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ON SEQUENTIAL OPTIMALITY CONDITIONS FOR ROBUST MULTIOBJECTIVE CONVEX OPTIMIZATION PROBLEMS

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Abstract. An uncertain multiobjective convex optimization problem (UMP) and its robust counterpart (RUMP) of (UMP) are considered. We present sequential optimality conditions for (weakly, properly) robust efficient solutions of (RUMP) which hold without any constraint qualification, which are expressed in terms of sequences with subdifferentials and ϵ-subdifferentials for involved convex functions. The interesting feature of the Lagrange optimality conditions for (RUMP) is that the number of the Lagrangian multipliers coincides with the number of constraint functions. Moreover, we give a sufficient condition that a robust efficient solution of (RUMP) can be a properly robust efficient solution of (RUMP). We present examples illustrating our results.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. For a set A in $\mathbb{R}^n$, the closure (resp. convex hull) of A is denoted by $\mathrm{cl}(A)$ (resp. $\mathrm{co}A$). We say A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. The indicator function $\delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if} \quad x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

For an extended real-valued function f on $\mathbb{R}^n$, the effective domain and the epigraph are respectively defined by $\mathrm{dom}f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ and $\mathrm{epi}f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq r\}$. We say that f is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\mathrm{dom}f \neq \emptyset$. Moreover, if $\liminf\limits_{x' \to x} f(x') \geq f(x)$ for all $x \in \mathbb{R}^n$, we say f is a lower semicontinuous function. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\mu \in [0,1]$, $f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$. Moreover,

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we say f is concave if $-f$ is convex. The subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For any proper convex function f on $\mathbb{R}^n$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in \mathbb{R}^n$. For details of conjugate function, see [17]. Clearly, f^* is a proper lower semicontinuous convex function and $\lambda \text{epi} f^* = \text{epi}(\lambda f)^*$ for any $\lambda > 0$. If one of the convex functions f_1, f_2 is continuous, then we have

$$\text{epi}(f_1 + f_2)^* = \text{epi} f_1^* + \text{epi} f_2^*. \tag{1.1}$$

For details see [16].

Lemma 1.1 (cf. [9])**.** *Let I be an arbitrary index set and let f_i, $i \in I$, be proper lower semicontinuous convex functions on $\mathbb{R}^n$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} f_i(x_0) < \infty$. Then*

$$\text{epi}(\sup_{i \in I} f_i)^* = \text{cl}\Big(\text{co} \bigcup_{i \in I} \text{epi} f_i^*\Big),$$

where $\sup_{i \in I} f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i \in I} f_i)(x) = \sup_{i \in I} f_i(x)$ for all $x \in \mathbb{R}^n$.

We recall a version of the Brondsted-Rockafellar theorem which was established in [18].

Proposition 1.1 (Brondsted-Rockafellar Theorem [6,18])**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$ there exist $x_\epsilon \in \mathbb{R}^n$ and $x_\epsilon^* \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - \bar{x}\| \leq \sqrt{\epsilon}, \quad \|x_\epsilon^* - x^*\| \leq \sqrt{\epsilon} \quad \text{and} \quad |f(x_\epsilon) - x_\epsilon^*(x_\epsilon - \bar{x}) - f(\bar{x})| \leq 2\epsilon.$$

A standard form of multiobjective optimization problem is as follows:

$$\begin{array}{lll} \text{(MP)} & \min & (f_1(x), \ldots, f_l(x)) \\ & \text{s.t.} & g_j(x) \leq 0,\ j = 1, \ldots, m, \end{array}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, are functions.

A multiobjective optimization problem (MP) in the face of data uncertainty both in the objective and constraints can be captured by the problem

$$\begin{array}{lll} \text{(UMP)} & \min & (f_1(x, u_1), \ldots, f_l(x, u_l)) \\ & \text{s.t.} & g_j(x, v_j) \leq 0,\ j = 1, \ldots, m, \end{array}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^{q_1} \to \mathbb{R}$, $i = 1, \ldots, l$, and $g_j : \mathbb{R}^n \times \mathbb{R}^{q_2} \to \mathbb{R}$, $j = 1, \ldots, m$, are functions, $\mathcal{U}_i$, $i = 1, \ldots, l$, are nonempty subsets in $\mathbb{R}^{q_1}$ and $u_i \in \mathcal{U}_i$, $i = 1, \ldots, l$, and $\mathcal{V}_j$, $j = 1, \ldots, m$, are nonempty subsets in $\mathbb{R}^{q_2}$ and $v_j \in \mathcal{V}_j$, $j = 1, \ldots, m$.

Here we suppose that we do not know the exact values of u_i, $i = 1, \dots, l$ and v_j, $j = 1, \dots, m$, but we know that u_i, $i = 1, \dots, l$, belongs to some uncertainty sets $\mathcal{U}_i$, $i = 1, \dots, l$ and, v_j, $j = 1, \dots, m$, belongs to some uncertainty sets $\mathcal{V}_j$, $j = 1, \dots, m$. For the worst case of (UMP), the robust counterpart of (UMP) is given as follows (see [2]):

$$\begin{aligned} \text{(RUMP)} \quad & \min \quad \Big(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \max_{u_l \in \mathcal{U}_l} f_l(x, u_l) \Big) \\ & \text{s.t.} \quad g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \dots, m. \end{aligned}$$

Let the robust feasible set of (RUMP) defined by

$$F := \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \dots, m\}.$$

Then $\bar{x} \in F$ is said to be a robust efficient solution of (RUMP) if there does not exist a robust feasible solution x of (RUMP) such that

$$\begin{aligned} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) &\leq \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \ i = 1, \dots, l, \\ \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) &< \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \ \text{for some } k. \end{aligned}$$

Also, $\bar{x} \in F$ is called a weakly robust efficient solution of (RUMP) if there does not exist a robust feasible solution x of (RUMP) such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \ i = 1, \dots, l.$$

Also, $\bar{x} \in F$ is said to be a properly robust efficient solution of (RUMP) if it is an efficient robust solution of (RUMP) and there is a number $M > 0$ such that for all $i \in \{1, \dots, l\}$ and $x \in F$ satisfying $\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)$, there exists an index $k \in \{1, \dots, l\}$ such that $\max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) < \max_{u_k \in \mathcal{U}_k} f_i(x, u_k)$ and moreover

$$\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)}{\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) - \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k)} \leq M.$$

Convex programs that are affected by data uncertainty ([2–5, 12, 15]) have been intensively studied. Recently, the duality theory for convex programs under uncertainty via robust approach(worst-case approach) have been studied ([2, 12, 15]). It was shown that primal worst equals dual best ([2, 12]).

On the other hand, recently, new sequential Lagrange multiplier conditions characterizing optimality without any constraint qualification for convex programs are presented in terms of the convex subdifferentials and the ϵ-subdifferentials ([9, 10, 13, 14]). It was also shown how the sequential conditions are related to the standard Lagrange multiplier condition ([9, 14]).

In this paper, we present sequential optimality conditions for (weakly, properly) robust efficient solutions for (RUMP) which hold without any constraint qualification, which are expressed in terms of sequences with subdifferentials and ϵ-subdifferentials for convex functions. The interesting feature of the Lagrange optimality conditions is that the number of the Lagrangian multipliers coincides with the number of constraint functions. We give a sufficient condition that a robust efficient solution of (RUMP) can be a properly robust efficient solution of (RUMP).

Furthermore, we introduce a constraint qualification for (RUMP) and give Lagrange optimality conditions for (weakly, properly) robust efficient solutions for (RUMP) which hold under the constraint qualification. We present examples illustrating our results.

2. Sequential optimality conditions I

The following proposition, which describes the relationship between the epigraph of a conjugate function and the ϵ-subdifferential and which plays a key role in deriving the main results, was recently given in [8].

Proposition 2.1. *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function and let $a \in \mathrm{dom} f$. Then*

$$\mathrm{epi} h^* = \bigcup_{\epsilon \geq 0} \Big\{ \big(v, v^T a + \epsilon - h(a)\big) : v \in \partial_\epsilon h(a) \Big\}.$$

The following theorem, which is the robust version of an alternative theorem, can be obtained from Proposition 2.3 and Theorem 2.4 in [12]. For the sake of completeness, we give a short proof here.

Theorem 2.1 (Robust Theorem of the Alternative)**.** *Let $f_i : \mathbb{R}^n \times \mathbb{R}^{q_1} \to \mathbb{R}$, $i = 1, \ldots, l$, be continuous functions such that $f_i(\cdot, u_i)$ is a convex function for each $u_i \in \mathbb{R}^{q_1}$ and let $g_j : \mathbb{R}^n \times \mathbb{R}^{q_2} \to \mathbb{R}$, $j = 1, \ldots, m$, be continuous functions such that $g_j(\cdot, v_j)$ is a convex function for each $v_j \in \mathbb{R}^{q_2}$. Let $\mathcal{U}_i$ be a nonempty convex and compact subset of $\mathbb{R}^{q_1}$, $i = 1, \ldots, l$, and let $\mathcal{V}_j$ be a nonempty convex and compact subset of $\mathbb{R}^{q_2}$, $j = 1, \ldots, m$. Let $F := \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j,\ j = 1, \ldots, m\} \neq \emptyset$. Suppose that for each $x \in \mathbb{R}^n$, $g_j(x, \cdot)$ is a concave function. Then exact one of the following two statements holds:*

(i) $(\exists x \in \mathbb{R}^n)\ \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < 0,\ g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j,\ j = 1, \ldots, m;$

(ii) $(0, 0) \in \mathrm{epi}\Big(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \Big)^* + \mathrm{cl}\Big(\bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \mathrm{epi}\Big(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j) \Big)^* \Big).$

Proof. Suppose that (i) does not hold. Then for any $x \in F$,

$$\max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0,\ j = 1, \ldots, m \ \Rightarrow\ \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq 0.$$

So, we have $\inf_{x \in \mathbb{R}^n} \{ \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \delta_F(x) \} \geq 0$. By assumptions, $\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)$ is continuous. So, $(0, 0) \in \mathrm{epi}(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) + \delta_F)^* = \mathrm{epi}(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))^* + \mathrm{epi}\delta_F^*$. Since $\delta_F(x) = \sup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \sum_{j=1}^{m} \lambda_j g_j(x, v_j)$, it follows from Lemma 1.1 that

$$\mathrm{epi}\delta_F^* \ = \ \mathrm{epi}\Big(\sup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j) \Big)^*$$

$$= \operatorname{cl}\Big(\operatorname{co}\Big(\bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}} \operatorname{epi}\Big(\sum_{j=1}^m \lambda_j g_j(\cdot, v_j)\Big)^*\Big)$$

Moreover, we can check that the concavity assumption on the functions $g_j(x,\cdot)$ implies the convexity of the set $\bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}} \operatorname{epi}(\sum_{j=1}^m \lambda_j g_j(\cdot, v_j))^*$ (see the proof of Proposition 2.3 in [12]). Thus (ii) holds.

