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INFINITE PRODUCTS OF DISCONTINUOUS OPERATORS IN BANACH AND METRIC SPACES

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Dedicated to Professor Anthony To-Ming Lau on the occasion of his 70th birthday

Abstract. We study the convergence, either strong or weak, of sequences generated by methods for solving the convex feasibility problem. These methods involve infinite products of discontinuous operators and of their convex combinations. We consider Banach spaces, the Hilbert ball and CAT(0) spaces.

1. Introduction

The problem of finding a point in the nonempty intersection of convex and closed sets is referred to as the *convex feasibility problem* and has applications, for example, in the image recovery field. One of the first algorithms for solving it was proposed by J. von Neumann [21]. It concerned the intersection of two closed subspaces of a Hilbert space. Years later, I. Halperin [17] extended von Neumann's idea to the intersection of a finite number of subspaces. Since then, the interest in this problem has increased considerably and as a result, it has been studied in much more general settings, for instance, in certain Banach and metric spaces.

In this connection, E. Pustylnik and S. Reich have recently proved the following result [22]. Consider the orthogonal projections $\{P_{S_i} : 1 \leq i \leq m\}$ of a Hilbert space H onto its closed subspaces $\{S_i : 1 \leq i \leq m\}$. Consider also the possibly nonlinear operators $A_n^{(i)}$, $i = 1, 2, \ldots, m$; $n = 1, 2, \ldots$, and suppose that for all $x \in H$, the inequalities

$$\|A_n^{(i)} x - P_{S_i} x\| \leq \gamma_n \|x\|$$

hold for some positive numbers γ_n with $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, for each $x \in H$, there exists a point $\overline{x} \in S_1 \cap S_2 \cap \cdots \cap S_m$ such that

$$\lim_{n\to\infty} \left\| \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x - \overline{x} \right\| = 0.$$

Our aim in this paper is to extend the Pustylnik–Reich result to possibly discontinuous operators defined outside Hilbert space; more precisely, on Banach spaces,

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the Hilbert ball $\mathbb{B}$ and on CAT(0) spaces. To this purpose, we use norm-one projections, retractions and nearest point projections instead of orthogonal projections. These operators are then approximated by other, possibly nonlinear and even discontinuous operators.

In Section 2 we present some known definitions and results regarding Banach spaces and geodesic metric spaces, with more details in the case of the Hilbert ball and CAT(0) spaces. In Section 3 we deal with concepts such as exact and inexact orbits with summable errors of a nonexpansive operator and the relations among various convergence results. We devote Section 4 to Banach spaces, where we deal with the linear case, the nonlinear case and with weak convergence. In Section 5, we deal with the problem in which we are interested in the Hilbert ball setting. Finally, we consider CAT(0) spaces and prove convergence theorems in this framework in Section 6.

2. Preliminaries

In this section we collect several relevant definitions and results. We begin with Banach spaces, continue with geodesic metric spaces and then we study in more depth the particular cases of the Hilbert ball and more generally, CAT(0) spaces. Throughout the paper we let $\mathbb{N}$ stand for the set $\{1,2,3,\ldots\}$ of natural numbers. We denote by $\mathrm{Fix}(T)$ the set of all fixed points of an operator T.

2.1. **Banach spaces.** For basic information about Banach spaces we refer the reader to [6], [15] and [16]. Let E be a Banach space and $S \subset E$ a nonempty subset. Denote by $I : E \to E$ the identity operator. We say that an operator $R : E \to S$ is a *retraction* if the restriction $R|_S = I$. If, in addition, a retraction $P : E \to S$ is linear, then we call it a *projection*.

When E is uniformly convex and $\{P_{S_k} : 1 \le k \le m\}$ are norm-one projections of E onto its subspaces $\{S_k : 1 \le k \le m\}$, it is known ([10, Lemma 2.1]) that

$$\mathrm{Fix}\,(P_{S_m} P_{S_{m-1}} \cdots P_{S_1}) = \bigcap_{k=1}^{m} \mathrm{Fix}\,(P_{S_k}) = \bigcap_{k=1}^{m} S_k. \tag{2.1}$$

In addition, for numbers $a_1, \ldots, a_m \in (0,1)$ such that $a_1 + a_2 + \cdots + a_m = 1$, it follows from [26, Lemma 1.4] that

$$\mathrm{Fix}\left(\sum_{k=1}^{m} a_k P_{S_k}\right) = \bigcap_{k=1}^{m} \mathrm{Fix}\,(P_{S_k}) = \bigcap_{k=1}^{m} S_k. \tag{2.2}$$

Let both $C \subset E$ and $F \subset C$ be closed and convex subsets. Recall that a retraction $R : C \to F$ is called *sunny* [25] if $R\left((1-t)Rx + tx\right) = Rx$ for all $t \ge 0$ and $x, (1-t)Rx + tx \in C$.

If E is smooth and uniformly convex, and the subsets C and F are symmetric, that is, $C = -C$ and $F = -F$, and if F is a sunny nonexpansive retract of C, then the sunny nonexpansive retraction $R : C \to F$ is odd. This fact follows from the uniqueness of the sunny nonexpansive retraction from C onto F (see [9, Theorem 1]).

Remark 2.1. Although a sunny nonexpansive retraction $R : C \to F$ is neither linear nor bounded in general, when C and F are symmetric we do have $\|Rx\| \le \|x\|$ for all $x \in C$.

If E is uniformly convex and $\{R_{F_k} : 1 \le k \le m\}$ are sunny nonexpansive retractions of a closed and convex subset $C \subset E$ onto closed and convex subsets $\{F_k \subset C : 1 \le k \le m\}$, then by [10, Lemma 2.1],

$$\operatorname{Fix}(R_{F_m} R_{F_{m-1}} \cdots R_{F_1}) = \bigcap_{k=1}^{m} \operatorname{Fix}(R_{F_k}) = \bigcap_{k=1}^{m} F_k. \tag{2.3}$$

In addition, for numbers $a_1, \dots, a_m \in (0,1)$ such that $a_1 + a_2 + \cdots + a_m = 1$, it follows from [26, Lemma 1.4] that

$$\operatorname{Fix}\left(\sum_{k=1}^{m} a_k R_{F_k}\right) = \bigcap_{k=1}^{m} \operatorname{Fix}(R_{F_k}) = \bigcap_{k=1}^{m} F_k. \tag{2.4}$$

2.2. **Geodesic metric spaces.** Consider a metric space (X,d). A *geodesic path* joining $x \in X$ to $y \in X$ is a function $\gamma : [0,\ell] \subset \mathbb{R} \to X$ such that $\gamma(0) = x$, $\gamma(\ell) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0,\ell]$. The set $\gamma([0,\ell])$ is called a *geodesic segment* with *endpoints* x and y. We denote by $[x,y]$ the geodesic segment $\gamma([0,d(x,y)])$. A subset $C \subset X$ is *convex* if $[x,y] \subset C$ for all $x,y \in C$. The space (X,d) is called a (*uniquely*) *geodesic metric space* if every two points in X are joined by a (unique) geodesic.

For any two points $x,y \in X$, the geodesic segment $[x,y]$ is convex. A point $z \in [x,y]$ is denoted by $z := (1-t)x \oplus ty$, where $d(x,z) = td(x,y)$, and we say that z is a *convex combination* of x and y.

Given a geodesic metric space (X,d), we denote by $\Delta(x,y,z) \subset X$ the *geodesic triangle* with *vertices* $x,y,z \in X$ and geodesic segments (*edges*) $[x,y]$, $[y,z]$ and $[z,x]$. Let $\overline{\Delta}(x,y,z) \subset M_0^2$ (where M_0^2 is the Euclidean plane) be a *comparison triangle* for $\Delta(x,y,z)$ with vertices $\overline{x}, \overline{y}, \overline{z} \in M_0^2$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a *comparison point* in $\overline{\Delta}(x,y,z)$ for $p \in [x,y]$ if $d(x,p) = |\overline{x} - \overline{p}|$. Comparison triangles exist and are unique up to isometries (see [8, Chapter I]).

Hilbert and CAT(0) spaces are but two examples of (uniquely) geodesic metric spaces. For more information on this topic, we refer the reader to [2], [8] and [16].

2.3. **The Hilbert ball.** Given a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with induced norm $|\cdot|$, consider the open unit ball $\mathbb{B} := \{x \in H : |x| < 1\}$. The function $\rho : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ defined by $\rho(x,y) := \operatorname{argtanh}(1 - \sigma(x,y))^{1/2}$, where

$$\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1 - \langle x,y\rangle|^2} = \operatorname{argtanh}|\mathcal{M}_{-x}(y)|,$$

defines a metric on $\mathbb{B}$. The complete metric space $(\mathbb{B},\rho)$ is called the *Hilbert ball.* The operator $\mathcal{M}_u : \mathbb{B} \to \mathbb{B}$ denotes the *Möbius transformation* at $u \in \mathbb{B}$ (see [16, Section 14, page 97]). It is a weakly continuous automorphism of $\mathbb{B}$ (see [16, Theorem 14.1, page 98, and Lemma 21.3, page 115]). Hence $\mathcal{M}_u$ is invertible, $\mathcal{M}_{-u} \circ \mathcal{M}_u = I$ and $\mathcal{M}_{-u}(u) = 0$ for all $u \in \mathbb{B}$.

The metric ρ is topologically equivalent to the norm metric. This is true because for all $x, y \in \mathbb{B}$, the following inequalities hold:

$$\operatorname{argtanh}\left(\frac{|x-y|}{2}\right) \leq \rho(x,y) \leq \operatorname{argtanh}\left(\frac{|x-y|}{\operatorname{dist}(x,\partial\mathbb{B})}\right), \tag{2.5}$$

where $\operatorname{dist}(x,\partial\mathbb{B}) := \inf\{|x-y| : y \in \partial\mathbb{B}\}$ (see [16, Theorems 10.3 and 10.4, pages 89–90]). Another property of the metric ρ is stated below.

Proposition 2.2. *Suppose that $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \mathbb{B}$ converge weakly to $x, y \in \mathbb{B}$, respectively. Then*

$$\rho(x,y) \leq \liminf_{n\to\infty} \rho(x_n, y_n).$$

For a proof of this theorem we refer the reader to [19, Theorem 3.2].

In particular, $(\mathbb{B}, \rho)$ is a (uniquely) geodesic metric space (see [16, pages 68–70 and 102–103]). So a subset $C \subset \mathbb{B}$ is *ρ-convex* if the geodesic segment $[x,y] \subset C$ for all $x, y \in C$. For any $x, y \in \mathbb{B}$ and any $t \in [0,1]$, there exists a unique element $z \in \mathbb{B}$ (see [16, Section 2.17, page 103]) such that $\rho(x,z) = t\rho(x,y)$ and $\rho(y,z) = (1-t)\rho(x,y)$. We denote this point z by $(1-t)x \oplus ty$ and say that z is a *ρ-convex combination* of x and y. Note that this definition is compatible with the definition of convex combinations in general geodesic metric spaces. Given $a \in \mathbb{B}$, we have, for all $x, y \in \mathbb{B}$,

$$\mathcal{M}_a\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) = \frac{1}{2}\mathcal{M}_a(x) \oplus \frac{1}{2}\mathcal{M}_a(y). \tag{2.6}$$

Given two operators $T_1, T_2 : \mathbb{B} \to \mathbb{B}$, we define their ρ-convex combination $(1-t)T_1 \oplus tT_2$ by $((1-t)T_1 \oplus tT_2)x := (1-t)T_1x \oplus tT_2x$ for all $x \in \mathbb{B}$ and $t \in [0,1]$. We denote this operator by $C(T_1, T_2; (1-t), t)$.

