \ldots, \chi_{T^c}(N)a_N')\|_\psi \le 2,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\|(a_1', \ldots, a_N')\|_\psi &= \|(\chi_T(1)a_1', \ldots, \chi_T(N)a_N')\|_\psi \\
&= \|(\chi_{T^c}(1)a_1', \ldots, \chi_{T^c}(N)a_N')\|_\psi = 1.
\end{aligned}$$

This completes the proof. □

Now we have the following.

Theorem 4.3. *Let $\psi \in \Psi_N$. Then:*

(i) *The ψ-norm $\|\cdot\|_\psi$ has Property T_1^N if and only if $\psi \notin \Psi_N^{(1)}$.*
(ii) *The ψ-norm $\|\cdot\|_\psi$ has Property T_∞^N if and only if $\psi^* \notin \Psi_N^{(1)}$.*

Proof. (i) We have the following.

The ψ-norm $\|\cdot\|_\psi$ has property T_1^N
$\Longleftrightarrow$ For all $\boldsymbol{a} = (a_1, \ldots, a_N), \boldsymbol{b} = (b_1, \ldots, b_N) \in \mathbb{C}^N$ with $\operatorname{supp} \boldsymbol{a} \cap \operatorname{supp} \boldsymbol{b}$ $= \emptyset$ it does not hold that $\|\boldsymbol{a}\|_\psi = \|\boldsymbol{b}\|_\psi = \frac{1}{2}\|\boldsymbol{a} + \boldsymbol{b}\|_\psi = 1$.

$\iff$ There is no element $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ such that for some nonempty proper subset T of $\{1, \ldots, N\}$

$$\begin{aligned}\frac{1}{2}\|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\ &= \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi = 1\end{aligned}$$

$\iff$ $\psi \notin \Psi_N^{(1)}$ (by Theorem 3.4).

(ii) We have the following.

The ψ-norm $\|\cdot\|_\psi$ has property T_∞^N

$\iff$ For all $\boldsymbol{a} = (a_1, \ldots, a_N), \boldsymbol{b} = (b_1, \ldots, b_N) \in \mathbb{C}^N$ with supp $\boldsymbol{a} \cap$ supp $\boldsymbol{b}$ $= \emptyset$ it does not hold that $\|\boldsymbol{a}\|_\psi = \|\boldsymbol{b}\|_\psi = \|\boldsymbol{a}+\boldsymbol{b}\|_\psi = 1$

$\iff$ There is no element $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ such that for some nonempty proper subset T of $\{1, \ldots, N\}$

$$\begin{aligned}\|(a_1, \ldots, a_N)\|_\psi &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\psi \\ &= \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\psi = 1\end{aligned}$$

$\iff$ There is no element $(a_1, \ldots, a_N) \in \mathbb{R}_+^N$ such that

$$\begin{aligned}&\|(a_1, \ldots, a_N)\|_{\psi^*} \\ &= \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_{\psi^*} + \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_{\psi^*},\end{aligned}$$

where $\|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_{\psi^*} = \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_{\psi^*} = 1$ (by Proposition 4.2) $\iff$ $\psi^* \notin \Psi_N^{(1)}$ (by Theorem 3.4).

This completes the proof. □

By Theorems 3.5, 3.7 combined with Theorem 4.3 we have the following.

Corollary 4.4. *Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N^{(1)}$. Assume that the ψ-norm $\|\cdot\|_\psi$ (resp. ψ^*-norm $\|\cdot\|_{\psi^*}$) is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square.*
(ii) *All X_i are uniformly non-square and $\|\cdot\|_\psi$ has property T_1^N (resp. T_∞^N).*

From Proposition 3.9 and Theorem 4.3 the next result will be immediately derived.

Corollary 4.5. *Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. If $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square, the ψ-norm $\|\cdot\|_\psi$ has properties T_1^N and T_∞^N.*

Combining Theorems 3.11 and 4.3, we have the ψ-direct sum version of Dowling and Saejung's result (see Corollary 5.6 below):

Corollary 4.6. *Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N^{(1)}$. Assume that $\|\cdot\|_\psi$ or $\|\cdot\|_{\psi^*}$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square.*
(ii) *All X_i are uniformly non-square and $\|\cdot\|_\psi$ has both Properties T_1^N and T_∞^N.*

5. Z-DIRECT SUMS AND GENERAL DIRECT SUMS

As another notion of direct sum of Banach spaces a Z-direct sum is often discussed (cf. [6] etc.). Let Z be a finite dimensional normed space $(\mathbb{R}^N, \|\cdot\|_Z)$, where the Z-norm $\|\cdot\|_Z$ is monotone on $\mathbb{R}^N_+$, that is,

$$\|(a_1, \dots, a_N)\|_Z \le \|(b_1, \dots, b_N)\|_Z \text{ if } 0 \le a_j \le b_j \text{ for } 1 \le j \le N. \tag{5.1}$$

The Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \dots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_Z := \|(\|x_1\|, \cdots, \|x_N\|)\|_Z \tag{5.2}$$

for $(x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N$. In [6] the Z-norm $\|\cdot\|_Z$ is assumed to be absolute without loss of generality because of (5.2). On the other hand, according to Lemma 2.1 this is equivalent to the monotonicity of $\|\cdot\|_Z$. Thus in the above definition of Z-direct sum the condition (5.1) is superfluous and can be dropped.