Conversely, suppose that (ii) holds. Then $(0,0) \in \operatorname{epi}(\sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i) + \delta_F)^*$ and hence $\inf_{x\in\mathbb{R}^n}\{\sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(x,u_i) + \delta_F(x)\} \geq 0$. Thus for any $x \in F$, $\sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(x,u_i) \geq 0$. Hence (i) does not hold. $\square$

By using Proposition 2.1 and Theorem 2.1, we can obtain the following sequential optimality theorems:

Theorem 2.2. *Let $\mathcal{U}_i$ be a nonempty convex and compact subset of $\mathbb{R}^{q_1}$, $i = 1,\dots,l$, and let $\mathcal{V}_j$ be a nonempty convex and compact subset of $\mathbb{R}^{q_2}$, $j = 1,\dots,m$. Let $f_i : \mathbb{R}^n \times \mathbb{R}^{q_1} \to \mathbb{R}$, $i = 1,\dots,l$, be functions such that for each $u_i \in \mathcal{U}_i$, $f_i(\cdot,u_i)$ is a convex function on $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$, $f_i(x,\cdot)$ is a concave function on $\mathbb{R}^{q_1}$. Let $g_j : \mathbb{R}^n \times \mathbb{R}^{q_2} \to \mathbb{R}$, $j = 1,\dots,m$, be functions such that for each $v_j \in \mathcal{V}_j$, $g_j(\cdot,v_j)$ is a convex function on $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$, $g_j(x,\cdot)$ is a concave function on $\mathbb{R}^{q_2}$. Let $F := \{x \in \mathbb{R}^n \mid g_j(x,v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j,\ j = 1,\dots,m\} \neq \emptyset$. Let $\bar{x} \in F$. Then the following statements are equivalent:*

(i) *the point $\bar{x}$ is a robust efficient solution of* (RUMP)*;*

(ii)

$$\Big(0, -\sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big) \in \operatorname{epi}\Big(\sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\Big)^*$$
$$+\operatorname{cl}\Big[\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^l \mu_i f_i(\cdot,u_i)\Big)^* + \Big(0,\ \sum_{i=1}^l \mu_i \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)\Big)$$
$$+ \bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}} \operatorname{epi}\Big(\sum_{j=1}^m \lambda_j g_j(\cdot,v_j)\Big)^*\Big];$$

(iii) *there exist $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot,\bar{u}_i)(\bar{x})$, $\mu_i^n \geq 0$, $u_i^n \in \mathcal{U}_i$, $i = 1,\dots,l$, $\delta_n \geq 0$, $\xi_n \in \partial_{\delta_n}(\sum_{i=1}^l \mu_i^n f_i(\cdot,u_i^n))(\bar{x})$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1,\dots,m$, $\gamma_n \geq 0$ and $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^m \lambda_j^n g_j(\cdot,v_j^n))(\bar{x})$ such that*

$$\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i) = f_i(\bar{x},\bar{u}_i),\ i = 1,\dots,l,$$

$$0 = \sum_{i=1}^l \nu_i + \lim_{n\to\infty}(\xi_n + \zeta_n),\ \lim_{n\to\infty}\delta_n = 0,\ \lim_{n\to\infty}\gamma_n = 0,$$

$$\lim_{n\to\infty}[\mu_i^n f_i(\bar{x},u_i^n) - \mu_i^n f_i(\bar{x},\bar{u}_i)] = 0,\ i = 1,\dots,l \text{ and } \lim_{n\to\infty}\sum_{j=1}^m \lambda_j^n g_j(\bar{x},v_j^n) = 0.$$

Proof. Notice that $\bar{x}$ is a robust efficient solution of (RUMP) if and only if $\bar{x}$ is a solution of the following problem:

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \\ &\text{subject to} && \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \leq 0, \ i = 1, \ldots, l, \\ &&& \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m. \end{aligned}$$

Let $\bar{x}$ be a robust efficient solution of (RUMP). Let $\tilde{F} := \{x \in \mathbb{R}^n \mid \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \leq 0, \ i = 1, \ldots, l, \ \max_{v_j \in \mathcal{V}_j} g_j(x, v_j) \leq 0, \ j = 1, \ldots, m\}$. Then, for any $x \in F$, $\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \geq 0$. So, from Theorem 2.1,

$$\begin{aligned} &\Big(0, -\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) \in \operatorname{epi}\Big(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\Big)^* \\ &+ \operatorname{cl}\Big[\bigcup_{\substack{u_i \in \mathcal{U}_i, v_j \in \mathcal{V}_j \\ \mu_i \geq 0, \lambda_j \geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^{l} \mu_i (f_i(\cdot, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) + \sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j))^*\Big)\Big]. \end{aligned}$$

Hence, from (1.1), we have

$$\begin{aligned} &\Big(0, -\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) \in \operatorname{epi}\Big(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\Big)^* \\ &+ \operatorname{cl}\Big[\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^{l} \mu_i f_i(\cdot, u_i) - \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big)^*\Big) \\ &\quad + \bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}\Big(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j)\Big)^*\Big]. \end{aligned}$$

So, by the definition of epigraph, we have

$$\begin{aligned} &\Big(0, -\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) \in \operatorname{epi}\Big(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\Big)^* \\ &+ \operatorname{cl}[\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^{l} \mu_i f_i(\cdot, u_i)\Big)^* + \Big(0, \ \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big)\Big) \\ &\quad + \bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}\Big(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j)\Big)^*\Big], \end{aligned}$$

and so, by Proposition 2.1, we have

$$\begin{aligned}
&\Big(0, -\sum_{i=1}^{l} \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)\\
= &\bigcup_{\epsilon\geq 0}\Big\{\Big(\nu, \nu^T\bar{x}+\epsilon-\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big) \;\Big|\; \nu\in\partial_\epsilon\Big(\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\Big)(\bar{x})\Big\}\\
&+\mathrm{cl}\Big[\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}}\Big(\bigcup_{\delta\geq 0}\Big\{\Big(\xi,\xi^T\bar{x}+\delta-\sum_{i=1}^{l}\mu_i f_i(\bar{x},u_i)\Big)\;\Big|\;\xi\in\partial_\delta\Big(\sum_{i=1}^{l}\mu_i f_i(\cdot,u_i)\Big)(\bar{x})\Big\}\Big)\\
&+\Big(0,\sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)\\
&+\bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}}\Big(\bigcup_{\gamma\geq 0}\Big\{\Big(\zeta,\zeta^T\bar{x}+\gamma-\sum_{j=1}^{m}\lambda_j g_j(\bar{x},v_j)\Big)\;\Big|\;\zeta\in\partial_\gamma\Big(\sum_{j=1}^{m}\lambda_l g_j(\cdot,v_j)\Big)(\bar{x})\Big\}\Big)\Big].
\end{aligned}$$

Hence, there exist $\epsilon\geq 0$, $\nu\in\partial_\epsilon(\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i))(\bar{x})$, $u_i^n\in\mathcal{U}_i$, $\mu_i^n\geq 0$, $i=1,\ldots,l$, $\delta_n\geq 0$, $\xi_n\in\partial_{\delta_n}(\sum_{i=1}^{l}\mu_i f_i(\cdot,u_i^n))(\bar{x})$, $v_j^n\in\mathcal{V}_j$, $\lambda_j^n\geq 0$, $j=1,\ldots,m$, $\gamma_n\geq 0$ and $\zeta_n\in\partial_{\gamma_n}(\sum_{j=1}^{m}\lambda_j^n g_j(\cdot,v_j^n))(\bar{x})$ such that

$$\begin{aligned}
&\Big(0,-\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)\\
= &\Big(\nu,\nu^T\bar{x}+\epsilon-\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)+\lim_{n\to\infty}\Big[\Big(\xi_n,\xi_n^T\bar{x}+\delta_n-\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n)\Big)\\
&+\Big(0,\sum_{i=1}^{l}\mu_i^n\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)+\Big(\zeta_n,\zeta_n^T\bar{x}+\gamma_n-\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n)\Big)\Big].
\end{aligned}$$

Thus, there exist $\epsilon\geq 0$, $\nu\in\partial_\epsilon(\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i))(\bar{x})$, $u_i^n\in\mathcal{U}_i$, $\mu_i^n\geq 0$, $i=1,\ldots,l$, $\delta_n\geq 0$, $\xi_n\in\partial_{\delta_n}(\sum_{i=1}^{l}\mu_i f_i(\cdot,u_i^n))(\bar{x})$, $v_j^n\in\mathcal{V}_j$, $\lambda_j^n\geq 0$, $j=1,\ldots,m$, $\gamma_n\geq 0$ and $\zeta_n\in\partial_{\gamma_n}(\sum_{j=1}^{m}\lambda_j^n g_j(\cdot,v_j^n))(\bar{x})$ such that

$$\nu+\lim_{n\to\infty}(\xi_n+\zeta_n)$$

and

$$\epsilon+\lim_{n\to\infty}\Big[\delta_n+\gamma_n-\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n)+\sum_{i=1}^{l}\mu_i^n\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)-\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n)\Big].$$

Since $\epsilon\geq 0$, $\delta_n\geq 0$, $\gamma_n\geq 0$, $-\mu_i^n f_i(\bar{x},u_i^n)+\mu_i^n\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\geq 0$, $i=1,\ldots,l$, and $\lambda_j^n g_i(\bar{x},v_j^n)\leq 0$, $j=1,\ldots,m$, we have $\epsilon=0$, $\lim_{n\to\infty}\delta_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}[\mu_i^n f_i(\bar{x},u_i^n)-\mu_i^n\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)]=0$, $i=1,\ldots,l$, and $\lim_{n\to\infty}\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n)=0$. Notice that for each $i=1,\ldots,l$, by Lemma 2.1

in [11],

$$\partial\Big(\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\Big)(\bar{x}) = \bigcup_{\bar{u}_i\in\mathcal{U}_i}\Big\{\partial(f_i(\cdot,\bar{u}_i))(\bar{x}) \mid \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i) = f_i(\bar{x},\bar{u}_i)\Big\}. \tag{2.2}$$