For any $x, y, a, b \in \mathbb{B}$ and $t \in [0,1]$, the following inequality holds (see [16, Lemma 17.1, page 104]):

$$\rho((1-t)a \oplus tx, (1-t)b \oplus ty) \leq (1-t)\rho(a,b) + t\rho(x,y). \tag{2.7}$$

Given a ρ-closed and ρ-convex subset $D \subset \mathbb{B}$, we define the *nearest point projection* $P_D : \mathbb{B} \to D$ by assigning z to x, where $z \in D$ is the unique point in D satisfying $\rho(x,z) = \inf_{y\in D} \rho(x,y)$ (see [16, Theorem 19.1, page 108]). The operator P_D is ρ-nonexpansive ([16, Theorem 19.2, page 110]), that is,

$$\rho(P_Dx, P_Dy) \leq \rho(x,y) \quad \text{for all } x, y \in \mathbb{B}.$$

If $\{P_{K_i} : 1 \leq i \leq m\}$ are the nearest point projections of $\mathbb{B}$ onto ρ-closed and ρ-convex subsets $\{K_i : 1 \leq i \leq m\}$ with nonempty intersection, then

$$\operatorname{Fix}(P_{K_m}P_{K_{m-1}}\cdots P_{K_1}) = \bigcap_{i=1}^{m} K_i. \tag{2.8}$$

Moreover, when $m = 2$, we have

$$\operatorname{Fix}(tP_{K_1} \oplus (1-t)P_{K_2}) = K_1 \cap K_2 \tag{2.9}$$

for any $t \in (0,1)$. For a proof of (2.8) and (2.9) we refer the reader to [27, Lemma 3] and [5, Theorem 9.5], respectively.

In [18], E. Kopecká and S. Reich inductively defined the ρ-convex combination of more than two operators in the Hilbert ball, as presented below. Consider operators $T_1, \ldots, T_m : \mathbb{B} \to \mathbb{B}$ and numbers $a_1, \ldots, a_m \in (0,1)$ such that $\sum_{i=1}^m a_i = 1$. The ρ-convex combination of $T_1, \ldots, T_m$ is defined inductively by

$$C(T_1, T_2, \ldots, T_m; a_1, a_2, \ldots, a_m) := C(U, T_m; 1 - a_m, a_m)$$

where $U = C(T_1, T_2, \ldots, T_{m-1}; c_1, c_2, \ldots, c_{m-1})$ and $c_i = a_i/(1 - a_m)$ for $1 \le i \le m-1$. When $\mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m) \neq \emptyset$, it turns out that

$$\mathrm{Fix}\,(C(T_1, \ldots, T_m; a_1, \ldots, a_m)) = \bigcap_{i=1}^m \mathrm{Fix}\,(T_i). \tag{2.10}$$

For a proof of this equality we refer the reader to [18, Lemma 3.5]

2.4. **CAT(0) spaces.** Consider a geodesic metric space (X, d). A triangle $\Delta(x, y, z)$ is said to satisfy the CAT(0)-*inequality* if for all $p, q \in \Delta(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \overline{\Delta}(x, y, z)$, the inequality $d(p, q) \le |\overline{p} - \overline{q}|$ holds. A geodesic metric space (X, d) is said to be CAT(0) if all its geodesic triangles satisfy the CAT(0)-inequality.

It is known that any two points in a CAT(0) space (X, d) are joined by a *unique* geodesic segment (see [8, Proposition II.1.4]). In addition, for any $w, x, y, z \in X$ and any $t \in [0, 1]$, the following inequality holds ([8, Proposition II.2.2]):

$$d((1-t)w \oplus tx, (1-t)y \oplus tz) \le (1-t)d(w, y) + td(x, z). \tag{2.11}$$

Consider points $x, y, z \in X$ and their comparison points $\overline{x}, \overline{y}, \overline{z} \in M_0^2$. We denote by $\angle(x, y, z)$ the Alexandrov angle between the geodesic segments $[x, y]$ and $[y, z]$ (see [8, Section I.1]). We write $\overline{\angle}(x, y, z)$ to denote the comparison angle of $\angle(x, y, z)$ between the sides $[\overline{x}, \overline{y}]$ and $[\overline{y}, \overline{z}]$ in M_0^2.

Remark 2.3. (X, d) is a CAT(0) space if and only if the Alexandrov angle between the sides of any geodesic triangle in X, with distinct vertices, is no greater than the angle between the corresponding sides of its comparison triangle in M_0^2. For a proof of this fact we refer the reader to [8, Proposition II.1.7].

When (X, d) is a complete CAT(0) space, for each closed and convex subset $C \subset X$ and each $x \in X$, there exists a unique point $P_C x \in C$ such that $d(x, P_C x) = d(x, C) = \inf_{y \in C} d(x, y)$. The operator P_C is called the *nearest point projection* of X onto C. This operator is nonexpansive. In addition, given $x \notin C$ and $y \in C$ such that $y \neq P_C(x)$, we have $\angle(x, P_C x, y) \ge \pi/2$. For a proof of the existence and properties of nearest point projections in complete CAT(0) spaces we refer the reader to [8, Proposition II.2.4].

Remark 2.4. Since $\angle(x, P_C x, y) \ge \pi/2$, we also have $\overline{\angle}(x, P_C x, y) \ge \pi/2$ (see Remark 2.3).

Next, we delve into the study of some properties of nearest point projections in complete CAT(0) spaces.

Lemma 2.5. *Suppose (X, d) is a complete CAT(0) space and let $C \subset X$ be a closed and convex subset. If $d(q, P_C(x)) \ge \delta > 0$ for some $x \in X$ and $q \in C$, then $d(x, q) \ge \sqrt{\ell^2 + \delta^2}$, where $\ell := d(x, P_C x)$.*

Proof. Take $x \in X$ and $q \in C$ for which there exists $\delta > 0$ such that $d(q, P_C(x)) \geq \delta$. According to Remark 2.4, we have $\overline{\angle}(x, P_C x, q) \geq \pi/2$. Hence, using the cosine law and the inequality $d(q, P_C(x)) \geq \delta$, we get $d^2(q, x) \geq \ell^2 + d^2(q, P_C x) \geq \ell^2 + \delta^2$. Therefore $d(q, x) \geq \sqrt{\ell^2 + \delta^2}$. □

Corollary 2.6. *For each $L > 0$ and each $\varepsilon > 0$, there exists $\eta > 0$ such that for all complete CAT(0) spaces (X, d), for all closed and convex subsets C of X, for all $x \in X$ satisfying the inequality $d(x, C) \leq L$ and for all $q \in C$, we have*

$$|d(x, P_C x) - d(x, q)| < \eta \quad \Rightarrow \quad d(q, P_C x) < \varepsilon.$$

Proof. Suppose that there are $L > 0$ and $\varepsilon > 0$ for which no such $\eta > 0$ exists. Then, for each $n \in \mathbb{N}$, there exist a complete CAT(0) space (X_n, d_n), a closed and convex subset $C_n \subset X_n$, $x_n \in X_n$, and $q_n \in C_n$ such that

$$\lim_{n\to\infty} |d_n(x_n, P_{C_n} x_n) - d_n(x_n, q_n)| = 0, \tag{2.12}$$

where $d_n(q_n, P_{C_n} x_n) \geq \varepsilon > 0$ and $d_n(x_n, C_n) \leq L$ for all $n \in \mathbb{N}$. Hence by Lemma 2.5 we obtain

$$d_n^2(x_n, q_n) - d_n^2(x_n, P_{C_n} x_n) \geq \varepsilon^2 \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

Note that $d_n(x_n, P_{C_n} x_n) = d_n(x_n, C_n) \leq d_n(x_n, q_n)$ for all $n \in \mathbb{N}$, since P_{C_n} is the nearest point projection of X_n onto C_n and $q_n \in C_n$. Hence $d_n(x_n, q_n) - d_n(x_n, P_{C_n} x_n) \geq 0$ for all $n \in \mathbb{N}$. Since $d_n(x_n, C_n) \leq L$ for all $n \in \mathbb{N}$ and (2.12) holds, we see that the sequence $(d_n(x_n, q_n))_{n\in\mathbb{N}}$ is bounded too. Combining this fact with (2.12), we obtain

$$\lim_{n\to\infty} [d_n^2(x_n, q_n) - d_n^2(x_n, P_{C_n} x_n)] = 0,$$

but this contradicts (2.13). □

Theorem 2.7. *Suppose (X, d) is a complete CAT(0) space and consider a sequence $(y_k)_{k\in\mathbb{N}} \subset X$ and a point $y_* \in X$ such that $\lim_{k\to\infty} d(y_k, y_*) = 0$. For a fixed $z \in X$, consider the geodesic segments $[z, y_*]$ and $[z, y_k]$, $k \in \mathbb{N}$. If P_k and P_* are the nearest point projections of X onto $[z, y_k]$ and $[z, y_*]$, respectively, then for each $x \in X$,*

$$\lim_{k\to\infty} d(P_k x, P_* x) = 0.$$

Moreover, the sequence $(P_k)_{k\in\mathbb{N}}$ converges to P_* uniformly on bounded subsets of X.

Proof. Take $x \in X$. Since $P_k x \in [z, y_k]$, there exists $t_k \in [0, 1]$ such that $P_k x = (1 - t_k)z \oplus t_k y_k$ for each $k \in \mathbb{N}$. Moreover, since $(t_k)_{k\in\mathbb{N}}$ is a bounded sequence, there exists $t_* \in [0, 1]$ such that $\lim_{k\to\infty} t_k = t_*$, up to a subsequence. Now it is not difficult to see that

$$\lim_{k\to\infty} P_k x = \lim_{k\to\infty} (1 - t_k)z \oplus t_k y_k = (1 - t_*)z \oplus t_* y_* = P_* x.$$

So far we have proved pointwise convergence. To prove uniform convergence on bounded subsets, we argue as follows. Consider the triangle $\Delta(z, y_k, y_*)$. For each $k \in \mathbb{N}$, denote by $Q_k x$ the nearest point projection of $P_* x$ onto $[z, y_k]$. So by (2.11), for all $k \in \mathbb{N}$ we have

$$d(Q_k x, P_* x) \leq d((1 - t_*)z \oplus t_* y_k, (1 - t_*)z \oplus t_* y_*) \leq d(y_k, y_*). \tag{2.14}$$

Given $\varepsilon > 0$ and $L > 0$, let $\eta = \eta(\varepsilon, L)$ be determined by Corollary 2.6. Since $\lim_{k\to\infty} d(y_k, y_*) = 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$d(y_k, y_*) < \frac{\zeta}{2}, \tag{2.15}$$

where $\zeta = \min\{2\epsilon, \eta\}$. Let $\ell_k := d(x, [z, y_k])$ and $\ell_* := d(x, [z, y_*])$. Since $Q_k x \in [z, y_k]$, using the triangle inequality, (2.14) and (2.15), we see that for all $k \geq k_0$, $d(x, Q_k x) \leq d(x, P_* x) + d(P_* x, Q_k x) < \ell_* + \zeta/2$. Hence

$$0 \leq d(x, Q_k x) - \ell_k < \ell_* - \ell_k + \frac{\zeta}{2} \quad \text{for all } k \geq k_0. \tag{2.16}$$

Note that $\ell_* = d(x, (1 - t_*)z \oplus t_* y_*)$ and $\ell_k = d(x, (1 - t_k)z \oplus t_k y_k)$ for some $t_*, t_k \in [0, 1]$, $k \in \mathbb{N}$. By the triangle inequality, (2.11) and (2.15), we obtain, for all $k \geq k_0$,

$$\ell_k \leq d(x, (1 - t_*)z \oplus t_* y_k) \leq \ell_* + d(y_*, y_k) < \ell_* + \frac{\zeta}{2} \tag{2.17}$$

because $P_k x = (1 - t_k)z \oplus t_k y_k$ is the nearest point projection of X onto $[z, y_k]$. Similarly, from the triangle inequality, (2.11) and (2.15) we also get, for all $k \geq k_0$,

$$\ell_* \leq d(x, (1 - t_k)z \oplus t_k y_*) \leq \ell_k + d(y_*, y_k) < \ell_k + \frac{\zeta}{2}. \tag{2.18}$$

Combining (2.17) and (2.18), we see that for all $k \geq k_0$, $|\ell_* - \ell_k| < \zeta/2$. So $\lim_{k\to\infty} \ell_k = \ell_*$, uniformly on X. Combining this fact with (2.16), we get for all $k \geq k_0$,

$$|d(x, Q_k x) - d(x, P_k x)| = d(x, Q_k x) - \ell_k < \ell_* - \ell_k + \frac{\zeta}{2} < \zeta \leq \eta. \tag{2.19}$$

According to Corollary 2.6, where the convex subsets under consideration are the geodesic segments $[z, y_k]$, it follows from (2.19) that

$$d(Q_k x, P_k x) < \epsilon \quad \text{for all } k \geq k_0, \tag{2.20}$$

uniformly on bounded subsets of X. So by the triangle inequality, (2.14), (2.15) and (2.20), it is not difficult to see that for all $k \geq k_0$,

$$d(P_k x, P_* x) \leq d(P_k x, Q_k x) + d(Q_k x, P_* x) < 2\varepsilon$$

uniformly on bounded subsets of X. Since ε is arbitrary, the last inequality shows that P_k indeed converges to P_*, uniformly on bounded subsets of X, as asserted. □

Remark 2.8. Lemma 2.5, Corollary 2.6 and Theorem 2.7 are all due to M. R. Bridson [7].