The Z-direct sum is a more general notion than the ψ-direct sum; however they are equivalent as is mentioned in [6]. In fact we shall see that for any Z-direct sum there exists $\psi \in \Psi_N$ such that the Z-direct sum is isometrically isomorphic to the ψ-direct sum. This is true for a more general direct sum which are defined from an arbitrary norm on $\mathbb{R}^N$.

Definition 5.1. Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. We define A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ to be their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_A := \|(\|x_1\|, \cdots, \|x_N\|)\|_A \text{ for } (x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

We have the following.

Theorem 5.2. *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then, there exists $\psi \in \Psi_N$ such that $(X_1 \oplus \cdots \oplus X_N)_A$ is isometrically isomorphic to $(X_1 \oplus \cdots \oplus X_N)_\psi$.*

Proof. Take $e_j \in X_j$ with $\|e_j\| = 1$ for $1 \le j \le N$ and define a new norm $\|\cdot\|_B$ on $\mathbb{C}^N$ by

$$\|(z_1, \dots, z_N)\|_B = \|(z_1 e_1, \dots, z_N e_N)\|_A.$$

Then $\|\cdot\|_B$ is absolute. Indeed, for $(z_1, \dots, z_N) \in \mathbb{C}^N$ we have

$$\begin{aligned} \|(z_1, \dots, z_N)\|_B &= \|(z_1 e_1, \dots, z_N e_N)\|_A \\ &= \|(|z_1|, \dots, |z_N|)\|_A \\ &= \|(|z_1| e_1, \dots, |z_N| e_N)\|_A \\ &= \|(|z_1|, \dots, |z_N|)\|_B. \end{aligned}$$

Let

$$\|(x_1, \cdots, x_N)\|_B := \|(\|x_1\|, \cdots, \|x_N\|)\|_B \text{ for } (x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N.$$

Then we have

$$\begin{aligned} \|(x_1, \dots, x_N)\|_A &= \|(\|x_1\|, \dots, \|x_N\|)\|_A \\ &= \|(\|x_1\| e_1, \dots, \|x_N\| e_N)\|_A \\ &= \|(\|x_1\|, \dots, \|x_N\|)\|_B \\ &= \|(x_1, \dots, x_N)\|_B. \end{aligned}$$

Next let $a_j = \|(0, \dots, 0, \overset{j}{\overbrace{1}}, 0 \dots, 0)\|_B$ for $1 \le j \le N$. Define the norm $\|\cdot\|_C$ on $\mathbb{C}^N$ by

$$\|(z_1, \dots, z_N)\|_C = \|(z_1/a_1, \dots, z_N/a_N)\|_B \text{ for } (z_1, \dots, z_N) \in \mathbb{C}^N.$$

Then $\|\cdot\|_C$ is absolute and normalized, and

$$\|(z_1, \dots, z_N)\|_B = \|(a_1 z_1, \dots, a_N z_N)\|_C \text{ for } (z_1, \dots, z_N) \in \mathbb{C}^N.$$

Consequently we have

$$\|(x_1, \dots, x_N)\|_A = \|(a_1 x_1, \dots, a_N x_N)\|_C \text{ for } (x_1, \dots, x_N) \in X_1 \oplus \cdots \oplus X_N.$$

Thus $(X_1 \oplus \cdots \oplus X_N)_A$ is isometric to $(X_1 \oplus \cdots \oplus X_N)_C$. This completes the proof. □

As we have seen in the above proof, to construct the A-direct sum we may assume that the original norm $\|\cdot\|_A$ is absolute and normalized without loss of generality, which indicates that the ψ-direct sum is general enough. Owing to Theorem 5.2 with Theorem 4.3, from the previous Theorems 3.5, 3.7, and 3.11 for the ψ-direct sum we shall obtain a sequence of "general" results:

Theorem 5.3. *Let $X_1, \dots, X_N$ be Banach spaces and let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_A$ is uniformly non-square.*
(ii) *X_j's are uniformly non-square and the norm $\|\cdot\|_A$ has Property T_1^N.*

Theorem 5.4. *Let $X_1, \dots, X_N$ be Banach spaces and let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that the dual norm $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_A$ is uniformly non-square.*
(ii) *X_j's are uniformly non-square and the norm $\|\cdot\|_A$ has Property T_∞^N.*

Theorem 5.5. *Let $X_1, \dots, X_N$ be Banach spaces and let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ or $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_A$ is uniformly non-square.*
(ii) *X_i's are uniformly non-square and the norm $\|\cdot\|_A$ has Properties T_1^N and T_∞^N.*

In particular the next result by Dowling-Saejung [6] is a consequence of Theorem 5.5.

Corollary 5.6 (Dowling and Saejung [6]). *Let $X_1, \dots, X_N$ be Banach spaces and let $\|\cdot\|_Z$ be an absolute norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_Z$ or $\|\cdot\|_Z^*$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_Z$ is uniformly non-square.*
(ii) *X_i's are uniformly non-square and the Z-norm $\|\cdot\|_Z$ has Properties T_1^N and T_∞^N.*

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