Thus, there exist $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial(f_i(\cdot,\bar{u}_i))(\bar{x})$, $\mu_i^n \geq 0$, $i = 1,\ldots,l$, $\delta_n \geq 0$, $\xi_n \in \partial_{\delta_n}(\sum_{i=1}^{l}\mu_i^n f_i(\cdot,u_i^n))(\bar{x})$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1,\ldots,m$, $\gamma_n \geq 0$ and $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^{m}\lambda_j^n g_j(\cdot,v_j^n))(\bar{x})$ such that

$$\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i) = f_i(\bar{x},\bar{u}_i),\ i = 1,\ldots,l,$$

$$0 = \sum_{i=1}^{l}\nu_i + \lim_{n\to\infty}(\xi_n+\zeta_n),\ \lim_{n\to\infty}\delta_n = 0,\ \lim_{n\to\infty}\gamma_n = 0,$$

$$\lim_{n\to\infty}[\mu_i^n f_i(\bar{x},u_i^n) - \mu_i^n f_i(\bar{x},\bar{u}_i)] = 0,\ i = 1,\ldots,l \text{ and } \lim_{n\to\infty}\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n) = 0.$$

So, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Now we will prove that (iii) $\Rightarrow$ (i). Suppose that if the sequential condition holds. Then, from (2.2), for any $x \in \tilde{F}$,

$$\begin{aligned}
&\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(x,u_i) - \sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\\
\geq\ & \Big\langle \sum_{i=1}^{l}\nu_i, x-\bar{x}\Big\rangle = -\lim_{n\to\infty}\langle \xi_n+\zeta_n, x-\bar{x}\rangle\\
\geq\ & -\limsup_{n\to\infty}\Big[\sum_{i=1}^{l}\mu_i^n f_i(x,u_i^n) - \sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n) + \delta_n\\
&\qquad + \sum_{j=1}^{m}\lambda_j^n g_j(x,v_j^n) - \sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n) + \gamma_n\Big]\\
\geq\ & \liminf_{n\to\infty}\Big[\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n) - \sum_{i=1}^{l}\mu_i^n f_i(x,u_i^n)\Big]\\
& + \liminf_{n\to\infty}(-\delta_n) + \liminf_{n\to\infty}\Big(-\sum_{j=1}^{m}\lambda_j^n g_j(x,v_j^n)\Big)\\
& + \liminf_{n\to\infty}\Big(\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n)\Big) + \liminf_{n\to\infty}(-\gamma_n)\\
\geq\ & \liminf_{n\to\infty}\Big[\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n) - \sum_{i=1}^{l}\mu_i^n f_i(x,u_i^n)\Big]\\
\geq\ & \liminf_{n\to\infty}\Big[\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n) - \sum_{i=1}^{l}\mu_i^n \max_{u_i\in\mathcal{U}_i} f_i(x,u_i)\Big]
\end{aligned}$$

$$\begin{aligned}
&\geq \liminf_{n\to\infty}\left[\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n)-\sum_{i=1}^{l}\mu_i^n\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)\right]\\
&= \liminf_{n\to\infty}\left[\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},u_i^n)-\sum_{i=1}^{l}\mu_i^n f_i(\bar{x},\bar{u}_i)\right]\\
&= 0.
\end{aligned}$$

Hence, for any $x\in\tilde{F}$, $\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)\geq\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)$. So, $\bar{x}$ is a robust efficient solution of (RUMP). □

Remark 2.1. *Following the semi-infinite program approach, we can obtain the above type sequential optimality conditions with finite multipliers λ_j, but we can not fix the number of the multipliers as the constant m in the above Theorem 2.2 (see Theorem 5.2 in [9]).*

Theorem 2.3. *Let $\bar{x}\in F$. Under the assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a weakly robust efficient solution of* (RUMP)*;*

(ii) *there exist $\mu_i\geq 0$, $i=1,\ldots,l$, not all zero, such that*

$$\Big(0,-\sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)\Big)\in\operatorname{epi}\Big(\sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i}f_i(\cdot,u_i)\Big)^*+\operatorname{cl}\Big(\bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}}\operatorname{epi}\Big(\sum_{j=1}^{m}\lambda_j g_j(\cdot,v_j)\Big)^*\Big);$$

(iii) *there exist $\mu_i\geq 0$, not all zero, $\bar{u}_i\in\mathcal{U}_i$, $\nu_i\in\partial f_i(\cdot,\bar{u}_i)(\bar{x})$, $i=1,\ldots,l$, $v_j^n\in\mathcal{V}_j$, $\lambda_j^n\geq 0$, $j=1,\ldots,m$, $\gamma_n\geq 0$ and $\zeta_n\in\partial_{\gamma_n}(\sum_{j=1}^{m}\lambda_j^n g_j(\cdot,v_j^n))(\bar{x})$ such that*

$$\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)=f_i(\bar{x},\bar{u}_i),\ i=1,\ldots,l,$$
$$0=\sum_{i=1}^{l}\mu_i\nu_i+\lim_{n\to\infty}\zeta_n,\ \lim_{n\to\infty}\gamma_n=0 \text{ and } \lim_{n\to\infty}\sum_{j=1}^{m}\lambda_j^n g_j(\bar{x},v_j^n)=0.$$

Proof. Suppose that $\bar{x}\in F$ is a weakly robust efficient solution of (RUMP). Equivalently, there exist $\mu_i\geq 0$, $i=1,\ldots,l$, not all zero, such that $\bar{x}\in F$ is a solution of the following problem:

$$\begin{aligned}
&\text{minimize} && \sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)\\
&\text{subject to} && g_j(x,v_j)\leq 0,\ \forall v_j\in\mathcal{V}_j,\ j=1,\ldots,m.
\end{aligned}$$

By approaches similar to the proof of Theorem 2.2, we can obtain the desired results. □

Theorem 2.4. *Let $\bar{x}\in F$. Under the assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point* $\bar{x}$ *is a properly robust efficient solution of* (RUMP)*;*

(ii) *there exist* $\mu_i > 0$, $i = 1, \ldots, l$, *such that*

$$\Big(0, -\sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) \in \operatorname{epi}\Big(\sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\Big)^* + \operatorname{cl}\Big(\bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}\Big(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j)\Big)^*\Big);$$

(iii) *there exist* $\mu_i > 0$, $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $i = 1, \ldots, l$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$, $\gamma_n \geq 0$ *and* $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ *such that*

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l,$$

$$0 = \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n \to \infty} \zeta_n, \ \lim_{n \to \infty} \gamma_n = 0 \text{ and } \lim_{n \to \infty} \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n) = 0.$$

Proof. Suppose that $\bar{x} \in F$ is a properly robust efficient solution of (RUMP). Equivalently, there exist $\mu_i > 0$, $i = 1, \ldots, l$, not all zero, such that $\bar{x} \in F$ is a solution of the following problem:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \\ \text{subject to} & g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \ldots, m. \end{array}$$

By approaches similar to the proof of Theorem 2.2, we can obtain the desired results. □

Now, we give an example illustrating Theorem 2.2 and Theorem 2.4:

Example 2.1. Let $\mathcal{U}_1 = [-2, -1]$, $\mathcal{U}_2 = [0, 1]$, $\mathcal{V}_1 = [0, 1]$, $f_1(x, u_1) = u_1 x$ and $f_2(x, u_2) = u_2 x^2$. For any $v_1 \in \mathcal{V}_1$, we define $g_1(x, v_1) = 0$ if $x \geq 0$ and $g_1(x, v_1) = v_1 x^2$ if $x < 0$. Then for each $v_1 \in \mathcal{V}_1$, $g_1(\cdot, v_1)$ is convex and for each $x \in \mathbb{R}$, $g_1(x, \cdot)$ is affine. Moreover, for each $\lambda_1 \geq 0$ and $v_1 \in \mathcal{V}_1$,

$$(\lambda_1 g_1(\cdot, v_1))^* = \begin{cases} \frac{a^2}{4\lambda_1 v_1}, & \text{if } a \leq 0, \\ +\infty, & \text{elsewhere.} \end{cases}$$

So, we see that

$$\bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot, v_1))^* = \{(a, r) \in \mathbb{R}^2 \mid a < 0, \ r > 0\} \bigcup \{0\} \times [0, +\infty)$$

is not closed.

Consider the following multiobjective optimization problem with uncertainty (UMP):

$$\begin{array}{lll} \text{(UMP)} & \min & (f_1(x, u_1), f_2(x, u_2)) \\ & \text{s.t.} & g_1(x, v_1) \leq 0, \end{array}$$

where $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$ and $v_1 \in \mathcal{V}_1$. Its robust counterpart (RUMP) of (UMP):

$$\begin{aligned} \text{(RUMP)} \quad \min \quad & \Big(\max_{u_1 \in \mathcal{U}_1} f_1(x,u_1), \max_{u_2 \in \mathcal{U}_2} f_2(x,u_2) \Big) \\ \text{s.t.} \quad & g_1(x,v_1) \leq 0, \ \forall v_1 \in \mathcal{V}_1. \end{aligned}$$

Then the robust feasible set of (RUMP) is $[0,+\infty)$. So, we can easily see that the set of the robust efficient solutions and the set of the weakly robust efficient solutions of (RUMP) are the same set $[0,+\infty)$, and the set of the properly robust efficient solutions of (RUMP) is $(0,+\infty)$.

Let $\bar{x} = 0$. Then $\bar{x}$ is a robust efficient solution for (RUMP). Let $\bar{u}_1 = -1$ and $\bar{u}_2 = 1$. Then $\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i) = 0$. Now, we will check $\operatorname{epi}(\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(\cdot,u_i))^*$. Since for all $x \in \mathbb{R}$, $\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(x,u_i) = \max_{u_1 \in \mathcal{U}_1} u_1 x + \max_{u_2 \in \mathcal{U}_2} u_2 x^2$,

$$\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(x,u_i) = \begin{cases} -x + x^2, & \text{if } x \geq 0, \\ -2x + x^2, & \text{if } x < 0. \end{cases}$$

So, we see that for each $u_i \in \mathcal{U}_i$, $i = 1,2$,

$$\Big(\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(\cdot,u_i) \Big)^*(a) = \begin{cases} \frac{(a+1)^2}{4}, & a < -1, \\ 0, & -2 \leq a \leq -1, \\ \frac{(a+2)^2}{4}, & a > -2 \end{cases}$$

and

$$\begin{aligned} & \operatorname{epi}\Big(\sum_{i=1}^2 \max_{u_i \in \mathcal{U}_i} f_i(\cdot,u_i) \Big)^* = \Big\{ (a,r) \mid a < -2, \ \frac{(a+2)^2}{4} \leq r \Big\} \\ & \bigcup \{ (a,r) \mid -2 \leq a \leq -1, \ 0 \leq r \} \bigcup \Big\{ (a,r) \mid a > -1, \frac{(a+1)^2}{4} \leq r \Big\}. \end{aligned} \tag{2.3}$$