If (X, d) is a CAT(0) space, an operator $T : X \to X$ is said to be *firmly nonexpansive* if $d(Tx, Ty) \leq d((1 - t)x \oplus tTx, (1 - t)y \oplus tTy)$ for all $x, y \in X$ and $t \in [0, 1]$. For example, the nearest point projection operator is firmly nonexpansive (see [1, Proposition 3.1]). It is clear that every firmly nonexpansive operator is nonexpansive. If $\mathrm{Fix}\,(T) \neq \emptyset$ and T is nonexpansive, we say that T is *strongly nonexpansive* if for any d-bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and any $z \in \mathrm{Fix}\,(T)$ such that $\lim_{n\to\infty}[d(x_n, z) - d(Tx_n, z)] = 0$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. These definitions extend those introduced in the Hilbert ball setting (see [27]).

It is not difficult to see that in a CAT(0) space X, every firmly nonexpansive operator with fixed points is also strongly nonexpansive. To show this, we can proceed as in [27, Lemma 1]. Moreover, if $\{T_i : 1 \le i \le m\}$ are strongly nonexpansive operators and $\mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m) \neq \emptyset$, then

$$\mathrm{Fix}\,(T_m T_{m-1} \cdots T_1) = \bigcap_{i=1}^{m} \mathrm{Fix}\,(T_i). \tag{2.21}$$

To prove this fact, we may proceed as in [27, Lemmata 3 and 4].

To finish this subsection, we recall the concept of weak convergence for CAT(0) spaces, which was introduced by J. Jost (see [3] and references therein). Consider a complete CAT(0) space X. A point $x^* \in X$ is the *asymptotic center* of a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X if

$$\limsup_{n\to\infty} d(x_n, x^*) = \inf_{x\in X} \limsup_{n\to\infty} d(x_n, x).$$

For a complete CAT(0) space X, a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is said to converge weakly to a point $x \in X$ if x is the asymptotic center of each subsequence of $(x_n)_{n\in\mathbb{N}}$. In Hilbert spaces, this definition of weak convergence coincides with the classical weak convergence in those spaces.

Proposition 2.9. *Suppose X is a CAT(0) space and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence. Then $(x_n)_{n\in\mathbb{N}}$ weakly converges to $x \in X$ if and only if $\lim_{n\to\infty} d(x, P_\gamma(x_n)) = 0$ for any geodesic segment γ through x, that is, for any geodesic segment γ such that $x \in \gamma$.*

For a proof of this result we refer the reader to [14, Proposition 5.2].

Proposition 2.10. *Suppose (X, d) is a complete CAT(0) space. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be bounded sequences of X converging weakly to x and y, respectively. Then $d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n)$.*

Proof. We know that $\lim_{n\to\infty} d(x, P_{\gamma_x} x_n) = \lim_{n\to\infty} d(y, P_{\gamma_y} y_n) = 0$ for all geodesic segments γ_x through x and all geodesic segments γ_y through y (see Proposition 2.9). Since there exists a (unique) geodesic segment γ joining x and y, using the triangle inequality, we obtain

$$d(x, y) \le d(x, P_\gamma x_n) + d(P_\gamma x_n, P_\gamma y_n) + d(P_\gamma y_n, y).$$

From the last inequality and the nonexpansivity of P_γ it follows that

$$d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n).$$

This concludes the proof. □

Remark 2.11. A complete CAT(0) space is called an Hadamard space [2, page 6].

3. Exact and inexact orbits

In this section we recall several results connecting convergence properties of exact and inexact orbits of nonexpansive operators with summable errors in metric spaces. We also focus on the particular cases of Banach spaces, the Hilbert ball and CAT(0) spaces with the corresponding concepts of weak convergence defined in them.

Consider a metric space (E, d) and an operator $T : E \to E$.

(1) Any sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_1 := x$ and $y_{n+1} := T^n x = Ty_n$ for all $n \in \mathbb{N}$ is called an *exact orbit* of T with initial point $x \in E$.
(2) A sequence $(x_n)_{n\in\mathbb{N}} \subset E$ such that $\sum_{n\in\mathbb{N}} d(x_{n+1}, Tx_n) < \infty$ is said to be an *inexact orbit of T with summable errors.*

For information regarding this topic, see, for example, [23] and [24], and references therein. Boundedness properties of exact and inexact orbits of a nonexpansive operator are related, as we see in the following result.

Proposition 3.1. *Suppose (E, d) is a metric space and let $T : E \to E$ be nonexpansive. If all exact orbits of T with summable errors are bounded sequences, then all inexact orbits of T with summable errors are bounded too. The converse is also true.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an inexact orbit of T with summable errors. So, by definition, there is $M > 0$ such that $\sum_{n=1}^{\infty} d(x_{n+1}, Tx_n) \leq M$. Take $x \in E$ and consider the corresponding exact orbit. Denote it by $(y_n)_{n\in\mathbb{N}}$, where $y_1 = x$. For each $n \in \mathbb{N}$, we then have $d(y_{n+1}, x_{n+1}) \leq \sum_{j=1}^{n} d(x_{j+1}, Tx_j) + d(x, x_1)$. Therefore $d(x_n, y_n) \leq M + d(x, x_1)$ for all $n \in \mathbb{N}$. This shows that if $(y_n)_{n\in\mathbb{N}}$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ is also bounded, and *vice versa*. □

For a proof of the following result we refer the reader to [11, Theorem 4.2]. We continue to denote by Fix (T) the set of all fixed points of an operator T.

Theorem 3.2. *Suppose (E, d) is a complete metric space. Let $T : E \to E$ be a nonexpansive operator with* Fix $(T) \neq \emptyset$*. Then the following two statements are equivalent:*

(i) *All exact orbits of T converge in (E, d);*
(ii) *All inexact orbits of T with summable errors converge in (E, d) to fixed points of T.*

Note that convergent exact orbits of a nonexpansive operator converge to fixed points of this operator. For Banach spaces, a result analogous to Theorem 3.2, involving weak convergence, also holds. For a proof see [11, Theorem 4.1] and [12, Note added in proof]).

Theorem 3.3. *Suppose E is a Banach space. Let $S \subset E$ be a weakly closed subset and consider a nonexpansive operator $T : S \to S$ (with* Fix $(T) \neq \emptyset$*). Then the following two statements are equivalent:*

(i) *All exact orbits of T converge weakly (to fixed points of T);*
(ii) *All inexact orbits of T with summable errors converge weakly (to fixed points of T).*

When we consider the Hilbert ball $(\mathbb{B}, \rho)$, a version of Theorem 3.3 holds with respect to the weak convergence inherited from the ambient Hilbert space. We first mention the following fact.

Remark 3.4. Let $T : \mathbb{B} \to \mathbb{B}$ be a ρ-nonexpansive operator. If an exact orbit of T weakly converges to a point in $\mathbb{B}$, then this point is a fixed point of T.

To prove this fact, let $y \in \mathbb{B}$ and suppose that the sequence $(y_i)_{i\in\mathbb{N}} = (T^i y)_{i\in\mathbb{N}}$ converges weakly to $y^* \in \mathbb{B}$. We know that T has a fixed point by [16, Corollary 25.3, page 126]. Since T is ρ-nonexpansive, the exact orbit $(y_i)_{i\in\mathbb{N}}$ is ρ-bounded. So, we know (see [16, Proposition 21.4, page 117]) that y^* is the unique point satisfying

$$\limsup_{i\to\infty} \rho(y_i, y^*) = \min_{y\in\mathbb{B}} \limsup_{i\to\infty} \rho(y_i, y).$$

Now we note that, by the ρ-nonexpansivity of T,

$$\limsup_{i\to\infty} \rho(y_i, Ty^*) = \limsup_{i\to\infty} \rho(Ty_{i-1}, Ty^*) \le \min_{y\in\mathbb{B}} \limsup_{i\to\infty} \rho(y_i, y),$$

which proves that $Ty^* = y^*$.

Theorem 3.5. *Let $T : \mathbb{B} \to \mathbb{B}$ be a ρ-nonexpansive operator. The following two statements are equivalent:*

(i) *All exact orbits of T converge weakly to points in $\mathbb{B}$;*
(ii) *All inexact orbits of T with summable errors converge weakly to fixed points of T.*

Proof. It is clear that (ii)⇒(i) because all exact orbits are inexact orbits of T with summable errors.

(i)⇒(ii): We follow the ideas which were used to prove [11, Theorem 4.1]. Assume that all exact orbits of T converge weakly to points in $\mathbb{B}$. Let $(x_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ be an inexact orbit of T with summable errors and take a sequence $(r_k)_{k\in\mathbb{N}}$ of real numbers such that $\sum_{k\in\mathbb{N}} r_k < \infty$ and $\rho(x_{k+1}, Tx_k) \le r_k$ for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. By induction over i, we can see that

$$\rho(T^i x_k, x_{k+i}) \le \sum_{j=k}^{i+k-1} r_j \quad \text{for each } i \in \mathbb{N}. \tag{3.1}$$

By hypothesis, we know that all exact orbits of T converge weakly to points in $\mathbb{B}$. Hence there exists $y_k \in \mathbb{B}$ such that

$$\lim_{i\to\infty} T^i x_k = y_k \quad \text{weakly.} \tag{3.2}$$

Let $q \in \mathbb{N}$ be fixed. By (3.1) and since T is ρ-nonexpansive, we obtain

$$\rho(T^{q+i} x_k, T^i x_{k+q}) \le \rho(T^q x_k, x_{k+q}) \le \sum_{j=k}^{\infty} r_j \quad \text{for each } i \in \mathbb{N}. \tag{3.3}$$

By Proposition 2.2, (3.2) and (3.3), we see that

$$\rho(y_k, y_{q+k}) \le \liminf_{i\to\infty} \rho(T^{q+i} x_k, T^i x_{k+q}) \le \sum_{j=k}^{\infty} r_j \tag{3.4}$$

for all $k, q \in \mathbb{N}$. Since $\sum_{j\in\mathbb{N}} r_j < \infty$, using (3.4), we conclude that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, so there exists $y^* \in \mathbb{B}$ such that

$$\lim_{k\to\infty} \rho(y_k, y^*) = 0. \tag{3.5}$$

So, when $q \to \infty$, it follows from (3.4) that $\rho(y_k, y^*) \leq \sum_{j=k}^{\infty} r_j$ for each $k \in \mathbb{N}$. Take $\psi \in H$ such that $|\psi| \leq 1$. Then by the triangle inequality,

$$|\langle \psi, y^* - x_{k+i} \rangle| \leq |y^* - y_k| + |\langle \psi, y_k - T^i x_k \rangle| + |T^i x_k - x_{k+i}| \tag{3.6}$$

for all $i, k \in \mathbb{N}$. By inequality (2.5) and since the hyperbolic tangent function is increasing, we have for all $i, k \in \mathbb{N}$,

$$\begin{cases} |y^* - y_k| \leq 2\tanh(\rho(y^*, y_k)) \\ \text{and } |T^i x_k - x_{k+i}| \leq 2\tanh(\rho(T^i x_k, x_{k+i})). \end{cases} \tag{3.7}$$

Combining (3.1), (3.6) and (3.7), we see that for all $i, k \in \mathbb{N}$,

$$\begin{aligned} |\langle \psi, y^* - x_{k+i} \rangle| &\leq 2\tanh(\rho(y^*, y_k)) + |\langle \psi, y_k - T^i x_k \rangle| \\ &\quad + 2\tanh\left(\sum_{j=k}^{\infty} r_j\right). \end{aligned} \tag{3.8}$$

Fix a positive number ε. By (3.2), we see that for each $k \in \mathbb{N}$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$|\langle \psi, y_k - T^i x_k \rangle| < \frac{\varepsilon}{3}. \tag{3.9}$$

In addition, since $\sum_{k \in \mathbb{N}} r_k < \infty$ and (3.5) holds, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\tanh\left(\sum_{j=k}^{\infty} r_j\right) < \frac{\varepsilon}{6} \quad \text{and} \quad \tanh(\rho(y^*, y_k)) < \frac{\varepsilon}{6}. \tag{3.10}$$

Thus by (3.8), (3.9) and (3.10), there exist $k_0 \in \mathbb{N}$ and a corresponding $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the inequality $|\langle \psi, y^* - x_{k_0+i} \rangle| < \varepsilon$ holds. Since ψ is arbitrary, we conclude that $\lim_{k\to\infty} x_k = y^*$ weakly.