Now we will calculate $\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} (\operatorname{epi}(\sum_{i=1}^2 \mu_i f_i(\cdot,u_i))^* + (0, \ \sum_{i=1}^2 \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i)))$. Since $\max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i) = 0$, $i = 1,2$, we see that $(0, \ \sum_{i=1}^2 \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i)) = (0,0)$ and

$$\begin{aligned} & \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^2 \mu_i f_i(\cdot,u_i) \Big)^* + \Big(0, \ \sum_{i=1}^2 \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x},u_i) \Big) \Big) \\ & = \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \Big(\operatorname{epi}\Big(\sum_{i=1}^2 \mu_i f_i(\cdot,u_i) \Big)^* \Big). \end{aligned}$$

Moreover, we see that

$$\Big(\sum_{i=1}^2 \mu_i f_i(\cdot,u_i) \Big)^* = \begin{cases} \text{if } \mu_2 \neq 0 \text{ and } u_2 \neq 0, & \frac{(a-\mu_1 u_1)^2}{4\mu_2 u_2}, \\ \text{if } \mu_2 = 0 \text{ or } u_2 = 0, & \begin{cases} 0, & a = \mu_1 u_1, \\ +\infty, & a \neq \mu_1 u_1. \end{cases} \end{cases}$$

and

$$\begin{aligned}
&\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \mathrm{epi}\Big(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i)\Big)^*\\
= &\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \Bigg(\Big\{(a,r)\ \Big|\ \frac{(a-\mu_1u_1)^2}{4\mu_2u_2}\leq r,\ \mu_2\neq 0,\ u_2\neq 0\Big\}\\
&\qquad \bigcup\{(a,r)\mid a=\mu_1u_1,\ 0\leq r,\ \mu_2=0 \text{ or } u_2=0\}\Bigg)\\
= &\Big\{(a,r)\mid a\leq 0,\ r\geq 0\}\bigcup\{(a,r)\mid a>0,\ r>0\Big\}.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \mathrm{epi}\Big(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i))^* + \bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}} \mathrm{epi}(\lambda_1 g_1(\cdot,v_1)\Big)^*\\
&\qquad\qquad = \Big\{(a,r)\mid a\leq 0,\ r\geq 0\}\bigcup\{(a,r)\mid a>0,\ r>0\Big\},
\end{aligned}$$

which is not closed. Since $\mathrm{cl}(\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \mathrm{epi}(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i))^* + \bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}}\mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*) = \mathbb{R}\times\mathbb{R}_+$, from (2.3), we see that

$$\begin{aligned}
(0,0) &= (0-\sum_{i=1}^{2}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i))\\
&\in \mathrm{epi}\Big(\sum_{i=1}^{2}\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\Big)^*\\
&\qquad +\mathrm{cl}\Big[\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}}\Big(\mathrm{epi}\Big(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i)\Big)^* + \Big(0,\ \sum_{i=1}^{2}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\Big)\Big)\\
&\quad + \bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}}\mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*\Big]\\
&= \mathbb{R}\times\mathbb{R}_+.
\end{aligned}$$

Now, we calculate $\partial f_i(\cdot,\bar{u}_i)(\bar{x})$, $i = 1,2$, $\partial_{\delta_n}(\sum_{i=1}^{2}\mu_i^n f_i(\cdot,u_i^n))(\bar{x})$ and $\partial_{\gamma_n}(\lambda_1^n g_1(\cdot,v_1^n))(\bar{x})$. We can easily calculate that $\partial f_1(\cdot,\bar{u}_1)(\bar{x}) = \{-1\}$ and $\partial f_2(\cdot,\bar{u}_2)(\bar{x}) = \{0\}$. Moreover, we can easily check that

$$\begin{aligned}
&\partial_{\delta_n}\Big(\sum_{i=1}^{2}\mu_i^n f_i(\cdot,u_i^n)\Big)(\hat{x})\\
&\qquad = \{\xi\in\mathbb{R}\mid \mu_1^nu_1^n-2\sqrt{\delta_n\mu_2^nu_2^n}\leq\xi\leq\mu_1^nu_1^n+2\sqrt{\delta_n\mu_2^nu_2^n}\}
\end{aligned}$$

and

$$\partial_{\gamma_n}(\lambda_1^n g_1(\cdot, v_1^n))(\bar{x}) = \{\zeta \in \mathbb{R} \mid -2\sqrt{\lambda_1^n v_1^n \gamma_n} \leq \zeta \leq 0\}.$$

Let $\bar{\mu}_1^n = \frac{1}{n}$, $\bar{\mu}_2^n = n$, $\bar{u}_1^n = -1 - \frac{1}{n}$, $\bar{u}_2^n = 1 - \frac{1}{n}$, $\bar{\delta}_n = \frac{1}{4n}$, $\bar{v}_1^n = \frac{1}{n}$, $\bar{\lambda}_1^n = n$ and $\bar{\gamma}_n = \frac{1}{4n^2}$. Then we have

$$\partial_{\bar{\delta}_n}\bigg(\sum_{i=1}^{2} \bar{\mu}_i^n f_i(\cdot, \bar{u}_i^n)\bigg)(\bar{x})$$
$$= \bigg\{\xi \in \mathbb{R} \mid -\tfrac{1}{n} - \tfrac{1}{n^2} - \sqrt{1 - \tfrac{1}{n}} \leq \xi \leq -\tfrac{1}{n} - \tfrac{1}{n^2} + \sqrt{1 - \tfrac{1}{n}}\bigg\}$$

and

$$\partial_{\bar{\gamma}_n}(\bar{\lambda}_1^n g_1(\cdot, \bar{v}_1^n))(\bar{x}) = \{\zeta \in \mathbb{R} \mid -\tfrac{1}{n} \leq \zeta \leq 0\}.$$

Let $\nu_1 = -1$, $\nu_2 = 0$, $\xi_n = -\frac{1}{n} - \frac{1}{n^2} + \sqrt{1 - \frac{1}{n}}$ and $\zeta_n = -\frac{1}{n}$. Then, we have

$$\begin{aligned}
&\max_{u_1 \in \mathcal{U}_1} f_1(\bar{x}, u_1) = 0 = \bar{u}_1 \bar{x} = f_1(\bar{x}, \bar{u}_1),\\
&\max_{u_2 \in \mathcal{U}_2} f_2(\bar{x}, u_2) = 0 = \bar{u}_2 \bar{x}^2 = f_2(\bar{x}, \bar{u}_2),\\
&\nu_1 + \nu_2 + \lim_{n\to\infty}(\xi_n + \zeta_n) = 0, \ \lim_{n\to\infty} \delta_n = 0, \ \lim_{n\to\infty} \gamma_n = 0,\\
&\lim_{n\to\infty}[\mu_1^n f_1(\bar{x}, u_1^n) - \mu_1^n f_1(\bar{x}, \bar{u}_1)] = 0,\\
&\lim_{n\to\infty}[\mu_2^n f_2(\bar{x}, u_2^n) - \mu_2^n f_2(\bar{x}, \bar{u}_2)] = 0 \text{ and } \lim_{n\to\infty} \lambda_1^n g_1(\bar{x}, v_1^n) = 0.
\end{aligned}$$

So, Theorem 2.2 holds.

Let $\hat{x} = 1$. Then $\hat{x}$ is a properly robust efficient solution for (RUMP). Let $\hat{u}_1 = -1$, $\hat{u}_2 = 1$, $\hat{\mu}_1 = 2$ and $\hat{\mu}_2 = 1$. Then $\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) = -1$. Now, we will check $\text{epi}(\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))^*$. Since for all $x \in \mathbb{R}$, $\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = 2\max_{u_1 \in \mathcal{U}_1} u_1 x + \max_{u_2 \in \mathcal{U}_2} u_2 x^2$,

$$\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = \begin{cases} -2x + x^2, & \text{if } x \geq 0,\\ -4x + x^2, & \text{if } x < 0.\end{cases}$$

So, we see that for each $u_i \in \mathcal{U}_i$, $i = 1, 2$,

$$\bigg(\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\bigg)^*(a) = \begin{cases} \frac{(a+2)^2}{4}, & a > -2,\\ 0, & -4 \leq a \leq -2,\\ \frac{(a+4)^2}{4}, & a < -4.\end{cases}$$

and

$$\text{epi}\bigg(\sum_{i=1}^{2} \hat{\mu}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i)\bigg)^* = \bigg\{(a, r) \mid a < -4, \frac{(a+4)^2}{4} \leq r\bigg\}$$
$$\bigcup\{(a, r) \mid -4 \leq a \leq -2, \ 0 \leq r\}\bigcup\bigg\{(a, r) \mid a > -2, \frac{(a+2)^2}{4} \leq r\bigg\}.$$

Since $\mathrm{cl}(\bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}} \mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*) = -\mathbb{R}_+\times\mathbb{R}_+$, we have

$$\begin{aligned}(0,1) &= \left(0,-\sum_{i=1}^{2}\hat{\mu}_i \max_{u_i\in\mathcal{U}_i} f_i(\hat{x},u_i)\right)\\ &\in \mathrm{epi}\left(\sum_{i=1}^{2}\hat{\mu}_i \max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\right)^* + \mathrm{cl}\left(\bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}} \mathrm{epi}(\lambda_1 g_1(\cdot,v_1))^*\right)\\ &= \{(a,r)\mid a\leq -2,\ 0\leq r\}\bigcup\left\{(a,r)\ \middle|\ a>-2, \frac{(a+2)^2}{4}\leq r\right\}.\end{aligned}$$

Now, we calculate $\partial f_i(\cdot,\hat{u}_i)(\hat{x})$, $i=1,2$ and $\partial_{\gamma_n}(\lambda_1^n g_1(\cdot,v_1^n))(\hat{x})$. We can easily calculate that $\partial f_1(\cdot,\hat{u}_1)(\hat{x})=\{-1\}$ and $\partial f_2(\cdot,\hat{u}_2)(\hat{x})=\{2\}$. Moreover, we can easily check that $\partial_{\gamma_n}(\lambda_1^n g_1(\cdot,v_1^n))(\hat{x})=\{0\}$. Let $\hat{v}_1^n=\frac{1}{n}$, $\hat{\lambda}_1^n=n$ and $\hat{\gamma}_n=\frac{1}{n}$, $\hat{\nu}_1=-1$, $\hat{\nu}_2=2$ and $\zeta_n=0$. Then, we have

$$\begin{aligned}&\max_{u_1\in\mathcal{U}_1} f_1(\hat{x},u_1)=\max_{u_1\in\mathcal{U}_1} u_1=-1=\hat{u}_1=\hat{u}_1\hat{x}=f_1(\hat{x},\hat{u}_1),\\ &\max_{u_2\in\mathcal{U}_2} f_2(\hat{x},u_2)=\max_{u_2\in\mathcal{U}_2} u_2=1=\hat{u}_2=\hat{u}_2\hat{x}^2=f_2(\hat{x},\hat{u}_2),\\ &\hat{\mu}_1\nu_1+\hat{\mu}_2\nu_2+\lim_{n\to\infty}\zeta_n=0,\ \lim\gamma_n=0 \text{ and } \lim_{n\to\infty}\lambda_1^n g_1(\hat{x},v_1^n)=0.\end{aligned}$$

So, Theorem 2.4 holds.