To conclude, we note that $y^* \in \operatorname{Fix}(T)$. Indeed, by Remark 3.4 each y_k belongs to $\operatorname{Fix}(T)$. Since $\operatorname{Fix}(T)$ is ρ-closed by [16, Theorem 23.2, page 120], (3.5) implies that y^* also belongs to $\operatorname{Fix}(T)$. □

Theorem 3.5 can be extended to more general complete CAT(0) spaces and the weak convergence defined in them.

Theorem 3.6. *Let (X, d) be a complete CAT(0) space and consider a nonexpansive operator $T : X \to X$. Then the following two statements are equivalent:*

(i) *All exact orbits of T converge weakly;*
(ii) *All inexact orbits of T with summable errors converge weakly to fixed points of T.*

Proof. The implication (ii)⇒(i) is obvious. In order to prove (i)⇒(ii), suppose that all exact orbits of T converge weakly. Let $(x_k)_{k\in\mathbb{N}}$ be an inexact orbit of T with summable errors. By Proposition 3.1 and since all exact orbits of T are weakly convergent (in particular, bounded), $(x_k)_{k\in\mathbb{N}}$ is bounded. Moreover, there is a sequence $(r_k)_{k\in\mathbb{N}}$ of real numbers such that $d(Tx_k, x_{k+1}) \leq r_k$ for each $k \in \mathbb{N}$, where $\sum_{k\in\mathbb{N}} r_k < \infty$. Fix a positive integer k. By hypothesis, there is $y_k \in X$ such that $\lim_{i\to\infty} T^i x_k = y_k$ weakly; or equivalently, by Proposition 2.9,

$$\lim_{i\to\infty} d(y_k, P_{\gamma_k}(T^i x_k)) = 0 \quad \text{for all geodesics } \gamma_k \text{ through } y_k. \tag{3.11}$$

For each $i \in \mathbb{N}$, we have $d(T^i x_k, x_{k+i}) \leq \sum_{j=k}^{i+k-1} r_j$ (compare with (3.1)). Hence, for each fixed $q \in \mathbb{N}$, we have $d(T^q x_k, x_{k+q}) \leq \sum_{j=k}^{\infty} r_j$. This implies that

$$d(T^{q+i} x_k, T^i x_{k+q}) \leq d(T^q x_k, x_{k+q}) \leq \sum_{j=k}^{\infty} r_j \quad \text{for all } i \in \mathbb{N} \tag{3.12}$$

because T is nonexpansive. By Proposition 2.10 and (3.12), the following inequalities hold:

$$d(y_k, y_{k+q}) \leq \liminf_{i\to\infty} d(T^{q+i} x_k, T^i x_{k+q}) \leq \sum_{j=k}^{\infty} r_j. \tag{3.13}$$

Since $\sum_{j\in\mathbb{N}} r_j < \infty$ and since inequality (3.13) holds for each pair of positive integers (q, k), it follows that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. So there exists a point $y_* \in X$ such that

$$\lim_{k\to\infty} d(y_k, y_*) = 0. \tag{3.14}$$

To conclude, we need to prove that $(x_k)_{k\in\mathbb{N}}$ weakly converges to y_*. Let γ_* be a geodesic segment through y_*. Fix $i, k \in \mathbb{N}$. For all geodesic segments γ_k through y_k, by the triangle inequality and since P_{γ_k} is nonexpansive, we have

$$\begin{aligned} d(y_*, P_{\gamma_*}(x_{k+i})) \quad &\leq \quad d(y_*, y_k) + d(y_k, P_{\gamma_k}(T^i x_k)) \\ &\quad +d(T^i x_k, x_{k+i}) + d(P_{\gamma_k}(x_{k+i}), P_{\gamma_*}(x_{k+i})). \end{aligned} \tag{3.15}$$

Let $\varepsilon > 0$ be fixed. By (3.14), there exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$,

$$d(y_*, y_k) < \frac{\varepsilon}{4}. \tag{3.16}$$

By (3.11), there exists $i_0 \in \mathbb{N}$ such that for each $i \geq i_0$,

$$d(y_k, P_{\gamma_k}(T^i x_k)) < \frac{\varepsilon}{4}. \tag{3.17}$$

On the other hand, by (3.12) we know that

$$d(T^i x_k, x_{k+i}) \leq \sum_{j=k}^{\infty} r_j. \tag{3.18}$$

Fix $z \in \gamma_*$. For each $k \in \mathbb{N}$, let γ_k be the unique geodesic segment joining y_k and z. Since (3.14) holds, Theorem 2.7 implies that P_{γ_k} converges to P_{γ_*}, uniformly over $(x_k)_{k\in\mathbb{N}}$, which is a bounded subset of X. Therefore, for all $k \geq k_0$,

$$d(P_{\gamma_k}(x_{k+i}), P_{\gamma_*}(x_{k+i})) < \frac{\varepsilon}{4} \quad \text{for all } i \in \mathbb{N}. \tag{3.19}$$

Hence by (3.15), (3.16), (3.17), (3.18) and (3.19), there exist $k_0 \in \mathbb{N}$ and a corresponding $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$d(y_*, P_{\gamma_*}(x_{k_0+i})) < \varepsilon.$$

Since ϵ is arbitrary, this implies that $\lim_{k\to\infty} d(y_*, P_{\gamma_*}(x_k)) = 0$ for all geodesic segments γ_* through y_*, that is, $(x_k)_{k\in\mathbb{N}}$ converges weakly to y_* (see Proposition 2.9), as required. To conclude, note that by definition each y_k is the asymptotic center of the sequence $(T^i x_k)_{i\in\mathbb{N}}$. From the nonexpansivity of T, it follows that Ty_k is also the asymptotic center of this sequence, and consequently, $y_k = Ty_k$ for each $k \in \mathbb{N}$. Now (3.14) implies that $y^* \in \operatorname{Fix}(T)$. □

4. Banach spaces

In this section we use known results regarding the convergence of exact orbits of certain operators defined on Banach spaces to prove the convergence, either strong or weak, of some infinite products associated with these operators.

4.1. **The linear case.** Suppose E is a uniformly convex Banach space and $\{P_{S_k} : 1 \le k \le m\}$ are norm-one projections of E onto subspaces $\{S_k : 1 \le k \le m\}$. It is known that the strong

$$\lim_{n\to\infty} (P_{S_m} P_{S_{m-1}} \cdots P_{S_1})^n x = Px \tag{4.1}$$

exists for all $x \in E$. In addition, if $a_1, a_2, \dots, a_m \in (0,1)$ are numbers such that $a_1 + a_2 + \cdots + a_m = 1$, then the strong

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m} a_k P_{S_k} \right)^n x = Qx \tag{4.2}$$

also exists for all $x \in E$. Both P and Q define norm-one projections of E onto $S_1 \cap S_2 \cap \cdots \cap S_m$. For proofs of (4.1) and (4.2) we refer the reader to [10, Theorem 2.1] and [26, Theorem 1.7], respectively.

When these norm-one projections are approximated by certain possibly nonlinear, even discontinuous operators, their infinite products converge, as we state and prove below.

Theorem 4.1. *Suppose E is a uniformly convex Banach space and let $\{P_{S_k} : 1 \le k \le m\}$ be norm-one projections of E onto subspaces $\{S_k : 1 \le k \le m\}$. Let the given, possibly nonlinear operators $A_n^{(k)} : E \to E$, $k = 1, 2, \dots, m$; $n \in \mathbb{N}$, satisfy for all $x \in E$ the inequalities*

$$\|A_n^{(k)} x - P_{S_k} x\| \le \gamma_n \|x\| \tag{4.3}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then, for each $x \in E$, there exists a point $\overline{x} = \overline{x}(x) \in S_1 \cap S_2 \cap \cdots \cap S_m$ such that

$$\lim_{n\to\infty} \left\| \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x - \overline{x} \right\| = 0.$$

Proof. Given $x \in E$, consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined as follows:

$$x_1 = x \quad \text{and} \quad x_{n+1} = A_n^{(m)} A_n^{(m-1)} \cdots A_n^{(1)} x_n \quad \text{for all } n \in \mathbb{N}.$$

Using (4.3) and the fact that each operator P_{S_k} is linear and a norm-one projection, we can proceed as in [22, Theorem 2.2] and prove that $(x_n)_{n\in\mathbb{N}}$ is an inexact orbit of $P_{S_m} \cdots P_{S_1}$ with summable errors. Since all exact orbits of $P_{S_m} \cdots P_{S_1}$ converge to points in $S_1 \cap \cdots \cap S_m$ (see (4.1)), it follows from Theorem 3.2 and (2.1) that there exists a point $\overline{x} = \overline{x}(x) \in S_1 \cap S_2 \cap \cdots \cap S_m$ such that $\lim_{n\to\infty} \|x_n - \overline{x}\| = 0$. This concludes the proof. □

We say that Theorem 4.1 is a "linear case" since in (4.3) we use the operators $(A_n^{(k)})_{n\in\mathbb{N}}$ to approximate the linear norm-one projections P_{S_k}.

We now provide an analog of Theorem 4.1, where the products of convex combinations of these possibly nonlinear operators are considered.

Theorem 4.2. *Suppose E is a uniformly convex Banach space and let $\{P_{S_k} : 1 \le k \le m\}$ be norm-one projections of E onto subspaces $\{S_k : 1 \le k \le m\}$. Let the given, possibly nonlinear operators $A_n^{(k)} : E \to E$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in E$ the inequalities*

$$\|A_n^{(k)}x - P_{S_k}x\| \le \gamma_n \|x\| \tag{4.4}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then, for each $x \in E$, there exists a point $\overline{x} = \overline{x}(x) \in S_1 \cap S_2 \cap \cdots \cap S_m$ such that

$$\lim_{n\to\infty} \left\| \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x - \overline{x} \right\| = 0,$$

where $a_k \in (0, 1)$ for each $k = 1, \ldots, m$ and $a_1 + a_2 + \cdots + a_m = 1$.