The following proposition, which is a generalization of the Isermann's result([7, 14]), gives a sufficient condition that a robust efficient solution of (RUMP) can be a properly robust efficient solution of (RUMP).

Proposition 2.2. *Let $\bar{x}\in F$. Suppose that the assumptions of Theorem 2.2 hold. Assume that*

$$\begin{aligned}&\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}}\left(\mathrm{epi}\left(\sum_{i=1}^{l}\mu_i f_i(\cdot,u_i)\right)^* + \left(0,\sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\right)\right)\\ &+\bigcup_{\substack{v_j\in\mathcal{V}_j\\ \lambda_j\geq 0}}\mathrm{epi}\left(\sum_{j=1}^{m}\lambda_j g_j(\cdot,v_j)\right)^*\end{aligned}$$

is closed. Then if $\bar{x}$ is a robust efficient solution of (RUMP)*, then $\bar{x}$ is a properly robust efficient solution of* (RUMP).

Proof. Let $\bar{x}$ be a robust efficient solution of (RUMP). Then, by Theorem 2.2,

$$\begin{aligned}&\left(0,-\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\right)\in\mathrm{epi}\left(\sum_{i=1}^{l}\max_{u_i\in\mathcal{U}_i} f_i(\cdot,u_i)\right)^*\\ &+\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}}\left(\mathrm{epi}\left(\sum_{i=1}^{l}\mu_i f_i(\cdot,u_i)\right)^* + \left(0,\ \sum_{i=1}^{l}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\right)\right)\end{aligned}$$

$$+ \bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}\Big(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j)\Big)^*.$$

By Proposition 2.1, there exist $\bar{u}_i \in \mathcal{U}_i$, $\mu_i \geq 0$, $i = 1, \ldots, l$, $\bar{v}_j \in \mathcal{V}_j$, $\lambda_j \geq 0$, $j = 1, \ldots, m$, $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, $\epsilon_3 \geq 0$, $\bar{\nu} \in \partial_{\epsilon_1}(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))(\bar{x})$, $\bar{\xi} \in \partial_{\epsilon_2}(\sum_{i=1}^{l} \mu_i f_i(\cdot, \bar{u}_i))(\bar{x})$ and $\bar{\zeta} \in \partial_{\epsilon_3}(\sum_{j=1}^{m} \lambda_j g_j(\cdot, \bar{v}_j))(\bar{x})$ such that

$$\begin{aligned}
&\Big(0, -\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) \\
= \;& \Big(\bar{\nu}, \bar{\nu}^T \bar{x} + \epsilon_1 - \sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) + \Big(\bar{\xi}, \bar{\xi}^T \bar{x} + \epsilon_2 - \sum_{i=1}^{l} \mu_i f_i(\bar{x}, \bar{u}_i)\Big) \\
& + \Big(0, \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\Big) + \Big(\bar{\zeta}, \bar{\zeta}^T \bar{x} + \epsilon_3 - \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j)\Big).
\end{aligned}$$

Since $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, $\epsilon_3 \geq 0$, $\mu_i(f_i(\bar{x}, u_i) - f_i(\bar{x}, \bar{u}_i)) \geq 0$, $i = 1, \ldots, l$, and $-\sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) \geq 0$, we have $\epsilon_1 = 0$, $\epsilon_2 = 0$, $\epsilon_3 = 0$, $\mu_i(f_i(\bar{x}, u_i) - f_i(\bar{x}, \bar{u}_i)) = 0$, $i = 1, \ldots, l$, and $-\sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) = 0$.

Notice that if $\mu_i \neq 0$, $\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i)$. So, there exist $\bar{u}_i \in \mathcal{U}_i$, $\mu_i \geq 0$, $i = 1, \ldots, l$, $\bar{v}_j \in \mathcal{V}_j$, $\lambda_j \geq 0$, $j = 1, \ldots, m$, $\bar{\nu} \in \partial(\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))(\bar{x})$, $\bar{\xi} \in \partial(\sum_{i=1}^{l} \mu_i f_i(\cdot, \bar{u}_i))(\bar{x})$ and $\bar{\zeta} \in \partial(\sum_{j=1}^{m} \lambda_j g_j(\cdot, \bar{v}_j))(\bar{x})$ such that

$$\begin{aligned}
&\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i) \text{ (if } \mu_i \neq 0), \\
&0 = \bar{\nu} + \bar{\xi} + \bar{\zeta}, \text{ and } \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) = 0.
\end{aligned}$$

Notice that for each $i = 1, \ldots, l$, by Lemma 2.1 in [11],

$$\partial(\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))(\bar{x}) = \bigcup_{\bar{u}_i \in \mathcal{U}_i} \{\partial(f_i(\cdot, \bar{u}_i))(\bar{x}) \mid \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i)\}.$$

So, there exist $\mu_i \geq 0$, $i = 1, \ldots, l$, $\bar{v}_j \in \mathcal{V}_j$, $\lambda_j \geq 0$, $j = 1, \ldots, m$, $\bar{\nu} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^n$ and $\bar{\zeta} \in \partial(\sum_{j=1}^{m} \lambda_j g_j(\cdot, \bar{v}_j))(\bar{x})$ such that $\bar{\nu} + \bar{\xi} \in \partial(\sum_{i=1}^{l}(1 + \mu_i) \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i))(\bar{x})$, $\bar{\nu} + \bar{\xi} + \bar{\zeta} = 0$ and $\sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) = 0$. Thus, for any $x \in F$,

$$\begin{aligned}
0 &\geq \sum_{j=1}^{m} \lambda_j g_j(x, \bar{v}_j) - \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) \\
&\geq \bar{\zeta}^T (x - \bar{x}) \\
&= -(\bar{\nu} - \bar{\xi})^T (x - \bar{x}) \\
&\geq \sum_{i=1}^{l} (1 + \mu_i) \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \sum_{i=1}^{l} (1 + \mu_i) \max_{u_i \in \mathcal{U}_i} f_i(x, u_i).
\end{aligned}$$

So, for any $x \in F$, $\sum_{i=1}^{l}(1+\mu_i)\max_{u_i\in\mathcal{U}_i} f_i(x,u_i) \geq \sum_{i=1}^{l}(1+\mu_i)\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)$. Hence $\bar{x}$ is a properly robust efficient solution of (RUMP). □

The following example illustrates that Proposition 2.2 holds.

Example 2.2. Let $\mathcal{U}_1 = [-2,-1]$, $\mathcal{U}_2 = [0,1]$, $\mathcal{V}_1 = [0,1]$, $f_1(x,u_1) = u_1 x$ and $f_2(x,u_2) = u_2 x^2$. For any $v_1 \in \mathcal{V}_1$, we define $g_1(x,v_1) = 0$ if $x \geq 0$ and $g_1(x,v_1) = v_1 x^2$ if $x < 0$. Then for each $v_1 \in \mathcal{V}_1$, $g_1(\cdot,v_1)$ is convex and for each $x \in \mathbb{R}$, $g_1(x,\cdot)$ is affine. Moreover we already checked that

$$\bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot,v_1))^* = \{(a,r)\in\mathbb{R}^2 \mid a<0,\ r>0\}\bigcup\{0\}\times[0,\infty)$$

is not closed in Example 2.1.

Consider the following multiobjective optimization problem with uncertainty (UMP):

$$\begin{aligned} \text{(UMP)}\quad \min\quad & (f_1(x,u_1), f_2(x,u_2))\\ \text{s.t.}\quad & g_1(x,v_1)\leq 0, \end{aligned}$$

where $u_1 \in \mathcal{U}_1$, $u_2 \in \mathcal{U}_2$ and $v_1 \in \mathcal{V}_1$. Its robust counterpart (RUMP) of (UMP):

$$\begin{aligned} \text{(RUMP)}\quad \min\quad & (\max_{u_1\in\mathcal{U}_1} f_1(x,u_1), \max_{u_2\in\mathcal{U}_2} f_2(x,u_2))\\ \text{s.t.}\quad & g_1(x,v_1)\leq 0,\ \forall v_1\in\mathcal{V}_1. \end{aligned}$$

Then the robust feasible set of (RUMP) is $[0,+\infty)$. So, we can easily see that the set of robust efficient solution of (RUMP) are $[0,+\infty)$, and the set of properly robust efficient solution of (RUMP) is $(0,+\infty)$.

Let $\hat{x} = 1$. Then $\hat{x}$ is a robust efficient solution for (RUMP). Moreover $\hat{x}$ is a properly robust efficient solution for (RUMP). Now we will calculate

$$\bigcup_{\substack{u_i\in\mathcal{U}_i\\ \mu_i\geq 0}} \left(\operatorname{epi}\left(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i)\right)^* + \left(0,\sum_{i=1}^{2}\mu_i \max_{u_i\in\mathcal{U}_i} f_i(\hat{x},u_i)\right)\right) + \bigcup_{\substack{v_1\in\mathcal{V}_1\\ \lambda_1\geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot,v_1))^*.$$

In Example 2.1, we already calculated

$$\begin{aligned} \operatorname{epi}\left(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i)\right)^* &= \left\{(a,r)\ \middle|\ \frac{(a-\mu_1u_1)^2}{4\mu_2u_2}\leq r,\ \mu_2\neq 0,\ u_2\neq 0\right\}\\ &\quad \bigcup\{(a,r)\mid a=\mu_1u_1,\ 0\leq r,\ \mu_2=0 \text{ or } u_2=0\}. \end{aligned}$$

Moreover, we can easily see that $(0,\sum_{i=1}^{2}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\hat{x},u_i)) = (0,\mu_2-\mu_1)$. So,

$$\begin{aligned} &\operatorname{epi}\left(\sum_{i=1}^{2}\mu_i f_i(\cdot,u_i)\right)^* + \left(0,\sum_{i=1}^{2}\mu_i\max_{u_i\in\mathcal{U}_i} f_i(\hat{x},u_i)\right)\\ &= \left\{(a,r)\ \middle|\ \frac{(a-\mu_1u_1)^2}{4\mu_2u_2}+\mu_2-\mu_1\leq r,\ \mu_2\neq 0,\ u_2\neq 0\right\}\\ &\quad \bigcup\{(a,r)\mid a=\mu_1u_1,\ -\mu_1\leq r,\ \mu_2=0 \text{ or } u_2=0\}. \end{aligned}$$