Proof. For $x \in E$, we inductively define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = x$ and $x_{n+1} = \sum_{k=1}^{m} a_k A_n^{(k)} x_n$ for all $n \in \mathbb{N}$. By the triangle inequality, (4.4) and by using the fact that P_{S_k} is a norm-one projection for each $k = 1, \ldots, m$, we obtain

$$\left\| \sum_{k=1}^{m} a_k A_n^{(k)} x \right\| \le \sum_{k=1}^{m} a_k (\|A_n^{(k)}x - P_{S_k}x\| + \|P_{S_k}x\|) \le (1 + \gamma_n)\|x\|.$$

This implies that

$$\|x_n\| = \left\| \prod_{j=1}^{n-1} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x \right\| \le \prod_{j=1}^{n-1} (1 + \gamma_j)\|x\| \tag{4.5}$$

for any $n \ge 2$. Since $\sum_{n\in\mathbb{N}} \gamma_n < \infty$, we know that $\prod_{n\in\mathbb{N}}(1 + \gamma_j) < \infty$ (see, for example, [13, Proposition VII.5.4]). Combining this fact with (4.5), we see that there exists a number $M > 0$ such that

$$\|x_n\| \le M\|x\| \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

Let $T = \sum_{k=1}^{m} a_k P_{S_k}$. By the triangle inequality, (4.4) and (4.6), we obtain

$$\|x_{n+1} - Tx_n\| \le \sum_{k=1}^{m} a_k \|A_n^{(k)} x_n - P_{S_k} x_n\| \le \gamma_n \|x_n\| \le \gamma_n M\|x\|,$$

which proves that the sequence $(x_n)_{n\in\mathbb{N}}$ is an inexact orbit of T with summable errors. By (4.2), we know that all exact orbits of T converge. By Theorem 3.2, this implies that all inexact orbits of T with summable errors converge to fixed points of T. Thus there exists a point $\overline{x} \in \operatorname{Fix}(T)$ such that

$$0 = \lim_{n\to\infty} \|x_n - \overline{x}\| = \lim_{n\to\infty} \left\| \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x - \overline{x} \right\|.$$

Using (2.2), we conclude that $\overline{x} \in S_1 \cap \cdots \cap S_m = \operatorname{Fix}(T)$. □

We see that in the proofs of both Theorem 4.1 and Theorem 4.2, neither continuity nor linearity of the operators $A_n^{(k)}$ are needed.

We finish this subsection with a case where the uniform convexity of the Banach space E is replaced with another assumption. Suppose $\{P_{S_k} : 1 \le k \le m\}$ are norm-one projections of E onto subspaces $\{S_k : 1 \le k \le m\}$. Consider the convex multiplicative semigroup generated by $P_{S_1}, \ldots, P_{S_m}$ and denote it by $\mathbf{S} = \mathbf{S}(P_{S_1}, \ldots, P_{S_m})$. In other words, $\mathbf{S}$ is the convex hull of the semigroup consisting of all products with factors from $\{P_{S_1}, \ldots, P_{S_m}\}$.

If E is a uniformly smooth complex Banach space, then for every operator $T \in \mathbf{S}(P_{S_1}, \ldots, P_{S_m})$, the strong

$$\lim_{n\to\infty} T^n x = Px \tag{4.7}$$

exists for all $x \in E$ and defines a norm-one projection P of E onto $\operatorname{Fix}(T)$ (see [4, Main Theorem]). Hence, if there exist a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ and a convergent series of positive numbers $\sum_{n\in\mathbb{N}} \gamma_n$ satisfying the inequalities

$$\|x_{n+1} - Tx_n\| \le \gamma_n \quad \text{for all } n \in \mathbb{N}, \tag{4.8}$$

then there exists a point $\overline{x} \in \operatorname{Fix}(T)$ such that

$$\lim_{n\to\infty} \|x_n - \overline{x}\| = 0. \tag{4.9}$$

This is true because by (4.8), we see that $(x_n)_{n\in\mathbb{N}}$ is an inexact orbit of T with summable errors. Since all the exact orbits of T converge by (4.7), we conclude by Theorem 3.2 that there is a point $\overline{x} \in \operatorname{Fix}(T)$ satisfying (4.9).

Below we consider a particular element of $\mathbf{S}(P_{S_1}, \ldots, P_{S_m})$, but the proof can be used as a blueprint for proving analogous results for any operator in $\mathbf{S}(P_{S_1}, \ldots, P_{S_m})$.

Theorem 4.3. *Suppose E is a uniformly smooth complex Banach space and let $\{P_{S_k} : 1 \le k \le m\}$ be norm-one projections of E onto subspaces $\{S_k : 1 \le k \le m\}$. Let the given, possibly nonlinear operators $A_n^{(k)} : E \to E$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in E$ the inequalities*

$$\|A_n^{(k)} x - P_{S_k} x\| \le \gamma_n \|x\| \tag{4.10}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then, for each $x \in E$, there exists a point $\overline{x} = \overline{x}(x) \in \operatorname{Fix}(\sum_{k=1}^{m-1} a_k P_{S_{k+1}} P_{S_k})$ such that

$$\lim_{n\to\infty} \left\| \prod_{j=1}^{n} \left(\sum_{k=1}^{m-1} a_k A_j^{(k+1)} A_j^{(k)} \right) x - \overline{x} \right\| = 0,$$

where $a_k \in (0, 1)$ for all $k = 1, \ldots, m-1$ and $a_1 + a_2 + \cdots + a_{m-1} = 1$.

Proof. Take $x \in E$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_1 = x \quad \text{and} \quad x_{n+1} = \left(\sum_{k=1}^{m-1} a_k A_n^{(k+1)} A_n^{(k)}\right) x_n \quad \text{for all } n \in \mathbb{N}.$$

Using the fact that P_{S_k} is a norm-one projection, the triangle inequality and (4.10), we get

$$\|A_n^{(k)} x\| \le \|A_n^{(k)} x - P_{S_k} x\| + \|P_{S_k} x\| \le (1+\gamma_n)\|x\|, \tag{4.11}$$

which implies that

$$\|A_n^{(k+1)} A_n^{(k)} x\| \le (1+\gamma_n)\|A_n^{(k)} x\| \le (1+\gamma_n)^2 \|x\|. \tag{4.12}$$

Since $a_1 + \cdots + a_m = 1$, using the triangle inequality and (4.12), we obtain

$$\left\|\sum_{k=1}^{m-1} a_k A_n^{(k+1)} A_n^{(k)} x\right\| \le (1+\gamma_n)^2 \|x\|. \tag{4.13}$$

Using (4.13), we arrive at the following inequality:

$$\|x_n\| \le \left(\prod_{j=1}^{n-1} (1+\gamma_j)^2\right) \|x\|. \tag{4.14}$$

When we use the triangle inequality, (4.10), (4.11) and the fact that P_{S_k} is nonexpansive, we get

$$\begin{aligned} & \|A_n^{(k)} A_n^{(k-1)} x_n - P_{S_k} P_{S_{k-1}} x_n\| \\ \le\ & \|A_n^{(k)} A_n^{(k-1)} x_n - P_{S_k} A_n^{(k-1)} x_n\| + \|P_{S_k} A_n^{(k-1)} x_n - P_{S_k} P_{S_{k-1}} x_n\| \\ \le\ & \gamma_n (1+\gamma_n)\|x_n\| + \gamma_n \|x_n\|. \end{aligned} \tag{4.15}$$

Let $T = \sum_{k=1}^{m-1} a_k P_{S_{k+1}} P_{S_k} \in \mathbf{S}(P_{S_1}, \ldots, P_{S_m})$. By the triangle inequality, (4.14) and (4.15), we get

$$\begin{aligned} \|x_{n+1} - T x_n\| &\le \sum_{k=1}^{m-1} a_k \left\|A_n^{(k+1)} A_n^{(k)} x_n - P_{S_{k+1}} P_{S_k} x_n\right\| \\ &\le \gamma_n (1+\gamma_n) \prod_{j=1}^{n-1} (1+\gamma_j)^2 \|x\| + \gamma_n \prod_{j=1}^{n-1} (1+\gamma_j)^2 \|x\|. \end{aligned} \tag{4.16}$$

Note that $\sum_{n\in\mathbb{N}} \gamma_n < \infty$ implies that $\prod_{n\in\mathbb{N}} (1+\gamma_j) < \infty$. Using this fact along with (4.16), we see that there exists a number $M > 0$ such that

$$\|x_{n+1} - T x_n\| \le \gamma_n M \|x\|.$$

Thus the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies (4.8) and consequently, there is a point $\overline{x} \in \operatorname{Fix}(T)$ such that

$$0 = \|x_n - \overline{x}\| = \lim_{n\to\infty} \left\|\prod_{j=1}^{n} \left(\sum_{k=1}^{m-1} a_k A_j^{(k+1)} A_j^{(k)}\right) x - \overline{x}\right\|.$$

□

4.2. **The nonlinear case.** Now suppose E is a smooth and uniformly convex Banach space. If $\{R_{F_k} : 1 \le k \le m\}$ are sunny nonexpansive retractions of a symmetric, closed and convex subset $C \subset E$ onto symmetric, closed and convex subsets $\{F_k \subset C : 1 \le k \le m\}$, then the strong

$$\lim_{n\to\infty} (R_{F_m} R_{F_{m-1}} \cdots R_{F_1})^n x = Rx \tag{4.17}$$

exists for all $x \in C$ (see [10, Theorem 2.2]). Moreover, when $a_1, a_2, \ldots, a_m \in (0,1)$ are numbers such that $a_1 + a_2 + \cdots + a_m = 1$, then the strong

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m} a_k R_{F_k} \right)^n x = Qx \tag{4.18}$$

also exists for all $x \in C$ (see [26, Theorem 2.3]). In (4.17) and (4.18), R and Q are nonexpansive retractions of C onto $F_1 \cap F_2 \cap \cdots \cap F_m$.

Below we replace norm-one projections by nonexpansive retractions to obtain a nonlinear analogue of Theorem 4.1.

Theorem 4.4. *Suppose E is a smooth and uniformly convex Banach space. Let C be a symmetric, closed and convex subset of E, and let $\{R_{F_k} : 1 \le k \le m\}$ be sunny nonexpansive retractions of C onto symmetric, closed and convex subsets $\{F_k \subset C : 1 \le k \le m\}$. Let the given, possibly nonlinear operators $A_n^{(k)} : C \to C$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in C$ the inequalities*

$$\|A_n^{(k)} x - R_{F_k} x\| \le \gamma_n \|x\| \tag{4.19}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then, for each $x \in C$, there exists a point $\overline{x} = \overline{x}(x) \in F_1 \cap F_2 \cap \cdots \cap F_m$ such that

$$\lim_{n\to\infty} \left\| \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x - \overline{x} \right\| = 0.$$

Proof. For $x \in C$, consider the sequence $(x_n)_{n\in\mathbb{N}}$, where

$$x_1 = x \quad \text{and} \quad x_{n+1} = A_n^{(m)} A_n^{(m-1)} \cdots A_n^{(1)} x_n \quad \text{for all } n \in \mathbb{N}.$$

By Remark 2.1, for each $k = 1, \ldots, m$ and $x \in C$, we have

$$\|R_{F_k} x\| \le \|x\|. \tag{4.20}$$

By the triangle inequality, (4.19) and (4.20), for all $n \in \mathbb{N}$ and $k = 1, \ldots, m$, we obtain

$$\|A_n^{(k)} A_n^{(k-1)} \cdots A_n^{(1)} x\| \le (1 + \gamma_n)^k \|x\| \quad \text{for all } n \in \mathbb{N}. \tag{4.21}$$

Since $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, we know that $\prod_{j\in\mathbb{N}} (1 + \gamma_j) < \infty$. When combined with (4.21) for $k = m$, this fact gives us a number $M > 0$ such that for all $n \ge 2$,

$$\|x_n\| \le \left(\prod_{j=1}^{n-1} (1 + \gamma_j)^m \right) \|x\| \le M \|x\|. \tag{4.22}$$

For $k = 1, \dots, m$ and $n \in \mathbb{N}$, consider the operator $\alpha_n^{(k)}(x) := A_n^{(k)}x - R_{F_k}x$ defined on C. By (4.19), this operator satisfies the following inequality:

$$\|\alpha_n^{(k)}(x)\| \le \gamma_n \|x\|. \tag{4.23}$$

Let $n \in \mathbb{N}$ be fixed. We consider $x_{n+1}^{(1)} := A_n^{(1)}x_n$ and $x_{n+1}^{(k+1)} := A_n^{(k+1)}x_{n+1}^{(k)} = \alpha_n^{(k+1)}(x_{n+1}^{(k)}) + R_{F_{k+1}}x_{n+1}^{(k)}$ for all $k = 1, \dots, m-1$. In particular, we see that

$$x_{n+1}^{(m)} = A_n^{(m)}A_n^{(m-1)} \cdots A_n^{(1)}x_n = x_{n+1}. \tag{4.24}$$

From the triangle inequality, (4.20) and (4.23), it follows that

$$\|x_{n+1}^{(k)}\| \le (1+\gamma_n)^k \|x_n\| \tag{4.25}$$

for each $k = 1, \dots, m$. So, for each $k = 2, \dots, m$, we obtain

$$\|R_{F_k}x_{n+1}^{(k-1)} - R_{F_k}R_{F_{k-1}} \cdots R_{F_1}x_n\| \le \sum_{j=2}^{k-1} \|\alpha_n^{(j)}(x_{n+1}^{(j-1)})\| + \|\alpha_n^{(1)}(x_n)\|. \tag{4.26}$$

Let $T = R_{F_m}R_{F_{m-1}} \cdots R_{F_1}$. By the triangle inequality, (4.23), (4.24), (4.25) and (4.26), we get

$$\begin{aligned} \|x_{n+1} - Tx_n\| &\le \|\alpha_n^{(m)}(x_{n+1}^{(m-1)})\| + \|R_{F_m}x_{n+1}^{(m-1)} - R_{F_m}R_{F_{m-1}} \cdots R_{F_1}x_n\| \\ &\le \gamma_n[(1+\gamma_n)^{m-1} + (1+\gamma_n)^{m-2} + \cdots + (1+\gamma_n) + 1]\|x_n\|. \end{aligned}$$

Without loss of generality, we may assume that $\gamma_n \le 1$ for each $n \in \mathbb{N}$, so by the above estimate we have

$$\|x_{n+1} - Tx_n\| \le \gamma_n[2^{m-1} + 2^{m-2} + \cdots + 2 + 1]\|x_n\| \le 2^m\gamma_n\|x_n\|.$$

By (4.22), it is clear that there is a number $M > 0$ such that

$$\|x_{n+1} - Tx_n\| \le 2^m\gamma_n M\|x\| \quad \text{for all } n \in \mathbb{N}.$$

This shows that $(x_n)_{n\in\mathbb{N}}$ is an inexact orbit of T with summable errors. By (4.17), all exact orbits of T converge; therefore by Theorem 3.2, there is a point $\overline{x} \in \operatorname{Fix}(T)$ such that

$$0 = \lim_{n\to\infty} \|x_n - \overline{x}\| = \left\| \left(\prod_{j=1}^{n} A_j^{(m)}A_j^{(m-1)} \cdots A_j^{(1)} \right) x - \overline{x} \right\|.$$

By (2.3), we know that $\overline{x} \in F_1 \cap \cdots \cap F_m$. This concludes the proof. □

Note that no expression of the type $\|A_n^{(k)}x - A_n^{(k)}y\|$ was involved in the proof of Theorem 4.4. So even if an operator $A_n^{(k)}$ is discontinuous at some point of C, our result remains true. Similarly, we can prove convergence of the products of convex combinations of these operators $A_n^{(k)}$.

Theorem 4.5. *Suppose E is a smooth and uniformly convex Banach space. Let C be a symmetric, closed and convex subset of E, and let $\{R_{F_k} : 1 \le k \le m\}$ be sunny nonexpansive retractions of C onto symmetric, closed and convex subsets*

$\{F_k \subset C : 1 \le k \le m\}$. *Let the given, possibly nonlinear operators* $A_n^{(k)} : C \to C$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, *satisfy for all* $x \in C$ *the inequalities*

$$\|A_n^{(k)} x - R_{F_k} x\| \le \gamma_n \|x\| \tag{4.27}$$

for some positive numbers γ_n *with* $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. *Then, for each* $x \in C$, *there exists a point* $\overline{x} = \overline{x}(x) \in F_1 \cap F_2 \cap \cdots \cap F_m$ *such that*

$$\lim_{n\to\infty} \left\| \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x - \overline{x} \right\| = 0,$$

where $a_k \in (0, 1)$ *for each* $k = 1, \ldots, m$ *and* $a_1 + a_2 + \cdots + a_m = 1$.

Proof. Given $x \in C$, consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = x$ and $x_{n+1} = \left(\sum_{k=1}^{m} a_k A_n^{(k)}\right) x_n$ for all $n \in \mathbb{N}$. Using (4.27), it is not difficult to prove that $(x_n)_{n\in\mathbb{N}}$ is an inexact orbit of $T = \sum_{k=1}^{m} a_k R_{F_k}$ with summable errors. By (4.18), we know that all exact orbits of T converge. Therefore, by Theorem 3.2, there exists a point $\overline{x} \in \operatorname{Fix}(T)$ such that

$$0 = \lim_{n\to\infty} \|x_n - \overline{x}\| = \lim_{n\to\infty} \left\| \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x - \overline{x} \right\|.$$

The assertion that $\overline{x} \in F_1 \cap F_2 \cap \cdots \cap F_m$ follows from (2.4). □

Note that also in this case no assumption concerning the continuity of the operators $A_n^{(k)}$ is needed.

4.3. **Weak convergence.** If we dispense with symmetry of the subsets $F_1, \ldots, F_m$ and C, at least weak convergence in (4.17) and (4.18) can be obtained. To this end, suppose that both E and E^* are uniformly convex Banach spaces. Recall that E^* is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable; see, for instance, [15, Theorem 9.9]. Let C be a closed and convex subset of E, and let $\{R_{F_k} : 1 \le k \le m\}$ be sunny nonexpansive retractions of C onto closed and convex subsets $\{F_k \subset C : 1 \le k \le m\}$. Assume that $F_1 \cap F_2 \cap \cdots \cap F_m \ne \emptyset$. Then the weak

$$\lim_{n\to\infty} (R_{F_m} R_{F_{m-1}} \cdots R_{F_1})^n x = Rx \tag{4.28}$$

exists for all $x \in C$ and defines a nonexpansive retraction R of C onto $F_1 \cap F_2 \cap \cdots \cap F_m$. Moreover, if $a_1, a_2, \ldots, a_m \in (0, 1)$ are numbers such that $a_1 + a_2 + \cdots + a_m = 1$, then the weak

$$\lim_{n\to\infty} \left(\sum_{k=1}^{m} a_k R_{F_k} \right)^n x = Qx \tag{4.29}$$

exists for each $x \in C$ and defines a nonexpansive retraction Q of C onto $F_1 \cap F_2 \cap \cdots \cap F_m$. For a proof of (4.28) and (4.29) we refer the reader to [26, Proposition 2.4] and [20, Theorem 4.8], respectively.

Theorem 4.6. *Suppose E and E^* are uniformly convex Banach spaces. Let C be a closed and convex subset of E, and let $\{R_{F_k} : 1 \le k \le m\}$ be sunny nonexpansive retractions of C onto closed subsets $\{F_k \subset C : 1 \le k \le m\}$ such that $F_1 \cap F_2 \cap \cdots \cap F_m \neq \emptyset$. Let the given, possibly nonlinear operators $A_n^{(k)} : C \to C$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in C$ the inequalities*

$$\|A_n^{(k)} x - R_{F_k} x\| \le \gamma_n \|x\| \tag{4.30}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then for each $x \in C$, there is a point $\overline{x} = \overline{x}(x) \in F_1 \cap F_2 \cap \cdots \cap F_m$ such that

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x = \overline{x} \quad \textit{weakly.}$$

Proof. Fix $z \in F_1 \cap F_2 \cap \cdots \cap F_m = \operatorname{Fix}(R_{F_m} \cdots R_{F_1})$. For each $k = 1, \ldots, m$ and $n \in \mathbb{N}$, define the subsets $\widetilde{F}_k := F_k - z \subset \widetilde{C} := C - z$. Consider also the operators $\widetilde{A}_n^{(k)} w := A_n^{(k)} x - z$ and $R_{\widetilde{F}_k} w := R_{F_k} x - z$, where $w \in \widetilde{C}$ and $x \in C$ are such that $w = x - z$. So $0 \in \widetilde{F}_k = \operatorname{Fix}(R_{\widetilde{F}_k})$ for each $k = 1, \ldots, m$. Since R_{F_k} is a nonexpansive retraction, so is $R_{\widetilde{F}_k}$. In particular,

$$\|R_{\widetilde{F}_k} w\| = \|R_{\widetilde{F}_k} w - R_{\widetilde{F}_k} 0\| \le \|w\| \quad \text{for all} \quad w \in \widetilde{C}. \tag{4.31}$$

It follows from (4.30), (4.31) and the triangle inequality that for each $k = 1, \ldots, m$ and $n \in \mathbb{N}$,

$$\|\widetilde{A}_n^{(k)} w - R_{\widetilde{F}_k} w\| = \|A_n^{(k)} x - R_{F_k} x\| \le \gamma_n \|w\| + \gamma_n \|z\| \tag{4.32}$$

for all $w \in \widetilde{C}$ and $x \in C$ such that $w = x - z$. Given $x \in C$, consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_1 = x$ and $x_{n+1} = A_n^{(m)} A_n^{(m-1)} \cdots A_n^{(1)} x_n$ for all $n \in \mathbb{N}$. We also define a sequence $(w_n)_{n\in\mathbb{N}} \subset \widetilde{C}$ by setting

$$w_1 = w = x - z \quad \text{and} \quad w_{n+1} = \widetilde{A}_n^{(m)} \widetilde{A}_n^{(m-1)} \cdots \widetilde{A}_n^{(1)} w_n \quad \text{for all } n \in \mathbb{N}.$$

Note that $w_n = x_n - z$ for all $n \in \mathbb{N}$. So by (4.31), (4.32) and the triangle inequality, we see that

$$\|\widetilde{A}_n^{(m)} \widetilde{A}_n^{(m-1)} \cdots \widetilde{A}_n^{(1)} w\| \le (1 + \gamma_n)^m \|w\| + \gamma_n \|z\| \sum_{i=0}^{m-1} (1 + \gamma_n)^i. \tag{4.33}$$

Using induction over n and inequality (4.33), we get

$$\begin{aligned} \|w_n\| \;&\le\; \prod_{j=1}^{n-1} (1 + \gamma_j)^m \|w\| + \gamma_{n-1} \|z\| \sum_{i=0}^{m-1} (1 + \gamma_{n-1})^i \\ &\quad + \|z\| \sum_{\ell=1}^{n-2} \gamma_\ell \prod_{j=\ell+1}^{n-1} (1 + \gamma_j)^m \sum_{i=0}^{m-1} (1 + \gamma_\ell)^i. \end{aligned} \tag{4.34}$$

Since $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, we know that $\prod_{j\in\mathbb{N}} (1 + \gamma_j) < \infty$. Combining these facts with (4.34), we see that there exist numbers $M_1, M_2, M_3 > 0$ satisfying $\prod_{j=p}^{n-1} (1 + \gamma_j)^m \le$

M_1 for all $n \geq p+1$, $\sum_{i=0}^{m-1}(1+\gamma_n)^i \leq M_2$ and $\sum_{i=1}^{n-1}\gamma_i \leq M_3$ for all $n \geq 2$, so that $\|w_n\| \leq M_1\|w\| + M_1M_2M_3\|z\|$. Hence there exists a number $M > 0$ such that

$$\|w_n\| \leq M(\|w\| + \|z\|) \quad \text{for all } n \in \mathbb{N}. \tag{4.35}$$