Since $\mu_1 \geq 0$ and $u_1 \in [-2,-1]$,

$$\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \{(a,r) \mid a = \mu_1 u_1,\ -\mu_1 \leq r,\ \mu_2 = 0 \text{ or } u_2 = 0\} = \{(a,r) \mid a \leq 0,\ a \leq r\}. \tag{2.4}$$

Now we will calculate

$$\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left\{(a,r) \;\middle|\; \frac{(a-\mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0\right\}.$$

Define $\phi(a) = \frac{(a-\mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1$, where $\mu_1 \geq 0$, $\mu_2 > 0$, $u_1 \in [-2,-1]$ and $u_2 \in (0,1]$. Let $L = \{(a,y) \in \mathbb{R}\times\mathbb{R} \mid y = \phi(a_0) + \phi'(a_0)(a-a_0)\}$. Then, by Theorem 3.2.5 in [1], the line L supports epiϕ at $(a_0, \phi(a_0))$. Moreover, since $u_1 \in [-2,-1]$ and $u_2 \in (0,1]$, we see that the $\phi(a)$-intercept is

$$\phi(0) = \frac{(\mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \geq \frac{\mu_1^2}{4\mu_2} + \mu_2 - \mu_1 = \frac{1}{4\mu_2}(\mu_1 - 2\mu_2)^2 \geq 0.$$

In particular, $\phi(0) = 0$ as $u_1 = -1$, $u_2 = 1$ and $\mu_1 = 2\mu_2$. So, we see that $(0,0)$ belongs to the line L. Hence we have

$$0 = \phi(a_0) + \phi'(0)(0-a_0) = \frac{(a_0 - \mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 - \frac{2(a_0 - \mu_1 u_1)}{4\mu_2 u_2} a_0,$$

and so, we have $a_0 = \pm\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2}$. Thus, we have

$$y = \frac{\pm\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1}{2\mu_2 u_2} a.$$

First, we consider $y = \frac{\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1}{2\mu_2 u_2} a$, i.e., $a_0 = \sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2}$. Then, we have

$$\begin{aligned} y &= \frac{\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1}{2\mu_2 u_2} a \\ &= \left(\sqrt{\left(\frac{\mu_1 u_1}{2\mu_2 u_2}\right)^2 - \frac{\mu_1}{\mu_2 u_2} + \frac{1}{u_2}} - \frac{\mu_1 u_1}{2\mu_2 u_2}\right) a. \end{aligned}$$

Let $\rho = \frac{\mu_1}{\mu_2}$. Then $y = (\sqrt{(\frac{u_1}{2u_2}\rho)^2 - \frac{1}{u_2}\rho + \frac{1}{u_2}} - \frac{u_1}{2u_2}\rho)a$. So, we see that the slope of y is $\sqrt{\frac{1}{u_2}}$ as $\rho \to 0$ and the slope of y is $+\infty$ as $\rho \to +\infty$. Since $1 \leq \sqrt{\frac{1}{u_2}} < +\infty$, the infimum of the slope of y is 1. It means that the line $L_1 = \{(a,y) \in \mathbb{R}\times\mathbb{R} \mid y = a\}$ supports epiϕ at all $(a_0, \phi(a_0))$, where $a_0 = \sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2}$. Secondly, we consider $y = -\frac{\mu_1 u_1}{2\mu_2 u_2} a$, i.e., $a_0 = 0$, that is, $u_1 = -1$, $u_2 = 1$ and $\mu_1 = 2\mu_2$. So, we have $y = a$. It means that the line $L_1 = \{(a,y) \in \mathbb{R}\times\mathbb{R} \mid y = a\}$ supports epiϕ at $(0,0)$. Finally, we consider $y = \frac{-\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1}{2\mu_2 u_2} a$, i.e., $a_0 = -\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2}$. Then, we have

$$y = \frac{-\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1}{2\mu_2 u_2} a$$

$$= \frac{2\mu_1 - 2\mu_2}{\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2} - \mu_1 u_1} a$$
$$= \frac{2 - 2\frac{\mu_2}{\mu_1}}{\sqrt{(u_1)^2 - 4\left(\frac{\mu_2}{\mu_1}\right)u_2 + 4\left(\frac{\mu_2}{\mu_1}\right)^2 u_2} - u_1} a.$$

So, we can easily see that the slope of y is 0 as $\mu_1 = \mu_2$ and the slope of y is small than 0 as $\mu_1 < \mu_2$. Now, we consider the slope of y as $\mu_1 > \mu_2$. Let $\sigma = \frac{u_2}{u_1}$. Then $0 < \sigma < 1$ and $y = \frac{2-2\sigma}{\sqrt{u_1^2 - 4\sigma u_2 + 4\sigma^2 u_2} - u_1} a$. So, we see that the slope of y is 0 as $\sigma \to 1$ and the slope of y is $-\frac{1}{u_1}$ as $\sigma \to 0$. Since $\frac{1}{2} \leq -\frac{1}{u_1} \leq 1$, the supremum of the slope of y is 1. It means that the line $L_1 = \{(a, y) \in \mathbb{R} \times \mathbb{R} \mid y = a\}$ supports epiϕ at all $(a_0, \phi(a_0))$, where $a_0 = -\sqrt{(\mu_1 u_1)^2 - 4\mu_1\mu_2 u_2 + 4\mu_2^2 u_2}$. Hence, we see that

$$(2.5) \quad \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left\{ (a, r) \;\middle|\; \frac{(a - \mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0 \right\} \subset \{(a, r) \mid a \leq r\}.$$

Let $(a_0, y_0) = (2\mu_2 - \mu_1, 2\mu_2 - \mu_1) \in L_1$ be any fixed. Let $u_1 = -1$ and $u_2 = 1$. Then $\phi(a_0) = a_0$. So,

$$\begin{aligned}(a_0, y_0) &\in \left\{ (a, r) \;\middle|\; \frac{(a + \mu_1)^2}{4\mu_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0 \right\} \\ &\subset \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left\{ (a, r) \;\middle|\; \frac{(a - \mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0 \right\}.\end{aligned}$$

Hence, we see that

$$(2.6) \quad \{(a, r) \mid a \leq r\} \subset \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \{(a, r) \mid \tfrac{(a - \mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0\}.$$

So, from (2.5) and (2.6),

$$(2.7) \quad \bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left\{ (a, r) \;\middle|\; \frac{(a - \mu_1 u_1)^2}{4\mu_2 u_2} + \mu_2 - \mu_1 \leq r,\ \mu_2 \neq 0,\ u_2 \neq 0 \right\} = \{(a, r) \mid a \leq r\}.$$

Thus, from (2.4) and (2.7),

$$\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left(\text{epi}\left(\sum_{i=1}^{2} \mu_i f_i(\cdot, u_i) \right)^* + \left(0, \sum_{i=1}^{2} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right) \right) = \{(a, r) \mid a \leq r\}.$$

Consequently, we see that

$$\bigcup_{\substack{u_i \in \mathcal{U}_i \\ \mu_i \geq 0}} \left(\text{epi}\left(\sum_{i=1}^{2} \mu_i f_i(\cdot, u_i) \right)^* + \left(0, \sum_{i=1}^{2} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \right) \right)$$

$$+ \bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot, v_1))^* = \{(a, r) \mid a \leq r\}$$

is closed, and so, Proposition 2.2 holds.

Now we consider the closed cone constraint qualification which requires that the cone,

$$\bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}\Bigg(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j)\Bigg)^*,$$

is closed.

Remark 2.2. *Let $g_j : \mathbb{R}^n \times \mathbb{R}^{q_2} \to \mathbb{R}$, $j = 1, \ldots, m$, be continuous functions such that for all $v_j \in \mathbb{R}^{q_2}$, $g_j(\cdot, v_j)$ is a convex function. Suppose that each $\mathcal{V}_j$, $j = 1, \ldots, m$, is compact and convex, and the Slater type condition holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_j(x_0, v_j) < 0$, $\forall v_j \in \mathcal{V}_j$, $j = 1, \ldots, m$. Then $\bigcup_{\substack{v_j \in \mathcal{V}_j \\ \lambda_j \geq 0}} \operatorname{epi}(\sum_{j=1}^{m} \lambda_j g_j(\cdot, v_j))^*$ is closed* ([12]).

Following the proof in [13], we can obtain the following Kuhn-Tucker theorems for (RUMP) under the closed cone constraint qualification:

Theorem 2.5. *Let $\bar{x} \in F$. Assume that the closed cone constraint qualification holds. Under assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a weakly robust efficient solution of* (RUMP)*;*
(ii) *there exist $\mu_i \geq 0$, not all zero, $\bar{u}_i \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\lambda_j \geq 0$ and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$, such that*

$$0 \in \sum_{i=1}^{l} \mu_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial g_j(\cdot, \bar{v}_j)(\bar{x}),$$

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l, \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) = 0.$$

Theorem 2.6. *Let $\bar{x} \in F$. Assume that the closed cone constraint qualification holds. Under assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a properly robust efficient solution of* (RUMP)*;*
(ii) *there exist $\mu_i > 0$, $\bar{u}_i \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\lambda_j \geq 0$ and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$, such that*

$$0 \in \sum_{i=1}^{l} \mu_i \partial f_i(\cdot, \bar{u}_i)(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial g_j(\cdot, \bar{v}_j)(\bar{x}),$$

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l, \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j g_j(\bar{x}, \bar{v}_j) = 0.$$

The following example illustrates that Theorem 2.5 and Theorem 2.6 holds.