Now on $\widetilde{C}$, consider the operator $\widetilde{\alpha}_n^{(k)}(w) := \widetilde{A}_n^{(k)}w - R_{\widetilde{F}_k}w$ for $k = 1, \ldots, m$ and $n \in \mathbb{N}$. By (4.32), we see that the inequality

$$\|\widetilde{\alpha}_n^{(k)}(w)\| \leq \gamma_n\|w\| + \gamma_n\|z\| \tag{4.36}$$

holds. For $n \in \mathbb{N}$ fixed, we now define $w_{n+1}^{(1)} := \widetilde{A}_n^{(1)}w_n = \widetilde{\alpha}_n^{(1)}(w_n) + R_{\widetilde{F}_1}w_n$ and $w_{n+1}^{(k+1)} := \widetilde{A}_n^{(k+1)}w_n^{(k)} = \widetilde{\alpha}_n^{(k+1)}(w_{n+1}^{(k)}) + R_{\widetilde{F}_{k+1}}w_{n+1}^{(k)}$ for $k = 1, \ldots, m-1$. In particular, $w_{n+1}^{(m)} = w_{n+1}$. So by the triangle inequality, (4.31) and (4.36), we get the following inequality:

$$\|w_{n+1}^{(k)}\| \leq (1+\gamma_n)^k\|w_n\| + \gamma_n\left(\sum_{i=0}^{k-1}(1+\gamma_n)^i\right)\|z\| \tag{4.37}$$

for all $k = 1, \ldots, m$. Since $R_{\widetilde{F}_k}$ is nonexpansive for each $k = 2, \ldots, m$, the following inequality holds:

$$\|R_{\widetilde{F}_k}w_{n+1}^{(k-1)} - R_{\widetilde{F}_k}\cdots R_{\widetilde{F}_1}w_n\| \leq \sum_{j=2}^{k-1}\|\widetilde{\alpha}_n^{(j)}(w_{n+1}^{(j-1)})\| + \|\widetilde{\alpha}_n^{(1)}(w_n)\|. \tag{4.38}$$

Let $\widetilde{T} = R_{\widetilde{F}_m}R_{\widetilde{F}_{m-1}}\cdots R_{\widetilde{F}_1}$. By the triangle inequality, (4.36), (4.37), (4.38) and since each $R_{\widetilde{F}_k}$ is nonexpansive, we have

$$\begin{aligned}
&\|w_{n+1} - \widetilde{T}w_n\| \\
\leq\ & \|\widetilde{\alpha}_n^{(m)}(w_{n+1}^{(m-1)})\| + \|R_{\widetilde{F}_m}w_{n+1}^{(m-1)} - R_{\widetilde{F}_m}R_{\widetilde{F}_{m-1}}\cdots R_{\widetilde{F}_1}w_n\| \\
\leq\ & \gamma_n\sum_{j=0}^{m-1}(1+\gamma_n)^j\|w_n\| + \gamma_n\left(\gamma_n\sum_{j=1}^{m-1}\sum_{i=0}^{j-1}(1+\gamma_n)^i + m\right)\|z\|.
\end{aligned}$$

Without loss of generality, we may assume that $\gamma_n \leq 1$ for all $n \in \mathbb{N}$. So by the above inequality we obtain

$$\begin{aligned}
\|w_{n+1} - \widetilde{T}w_n\| &\leq \gamma_n 2^{m-1}\|w_n\|\sum_{j=0}^{m-1}\frac{1}{2^j} + \gamma_n m\left(2^{m-1}\sum_{i=0}^{m-1}\frac{1}{2^i} + 1\right)\|z\| \\
&\leq \gamma_n 2^m\|w_n\| + \gamma_n m\,(2^m + 1)\,\|z\|.
\end{aligned} \tag{4.39}$$

Hence by (4.35) and (4.39), there is a number $M > 0$ such that

$$\|w_{n+1} - \widetilde{T}w_n\| \leq \gamma_n[2^mM\|w\| + (2^mM + m2^m + m)\|z\|],$$

which proves that $(w_n)_{n\in\mathbb{N}}$ is an inexact orbit of $\widetilde{T}$ with summable errors. By (4.28), we know that all exact orbits of $\widetilde{T}$ converge weakly to fixed points of $\widetilde{T}$. So, by Theorem 3.3 there exists a point $\overline{w} \in \operatorname{Fix}(\widetilde{T})$ such that $\lim_{n\to\infty} w_n = \overline{w}$ weakly.

From (2.3) it follows that $\overline{w} \in \widetilde{F}_1 \cap \cdots \cap \widetilde{F}_m$. Since $(w_n)_{n\in\mathbb{N}} = (x_n - z)_{n\in\mathbb{N}}$, there exists a point $\overline{x} \in F_1 \cap \cdots \cap F_m$ such that

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x = \overline{x} \quad \text{weakly},$$

as asserted. □

To conclude this section, we prove that weak convergence of infinite products of convex combinations of nonlinear operators also holds under the conditions of Theorem 4.6.

Theorem 4.7. *Suppose E and E^* are uniformly convex Banach spaces. Let C be a closed and convex subset of E, and let $\{R_{F_k} : 1 \le k \le m\}$ be sunny nonexpansive retractions of C onto closed and convex subsets $\{F_k \subset C : 1 \le k \le m\}$. Assume that $F_1 \cap F_2 \cap \cdots \cap F_m \neq \emptyset$. Let the given, possibly nonlinear operators $A_n^{(k)} : C \to C$, $k = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in C$ the inequalities*

$$\|A_n^{(k)} x - R_{F_k} x\| \le \gamma_n \|x\| \tag{4.40}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then for each $x \in C$, there exists a point $\overline{x} = \overline{x}(x) \in F_1 \cap F_2 \cap \cdots \cap F_m$ such that

$$\lim_{n\to\infty} \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x = \overline{x} \quad \textit{weakly},$$

where $a_k \in (0,1)$ for each $k = 1, \ldots, m$ and $a_1 + a_2 + \cdots + a_m = 1$.

Proof. Let $z \in F_1 \cap F_2 \cap \cdots \cap F_m$ be fixed. For each $k = 1, \ldots, m$, consider the subsets $\widetilde{F}_k := F_k - z \subset \widetilde{C} := C - z$, and the operators $R_{\widetilde{F}_k}$ and $\widetilde{A}_n^{(k)}$ defined as before. By (4.40) and the triangle inequality, we get

$$\|\widetilde{A}_n^{(k)} w - R_{\widetilde{F}_k} w\| \le \gamma_n \|w\| + \gamma_n \|z\| \tag{4.41}$$

for all $w \in \widetilde{C}$, $k = 1, \ldots, m$ and $n \in \mathbb{N}$. Take $x \in C$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_1 = x$ and $x_{n+1} = \sum_{k=1}^{m} a_k A_n^{(k)} x_n$ for all $n \in \mathbb{N}$. Define also the sequence $(w_n)_{n\in\mathbb{N}}$ by $w_1 = w = x - z$ and $w_{n+1} = \sum_{k=1}^{m} a_k \widetilde{A}_n^{(k)} w_n$ for all $n \in \mathbb{N}$. Note that $w_n = x_n - z$ for all $n \in \mathbb{N}$. Using the triangle inequality, (4.31) and (4.41), we obtain

$$\|\widetilde{A}_n^{(k)} w\| \le \|\widetilde{A}_n^{(k)} w - R_{\widetilde{F}_k} w\| + \|R_{\widetilde{F}_k} w\| \le (1 + \gamma_n)\|w\| + \gamma_n \|z\|. \tag{4.42}$$

Since $a_1 + a_2 + \cdots + a_m = 1$, by (4.42) and the triangle inequality, we get

$$\left\| \sum_{k=1}^{m} a_k \widetilde{A}_n^{(k)} w \right\| \le \sum_{k=1}^{m} a_k \|\widetilde{A}_n^{(k)} w\| \le (1 + \gamma_n)\|w\| + \gamma_n \|z\|.$$

Hence we obtain by induction the following inequality:

$$\|w_n\| \le \prod_{j=1}^{n-1} (1 + \gamma_j)\|w\| + \sum_{i=1}^{n-2} \gamma_i \|z\| \prod_{j=i+1}^{n-1} (1 + \gamma_j) + \gamma_{n-1} \|z\|, \tag{4.43}$$

where $n \geq 2$. Since $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, we know that $\prod_{j\in\mathbb{N}}(1+\gamma_j) < \infty$. Combining these facts with (4.43), we can find a number $M > 0$ such that

$$\|w_n\| \leq M(\|w\| + \|z\|) \quad \text{for all } n \in \mathbb{N}. \tag{4.44}$$

Now consider the operator $\widetilde{T} = a_1 R_{\widetilde{F}_1} + a_2 R_{\widetilde{F}_2} + \cdots + a_m R_{\widetilde{F}_m}$. By the triangle inequality, (4.41) and (4.44), we get, for all $n \in \mathbb{N}$,

$$\|w_{n+1} - \widetilde{T} w_n\| \leq \gamma_n (M\|w\| + (M+1)\|z\|),$$

which proves that $(w_n)_{n\in\mathbb{N}}$ is an inexact orbit of $\widetilde{T}$ with summable errors. By Theorem (4.29), we know that the exact orbits of $\widetilde{T}$ converge weakly, so Theorem 3.3 implies that there exists $\widetilde{w} \in \operatorname{Fix}(\widetilde{T})$ such that $\lim_{n\to\infty} w_n = \overline{w}$ weakly. From (2.3) we see that $\widetilde{w} \in \widetilde{F}_1 \cap \cdots \cap \widetilde{F}_m$. Thus there exists a point $\overline{x} = \overline{w} + z \in F_1 \cap F_2 \cap \cdots \cap F_m$ such that

$$\lim_{n\to\infty} \prod_{j=1}^{n} \left(\sum_{k=1}^{m} a_k A_j^{(k)} \right) x = \overline{x} \quad \text{weakly,}$$

as asserted. □

It is important to note that continuity, either strong or weak, of the operators $A_n^{(k)}$ is not relevant to the results in this subsection.

5. The Hilbert ball

Let $\{P_{K_i} : 1 \leq i \leq m\}$ be the nearest point projections of $\mathbb{B}$ onto ρ-closed and ρ-convex subsets $\{K_i \subset \mathbb{B} : 1 \leq i \leq m\}$. If $K_1 \cap K_2 \cap \cdots \cap K_m \neq \emptyset$, then the weak

$$\lim_{n\to\infty} (P_{K_m} P_{K_{m-1}} \cdots P_{K_1})^n x = Px \tag{5.1}$$

exists for all $x \in \mathbb{B}$ and defines a ρ-nonexpansive retraction P of $\mathbb{B}$ onto $K_1 \cap K_2 \cap \cdots \cap K_m$ ([27, Main Theorem]). When $m = 2$, the sequence

$$\left(\left(\frac{1}{2} P_{K_1} \oplus \frac{1}{2} P_{K_2} \right)^n x \right)_{n\in\mathbb{N}} \tag{5.2}$$

also converges weakly for each $x \in \mathbb{B}$ to a point in $K_1 \cap K_2$ ([5, Corollary 9.6]).

Now we present similar results to those obtained in the previous section for the infinite products of operators, where the setting is the Hilbert ball $\mathbb{B}$ instead of a Banach space.