Example 2.3. Consider the following multiobjective optimization problem problem with uncertainty (UMP):

$$\begin{array}{lll}\text{(UMP)} & \min & (u_1x_1, u_2x_2) \\ & \text{s.t.} & v_1x_1 + v_2x_2 - 1 \leq 0,\end{array}$$

where $u_1 \in \mathcal{U}_1 = [0,1]$, $u_2 \in \mathcal{U}_2 = [0,1]$ and $(v_1, v_2) \in \mathcal{V} := \{(v_1, v_2) \in \mathbb{R}_2 \mid v_1^2 + v_2^2 \leq 1\}$. Its robust counterpart (RUMP) of (UMP):

$$\begin{array}{lll}\text{(RUMP)} & \min & \left(\max\limits_{u_1\in[0,1]} u_1x_1,\ \max\limits_{u_2\in[0,1]} u_2x_2\right) \\ & \text{s.t.} & v_1x_1 + v_2x_2 - 1 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V}.\end{array}$$

Then the robust feasible set of (RUMP) is $\{(x_1, x_2) \mid \sqrt{x_1^2 + x_2^2} \leq 1\}$. We can easily see that the set of robust efficient solutions (RUMP) is

$$\left\{(x_1, x_2) \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq 1,\ x_1 \leq 0,\ x_2 \leq 0\right\},$$

the set of robust weakly efficient solutions of (RUMP) is

$$\left\{(x_1, x_2) \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq 1,\ x_1 \leq 0\right\} \bigcup \left\{(x_1, x_2) \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq 1,\ x_2 \leq 0\right\}$$

and the set of properly robust efficient solutions of (RUMP) is

$$\left\{(x_1, x_2) \;\middle|\; \sqrt{x_1^2 + x_2^2} \leq 1,\ x_1 \leq 0,\ x_2 \leq 0\right\}.$$

Let $f_1(x, u_1) = u_1x_1$, $f_2(x, u_2) = u_2x_2$ and $g_1(x, v_1, v_2) = v_1x_1 + v_2x_2 - 1$. Then we can easily find points which hold the Slater condition. Since for each $(v_1, v_2) \in \mathcal{V}$, $g_1(\cdot, v_1, v_2)$ is convex and $\mathcal{V}$ is compact, $\bigcup_{v\in\mathcal{V},\lambda_1\geq 0} \operatorname{epi}(\lambda_1 g_1(\cdot, v))^*$ is closed. In fact, for each $v \in \mathcal{V}$ and each $\lambda_1 \geq 0$,

$$\begin{aligned}(\lambda_1 g_1(\cdot, v))^*(a_1, a_2) &= \sup_{(x_1,x_2)\in\mathbb{R}^2} \{a_1x_1 + a_2x_2 - \lambda_1v_1x_1 - \lambda_1v_2x_2 + \lambda_1\} \\ &= \begin{cases} \lambda_1, & \text{if } a_1 = \lambda_1v_1,\ a_2 = \lambda_1v_2, \\ +\infty, & \text{elsewhere.}\end{cases}\end{aligned}$$

So, we see that

$$\begin{aligned}\bigcup_{\substack{v\in\mathcal{V}\\ \lambda_1\geq 0}} \operatorname{epi}(\lambda_1 g_1(\cdot, v))^* &= \bigcup_{\substack{v\in\mathcal{V}\\ \lambda_1> 0}} \operatorname{epi}(\lambda_1 g_1(\cdot, v))^* \bigcup (\{0\}\times\{0\}\times[0,\infty)) \\ &= \bigcup_{\substack{(v_1,v_2)\in\mathcal{V}\\ \lambda_1> 0}} \{(a_1, a_2, r) \mid a_1 = \lambda_1v_1,\ a_2 = \lambda_1v_2,\ \lambda_1 \leq r\} \\ &\qquad \bigcup (\{0\}\times\{0\}\times[0,\infty)) \\ &= \left\{(a_1, a_2, r) \;\middle|\; \sqrt{a_1^2 + a_2^2} \leq r\right\}\end{aligned}$$

is closed.

Let $(\hat{x}_1, \hat{x}_2) = (1, 0)$. Then $(\hat{x}_1, \hat{x}_2)$ is a weakly efficient solution for (RUMP). Let $\hat{u}_1 = 1$, $\hat{u}_2 = 0$, $\hat{\mu}_1 = 0$, $\hat{\mu}_2 = 1$, $(\hat{v}_1, \hat{v}_2) = (1, 0)$, $\hat{\lambda}_1 = 0$. Then we have

$$\max_{u_1 \in \mathcal{U}_1} f_1(\hat{x}, u_1) = \max_{u_1 \in \mathcal{U}_1} u_1 \hat{x}_1 = 1 = \hat{u}_1 = \hat{u}_1 \hat{x}_1 = f_1(\hat{x}_1, \hat{u}_1) \text{ and}$$
$$\max_{u_2 \in \mathcal{U}_2} f_2(\hat{x}, u_2) = \max_{u_2 \in \mathcal{U}_2} u_2 \hat{x}_2 = 0 = \hat{u}_2 = \hat{u}_2 \hat{x}_2 = f_2(\hat{x}_2, \hat{u}_2).$$

Moreover, we can easily calculate $\partial f_1(\cdot, \hat{u}_1)(\hat{x}) = \{(1, 0)\}$, $\partial f_2(\cdot, \hat{u}_2)(\hat{x}) = \{(0, 0)\}$ and $\partial g_1(\cdot, \hat{v})(\hat{x}) = \{(1, 0)\}$. So, we have

$$(0, 0) \in \hat{\mu}_1 \partial f_1(\cdot, \hat{u}_1)(\hat{x}) + \hat{\mu}_2 \partial f_2(\cdot, \hat{u}_2)(\hat{x}) + \hat{\lambda}_1 \partial g_1(\cdot, \hat{v})(\hat{x}) \text{ and } \hat{\lambda}_1 g_1(\hat{x}, \hat{v}) = 0.$$

So, Theorem 2.5 holds.

Let $(\tilde{x}_1, \tilde{x}_2) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then $(\tilde{x}_1, \tilde{x}_2)$ is a properly robust efficient solution for (RUMP). Let $\tilde{u}_1 = 0$, $\tilde{u}_2 = 0$, $\tilde{\mu}_1 = 1$, $\tilde{\mu}_2 = 1$, $(\tilde{v}_1, \tilde{v}_2) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $\tilde{\lambda}_1 = 0$. Then we have

$$\max_{u_1 \in \mathcal{U}_1} f_1(\tilde{x}, u_1) = \max_{u_1 \in \mathcal{U}_1} u_1 \tilde{x}_1 = 0 = \tilde{u}_1 = \tilde{u}_1 \tilde{x}_1 = f_1(\tilde{x}_1, \tilde{u}_1) \text{ and}$$
$$\max_{u_2 \in \mathcal{U}_2} f_2(\tilde{x}, u_2) = \max_{u_2 \in \mathcal{U}_2} u_2 \tilde{x}_2 = 0 = \tilde{u}_2 = \tilde{u}_2 \tilde{x}_2 = f_2(\tilde{x}_2, \tilde{u}_2).$$

Moreover, we can easily see that $\partial f_1(\cdot, \tilde{u}_1)(\tilde{x}) = \{(0, 0)\}$, $\partial f_2(\cdot, \tilde{u}_2)(\tilde{x}) = \{(0, 0)\}$ and $\partial g_1(\cdot, \tilde{v})(\tilde{x}) = \{(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. So, we have

$$(0, 0) \in \tilde{\mu}_1 \partial f_1(\cdot, \tilde{u}_1)(\tilde{x}) + \tilde{\mu}_2 \partial f_2(\cdot, \tilde{u}_2)(\tilde{x}) + \tilde{\lambda}_1 \partial g_1(\cdot, \tilde{v})(\tilde{x}) \text{ and } \tilde{\lambda}_1 g_1(\tilde{x}, \tilde{v}) = 0.$$

So, Theorem 2.6 holds.

3. Sequential Optimality Conditions II

By using Bronsted-Rockafellar theorem (Proposition 1.1), we can obtain the following sequential optimality theorems for (weakly, properly) robust efficient solutions of (RUMP):

Theorem 3.1. *Let $\bar{x} \in F$. Under assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a robust efficient solution of* (RUMP)*;*

(ii) *there exist $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $x_n \in \mathbb{R}^n$, $\mu_i^n \geq 0$, $u_i^n \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\tilde{\xi}_n \in \partial(\sum_{i=1}^{l} \mu_i^n f_i(\cdot, u_i^n))(x_n)$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$ and $\tilde{\zeta}_n \in \partial(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(x_n)$ such that*

$$\lim_{n \to \infty} x_n = \bar{x}, \ \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l,$$

$$0 = \sum_{i=1}^{l} \nu_i + \lim_{n \to \infty} (\tilde{\xi}_n + \tilde{\zeta}_n) \text{ and}$$

$$\lim_{n \to \infty} \left[\sum_{i=1}^{l} \mu_i^n f_i(x_n, u_i^n) - \sum_{i=1}^{l} \mu_i^n f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) \right] = 0.$$

Proof. Suppose that (i) holds. Then, from Theorem 2.2, there exist $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $\mu_i^n \geq 0$, $u_i^n \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\delta_n \geq 0$, $\xi_n \in \partial_{\delta_n}(\sum_{i=1}^l \mu_i^n f_i(\cdot, u_i^n))(\bar{x})$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$, $\gamma_n \geq 0$ and $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^m \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l,$$

$$0 = \sum_{i=1}^l \nu_i + \lim_{n \to \infty} (\xi_n + \zeta_n), \ \lim_{n \to \infty} \delta_n = 0, \ \lim_{n \to \infty} \gamma_n = 0,$$

$$\lim_{n \to \infty} [\mu_i^n f_i(\bar{x}, u_i^n) - \mu_i^n f_i(\bar{x}, \bar{u}_i)] = 0, \ i = 1, \ldots, l, \text{ and } \lim_{n \to \infty} \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n) = 0.$$

Notice that

$$\begin{aligned} \xi_n + \zeta_n &\in \partial_{\delta_n}\left(\sum_{i=1}^l \mu_i^n f_i(\cdot, u_i^n)\right)(\bar{x}) + \partial_{\gamma_n}\left(\sum_{j=1}^m \lambda_j^n g_j(\cdot, v_j^n)\right)(\bar{x}) \\ &\subset \partial_{\delta_n + \gamma_n}\left(\sum_{i=1}^l \mu_i^n f_i(\cdot, u_i^n) + \sum_{j=1}^m \lambda_j^n g_j(\cdot, v_j^n)\right)(\bar{x}). \end{aligned}$$

So, by Proposition 1.1, there exist $x_n \in \mathbb{R}^n$, $\tilde{\xi}_n \in \partial_{\delta_n}(\sum_{i=1}^l \mu_i^n f_i(\cdot, u_i^n))(\bar{x})$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$, and $\tilde{\zeta}_n \in \partial_{\gamma_n}(\sum_{j=1}^m \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that

$$\begin{aligned} &\|x_n - \bar{x}\| \leq \sqrt{\delta_n + \gamma_n}, \quad \|\xi_n - \tilde{\xi}_n + \zeta_n - \tilde{\zeta}_n\| \leq \sqrt{\delta_n + \gamma_n} \ \text{ and} \\ &\Bigg| \sum_{i=1}^l \mu_i^n f_i(x_n, u_i^n) + \sum_{j=1}^m \lambda_j^n g_j(x_n, v_j^n) - \langle \tilde{\xi}_n + \tilde{\zeta}_n, x_n - \bar{x} \rangle \\ &- \sum_{i=1}^l \mu_i^n f_i(\bar{x}, u_i^n) - \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n) \Bigg| \leq 2(\delta_n + \gamma_n). \end{aligned}$$

Since $\delta_n + \gamma_n \to 0$ as $n \to \infty$, $\|x_n - \bar{x}\| \to 0$ as $n \to \infty$ and $\|\xi_n - \tilde{\xi}_n + \zeta_n - \tilde{\zeta}_n\| \to 0$ as $n \to \infty$. Thus we have

$$0 = \sum_{i=1}^l \nu_i + \lim_{n \to \infty} (\tilde{\xi}_n + \tilde{\zeta}_n)$$

and

$$\lim_{n \to \infty} \left[\mu_i^n f_i(\bar{x}, u_i^n) - \mu_i^n f_i(\bar{x}, \bar{u}_i) - \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n)\right] = 0.$$

Since $\lim_{n \to \infty}[\mu_i^n f_i(\bar{x}, u_i^n) - \mu_i^n f_i(\bar{x}, \bar{u}_i)] = 0$, $i = 1, \ldots, l$, and $\lim_{n \to \infty} \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n) = 0$, we have $\lim_{n \to \infty}[\mu_i^n f_i(\bar{x}, u_i^n) - \mu_i^n f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n)] = 0$, $i = 1, \ldots, l$.

Now we suppose that there exist $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $x \in \mathbb{R}^n$, $\mu_i^n \geq 0$, $u_i^n \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\tilde{\xi}_n \in \partial_{\delta_n}(\sum_{i=1}^l \mu_i^n f_i(\cdot, u_i^n))(\bar{x})$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$, and $\tilde{\zeta}_n \in \partial_{\gamma_n}(\sum_{j=1}^m \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that

$$\lim_{n\to\infty} x_n = \bar{x},\ \max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i),\ i = 1, \ldots, l,$$

$$0 = \sum_{i=1}^l \nu_i + \lim_{n\to\infty} (\tilde{\xi}_n + \tilde{\zeta}_n) \text{ and}$$

$$\lim_{n\to\infty} \left[\sum_{i=1}^l \mu_i^n f_i(\bar{x}, u_i^n) - \sum_{i=1}^l \mu_i^n f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^m \lambda_j^n g_j(\bar{x}, v_j^n) \right] = 0.$$

Then, since $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$ and $\max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i)$, $i = 1, \ldots, l$, we have $\nu_i \in \partial(\max_{u_i\in\mathcal{U}_i} f_i(\cdot, u_i))(\bar{x})$, $i = 1, \ldots, l$. So, for any $x \in \tilde{F}$,

$$\begin{aligned} \sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(x, u_i) - \sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, u_i) &\geq \left\langle \sum_{i=1}^l \nu_i, x - \bar{x} \right\rangle \\ &= -\lim_{n\to\infty} \langle \tilde{\xi}_n + \tilde{\zeta}_n, x - \bar{x} \rangle. \end{aligned}$$

Since $\lim_{n\to\infty}(\tilde{\xi}_n + \tilde{\zeta}_n) = -\sum_{i=1}^l \nu_i$ and $\lim_{n\to\infty} x_n = \bar{x}$, we have

$$\begin{aligned} & -\lim_{n\to\infty} \langle \tilde{\xi}_n + \tilde{\zeta}_n, x - \bar{x} \rangle \\ = \ & -\lim_{n\to\infty} \langle \tilde{\xi}_n + \tilde{\zeta}_n, x - x_n + x_n - \bar{x} \rangle \\ = \ & -\lim_{n\to\infty} \langle \tilde{\xi}_n + \tilde{\zeta}_n, x - x_n \rangle \\ \geq \ & -\limsup_{n\to\infty} \left(\sum_{i=1}^l \mu_i^n f_i(x, u_i^n) - \sum_{i=1}^l \mu_i^n f_i(x_n, u_i^n) \right. \\ & \left. + \sum_{j=1}^m \lambda_j^n g_j(x, v_j^n) - \sum_{j=1}^m \lambda_j^n g_j(x_n, v_j^n) \right) \\ \geq \ & \liminf_{n\to\infty} \left(\sum_{i=1}^l \mu_i^n f_i(x_n, u_i^n) - \sum_{i=1}^l \mu_i^n f_i(x, u_i^n) + \sum_{j=1}^m \lambda_j^n g_j(x_n, v_j^n) \right) \\ \geq \ & \liminf_{n\to\infty} \left(\sum_{i=1}^l \mu_i^n f_i(x_n, u_i^n) - \sum_{i=1}^l \mu_i^n f_i(x, u_i^n) + \sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(x, u_i) \right. \\ & \left. - \sum_{i=1}^l \max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, u_i^n) + \sum_{j=1}^m \lambda_j^n g_j(x_n, v_j^n) \right) \\ \geq \ & \liminf_{n\to\infty} \left(\sum_{i=1}^l \mu_i^n f_i(x_n, u_i^n) - \sum_{i=1}^l \mu_i^n f_i(\bar{x}, u_i^n) + \sum_{j=1}^m \lambda_j^n g_j(x_n, v_j^n) \right) \\ = \ & 0. \end{aligned}$$

The last inequality holds since $\sum_{i=1}^l \mu_i^n \max_{u_i\in\mathcal{U}_i} f_i(x, u_i) - \sum_{i=1}^l \mu_i^n f_i(x, u_i^n) \geq 0$ and $\max_{u_i\in\mathcal{U}_i} f_i(\bar{x}, \bar{u}_i) = f_i(\bar{x}, \bar{u}_i)$, $i = 1, \ldots, l$. Hence, for any $x \in \tilde{F}$,

$\sum_{i=1}^{l} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq \sum_{i=1}^{l} \mu_i^n f_i(x, u_i^n)$. So, $\bar{x}$ is a robust efficient solution of (RUMP). □

Theorem 3.2. *Let $\bar{x} \in F$. Under assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a weakly robust efficient solution of* (RUMP)*;*

(ii) *there exist $\mu_i \geq 0$, not all zero, $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $i = 1, \dots, l$, $x_n \in \mathbb{R}^n$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \dots, m$, and $\tilde{\zeta}_n \in \partial(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that*

$$\lim_{n\to\infty} x_n = \bar{x}, \ \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \dots, l,$$

$$0 = \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \tilde{\zeta}_n \textit{ and } \lim_{n\to\infty} \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) = 0.$$

Proof. Suppose that (i) holds. Then, from Theorem 2.3, there exist $\mu_i^n \geq 0$, not all zero, $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $i = 1, \dots, l$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \dots, m$, $\gamma_n \geq 0$ and $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \dots, l,$$

$$0 = \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \zeta_n, \ \lim_{n\to\infty} \gamma_n = 0 \text{ and}$$

$$\lim_{n\to\infty} \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n) = 0.$$

Since $\zeta_n \in \partial_{\gamma_n}(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$, from Proposition 1.1, there exist $x_n \in \mathbb{R}^n$ and $\tilde{\zeta}_n \in \partial(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(x_n)$ such that

$$\|x_n - \bar{x}\| \leq \sqrt{\gamma_n}, \ \ \|\tilde{\zeta}_n - \zeta_n\| \leq \sqrt{\gamma_n} \ \text{ and}$$

$$\left| \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) - \langle \tilde{\zeta}_n, x_n - \bar{x} \rangle - \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n) \right| \leq 2\gamma_n.$$

Since $\lim_{n\to\infty} \gamma_n = 0$, we have $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty}(\tilde{\zeta}_n - \zeta_n) = 0$ and $\lim_{n\to\infty}[\sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) - \langle \tilde{\zeta}_n, x_n - \bar{x} \rangle - \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n)] = 0$. Since $\lim_{n\to\infty} \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \zeta_n = 0$ and $\lim_{n\to\infty}(\tilde{\zeta}_n - \zeta_n) = 0$, $\lim_{n\to\infty} \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \tilde{\zeta}_n = 0$, and hence $\lim_{n\to\infty}[\sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) - \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n)] = 0$. Since $\sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}, v_j^n) = 0$, $\lim_{n\to\infty} \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) = 0$. Thus

$$\lim_{n\to\infty} x_n = \bar{x}, \ \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \tilde{\zeta}_n = 0 \text{ and}$$

$$\lim_{n\to\infty} \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) = 0.$$

Now we suppose that (ii) holds. Then for any $x \in F$,

$$\begin{aligned}
& \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) \\
\geq \ & \left\langle \sum_{i=1}^{l} \mu_i \nu_i, x - \bar{x} \right\rangle = - \lim_{n\to\infty} \langle \tilde{\zeta}_n, x - \bar{x} \rangle = - \lim_{n\to\infty} \langle \tilde{\zeta}_n, x - x_n + x_n - \bar{x} \rangle \\
= \ & - \lim_{n\to\infty} \langle \tilde{\zeta}_n, x - x_n \rangle \geq - \limsup_{n\to\infty} \left(\sum_{j=1}^{m} \lambda_j^n g_j(x, v_j^n) - \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) \right) \\
\geq \ & \liminf_{n\to\infty} \left(\sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) \right) \\
= \ & 0.
\end{aligned}$$

Hence, for any $x \in \tilde{F}$, $\sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq \sum_{i=1}^{l} \mu_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)$. Thus, $\bar{x}$ is a weakly robust efficient solution of (RUMP). □

By using approaches similar to the proof of Theorem 3.2, we can obtain the following sequential optimality theorem for a properly robust efficient solution of (RUMP).

Theorem 3.3. *Let $\bar{x} \in F$. Under assumptions of Theorem 2.2, the following statements are equivalent:*

(i) *the point $\bar{x}$ is a properly robust efficient solution of* (RUMP)*;*

(ii) *there exist $\mu_i > 0$, $\bar{u}_i \in \mathcal{U}_i$, $\nu_i \in \partial f_i(\cdot, \bar{u}_i)(\bar{x})$, $i = 1, \ldots, l$, $x_n \in \mathbb{R}^n$, $v_j^n \in \mathcal{V}_j$, $\lambda_j^n \geq 0$, $j = 1, \ldots, m$, and $\tilde{\zeta}_n \in \partial_{\gamma_n}(\sum_{j=1}^{m} \lambda_j^n g_j(\cdot, v_j^n))(\bar{x})$ such that*

$$\lim_{n\to\infty} x_n = \bar{x}, \ \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i), \ i = 1, \ldots, l,$$

$$0 = \sum_{i=1}^{l} \mu_i \nu_i + \lim_{n\to\infty} \tilde{\zeta}_n \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=1}^{m} \lambda_j^n g_j(x_n, v_j^n) = 0.$$

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