Theorem 5.1. *Let $\{P_{K_i} : 1 \leq i \leq m\}$ be the nearest point projections of $\mathbb{B}$ onto ρ-closed and ρ-convex subsets $\{K_i \subset \mathbb{B} : 1 \leq i \leq m\}$ with $K_1 \cap K_2 \cap \cdots \cap K_m \neq \emptyset$. Let the given operators $A_n^{(i)} : \mathbb{B} \to \mathbb{B}$, $i = 1, 2, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in \mathbb{B}$ the inequalities*

$$\rho(A_n^{(i)} x, P_{K_i} x) \leq \gamma_n \rho(0, x) \tag{5.3}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then for each $x \in \mathbb{B}$, there exists a point $\overline{x} = \overline{x}(x) \in K_1 \cap K_2 \cap \cdots \cap K_m$ such that

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} A_j^{(m)} A_j^{(m-1)} \cdots A_j^{(1)} \right) x = \overline{x} \quad \textit{weakly.}$$

Proof. Fix $z \in K_1 \cap K_2 \cap \cdots \cap K_m$. For each $i = 1, \ldots, m$, consider the subsets $\widetilde{K}_i := \mathcal{M}_{-z}(K_i)$, and define the operators $P_{\widetilde{K}_i} w := \mathcal{M}_{-z}(P_{K_i} x)$ and $\widetilde{A}_n^{(i)} w := \mathcal{M}_{-z}(A_n^{(i)} x)$, where $w, x \in \mathbb{B}$ are such that $w = \mathcal{M}_{-z}(x)$. Note that $P_{\widetilde{K}_i}$ is the nearest point projection of $\mathbb{B}$ onto $\widetilde{K}_i$ for each $i = 1, \ldots, m$. Moreover, since $\mathcal{M}_{-z}$ is an automorphism and P_{K_i} is ρ-nonexpansive, so is $P_{\widetilde{K}_i}$ for all $i = 1, \ldots, m$. Since $z \in K_1 \cap K_2 \cap \cdots \cap K_m$, it is clear that $0 = \mathcal{M}_{-z}(z) \in \widetilde{K}_i$, so $P_{\widetilde{K}_i} 0 = 0$ for each $i = 1, \ldots, m$. Therefore the inequality

$$\rho(0, P_{\widetilde{K}_i} w) = \rho(P_{\widetilde{K}_i} 0, P_{\widetilde{K}_i} w) \le \rho(0, w) \tag{5.4}$$

holds, because $P_{\widetilde{K}_i}$ is ρ-nonexpansive. Using (5.3), we see that

$$\rho(\widetilde{A}_n^{(i)} w, P_{\widetilde{K}_i} w) \le \gamma_n \rho(-z, w) \le \gamma_n \rho(0, w) + \gamma_n \rho(0, -z) \tag{5.5}$$

for all points $w, x \in \mathbb{B}$ such that $w = \mathcal{M}_{-z}(x)$, $n \in \mathbb{N}$ and $i = 1, \ldots, m$. Now take $x \in \mathbb{B}$ and let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined by $x_1 = x$ and $x_{n+1} = A_n^{(m)} \cdots A_n^{(1)} x_n$ for each $n \in \mathbb{N}$. We also consider the sequence $(w_n)_{n \in \mathbb{N}}$ defined by $w_1 = w = \mathcal{M}_{-z}(x)$ and $w_{n+1} = \widetilde{A}_n^{(m)} \cdots \widetilde{A}_n^{(1)} w_n$ for each $n \in \mathbb{N}$. It is not difficult to see that $w_n = \mathcal{M}_{-z}(x_n)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $i = 1, \ldots, m$, using the triangle inequality, (5.4) and (5.5), we get

$$\rho(0, \widetilde{A}_n^{(i)} w) \le \rho(0, P_{\widetilde{K}_i} w) + \rho(P_{\widetilde{K}_i} w, \widetilde{A}_n^{(i)} w) \le (1 + \gamma_n) \rho(0, w) + \gamma_n \rho(0, -z).$$

Hence, for each $i = 2, \ldots, m$, it follows that

$$\rho(0, \widetilde{A}_n^{(i)} \cdots \widetilde{A}_n^{(1)} w) \le (1 + \gamma_n)^i \rho(0, w) + \gamma_n \sum_{k=0}^{i-1} (1 + \gamma_n)^k \rho(0, -z). \tag{5.6}$$

From (5.6), we deduce that

$$\begin{aligned} \rho(0, w_n) \quad \le \quad & \sum_{\ell=1}^{n-2} \gamma_\ell \prod_{j=\ell+1}^{n-1} (1 + \gamma_j)^m \sum_{i=0}^{m-1} (1 + \gamma_\ell)^i \rho(0, -z) \\ & + \gamma_{n-1} \sum_{i=0}^{m-1} (1 + \gamma_{n-1})^i \rho(0, -z) + \prod_{j=1}^{n-1} (1 + \gamma_j)^m \rho(0, w). \end{aligned} \tag{5.7}$$

Note that $\prod_{j \in \mathbb{N}} (1 + \gamma_j) < \infty$ because $\sum_{j \in \mathbb{N}} \gamma_j < \infty$. Combining these facts with (5.7), we get a number $M > 0$ such that

$$\rho(0, w_n) \le M[\rho(0, w) + \rho(0, -z)]. \tag{5.8}$$

Using induction over $i = 2, \ldots, m$, the triangle inequality, (5.5) and the ρ-nonexpansivity of $P_{\widetilde{K}_i}$, we obtain

$$\begin{aligned} & \rho(P_{\widetilde{K}_i} \widetilde{A}_n^{(i-1)} \cdots \widetilde{A}_n^{(1)} w_n, P_{\widetilde{K}_i} P_{\widetilde{K}_{i-1}} \cdots P_{\widetilde{K}_1} w_n) \\ \le \quad & \gamma_n \Big[\rho(0, \widetilde{A}_n^{(i-2)} \cdots \widetilde{A}_n^{(1)} w_n) + \rho(0, \widetilde{A}_n^{(i-3)} \cdots \widetilde{A}_n^{(1)} w_n) \\ & + \cdots + \rho(0, \widetilde{A}_n^{(1)} w_n) + \rho(0, w_n) \Big] + (i-1) \gamma_n \rho(0, -z). \end{aligned} \tag{5.9}$$

Now consider the operator $\widetilde{T} = P_{\widetilde{K}_m} P_{\widetilde{K}_{m-1}} \cdots P_{\widetilde{K}_1}$. By the triangle inequality, (5.5), (5.6) and (5.9), we have

$$\begin{aligned} & \rho(w_{n+1}, \widetilde{T}w_n) \\ \leq\ & \rho(\widetilde{A}_n^{(m)}\widetilde{A}_n^{(m-1)}\cdots\widetilde{A}_n^{(1)}w_n, P_{\widetilde{K}_m}\widetilde{A}_n^{(m-1)}\cdots\widetilde{A}_n^{(1)}w_n) \\ & +\rho(P_{\widetilde{K}_m}\widetilde{A}_n^{(m-1)}\cdots\widetilde{A}_n^{(1)}w_n, P_{\widetilde{K}_m}P_{\widetilde{K}_{m-1}}\cdots P_{\widetilde{K}_1}w_n) \\ \leq\ & \gamma_n\sum_{i=0}^{m-1}(1+\gamma_n)^i\rho(0,w_n) + \gamma_n m\sum_{i=0}^{m-1}(1+\gamma_n)^i\rho(0,-z) + m\gamma_n\rho(0,-z). \end{aligned}$$

Without loss of generality, we may assume that $\gamma_n \leq 1$ for each $n \in \mathbb{N}$. Hence

$$\begin{aligned} \rho(w_{n+1}, \widetilde{T}w_n) &\leq \gamma_n\sum_{i=0}^{m-1}2^i\rho(0,w_n) + \gamma_n m\sum_{i=0}^{m-1}2^i\rho(0,-z) + m\gamma_n\rho(0,-z) \\ &\leq \gamma_n[2^m\rho(0,w_n) + m(2^m+1)\rho(0,-z)]. \end{aligned} \tag{5.10}$$

By (5.8) and (5.10), we see that there exists a number $M > 0$ such that

$$\rho(w_{n+1}, \widetilde{T}w_n) \leq \gamma_n[2^m M(\rho(0,w) + (1 + m(2^m+1))\rho(0-z))].$$

This proves that the sequence $(w_n)_{n\in\mathbb{N}}$ is an inexact orbit of $\widetilde{T}$ with summable errors. By (5.1), we know that all the exact orbits of $\widetilde{T}$ converge weakly to fixed points of $\widetilde{T}$. Hence by Theorem 3.5, there exists a point $\overline{w} \in \widetilde{K}_1 \cap \widetilde{K}_2 \cap \cdots \cap \widetilde{K}_m$ such that $\lim_{n\to\infty} w_n = \overline{w}$ weakly. The fact that $\overline{w}$ belongs to $\widetilde{K}_1 \cap \widetilde{K}_2 \cap \cdots \cap \widetilde{K}_m$ follows from (2.8). Since $\mathcal{M}_z$ is weakly continuous and $(w_n)_{n\in\mathbb{N}} = (\mathcal{M}_{-z}(x_n))_{n\in\mathbb{N}}$, there exists a point $\overline{x} \in K_1 \cap K_2 \cap \cdots \cap K_m$ such that $\bar{x} = \mathcal{M}_z(\overline{w})$ and

$$\overline{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\prod_{j=1}^{n-1} A_j^{(m)}A_j^{(m-1)}\cdots A_j^{(1)}\right)x \quad \text{weakly.}$$

This concludes the proof. □

It is important to observe that continuity of the operators $A_n^{(i)}$ is irrelevant to the above proof. So Theorem 5.1 holds even for discontinuous operators. The infinite products of ρ-convex combinations of the operators $A_n^{(i)}$ also converge weakly. To prove our next weak convergence theorem, we use (5.2) and Theorem 3.5.

Theorem 5.2. *Let P_{K_1}, P_{K_2} be the nearest point projections of $\mathbb{B}$ onto ρ-closed and ρ-convex subsets $K_1, K_2 \subset \mathbb{B}$, respectively. Assume that $K_1 \cap K_2 \neq \emptyset$ and let the given operators $A_n^{(i)} : \mathbb{B} \to \mathbb{B}$, $i = 1, 2$; $n \in \mathbb{N}$, satisfy for all $x \in \mathbb{B}$, the inequalities*

$$\rho(A_n^{(i)}x, P_{K_i}x) \leq \gamma_n\rho(0,x) \tag{5.11}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}}\gamma_n < \infty$. Then for each $x \in \mathbb{B}$, there exists a point $\overline{x} = \overline{x}(x) \in K_1 \cap K_2$ such that

$$\lim_{n\to\infty}\left(\prod_{j=1}^{n}\frac{1}{2}A_j^{(1)} \oplus \frac{1}{2}A_j^{(2)}\right)x = \overline{x} \quad \textit{weakly.}$$

Now we use Kopecká–Reich definition of the ρ-convex combination of more than two operators to extend Theorem 5.2. To this end, we first recall [28] that a set-valued operator $T \subset \mathbb{B} \times \mathbb{B}$ with domain $\mathrm{Dom}\,(T)$ and range $\mathrm{Ran}\,(T)$ is said to be *coaccretive* if

$$\rho(x_1, x_2) \leq \rho((1+r)x_1 \ominus ry_1, (1+r)x_2 \ominus ry_2)$$

for all $y_1 \in Tx_1$, $y_2 \in Tx_2$ and $r > 0$. In addition, if $\mathrm{Ran}\,((1+r)I \ominus rT) = \mathbb{B}$ for all $r > 0$, T is said to be *m-coaccretive.* In particular, all ρ-nonexpansive operators are m-coaccretive (see [18]). If a set-valued operator T is coaccretive, then for each $r > 0$, the *resolvent* of T is the nonexpansive operator $J_r : \mathrm{Ran}\,((1+r)x \ominus rT) \to \mathrm{Dom}\,(T)$ defined by $J_r((1+r)x \ominus ry) = x$, where $x \in \mathrm{Dom}\,(T)$ and $y \in Tx$. We denote the fixed point set of T by $\mathrm{Fix}\,(T)$, that is, $\mathrm{Fix}\,(T) := \{x \in \mathbb{B} : (x, x) \in T\}$. Note that $\mathrm{Fix}\,(T) = \mathrm{Fix}\,(J_r)$ for each $r > 0$.

Consider the m-coaccretive operators $T_1, \ldots, T_m$ for which the intersection $\mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m) \neq \emptyset$. For each $i = 1, \ldots, m$, suppose that $r_i > 0$ and let J_{r_i} be the corresponding resolvent of T_i. If $a_1, \ldots, a_m \in (0, 1)$ are numbers such that $a_1 + a_2 + \cdots + a_m = 1$, then for each $x \in \mathbb{B}$, the weak limit

$$\lim_{n\to\infty} C(J_{r_1}, J_{r_2}, \ldots, J_{r_m}; a_1, a_2, \ldots, a_m)^n x = Px \tag{5.12}$$

exists and defines a ρ-nonexpansive retraction P of $\mathbb{B}$ onto $\mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m)$ (see [18, Theorem 3.8]).

Unlike Theorem 5.2, our next result is true for more general operators than nearest point projections. This generalization takes place in the framework of m-coaccretive operators.

Theorem 5.3. *Let $T_1, \ldots, T_m$ be m-coaccretive operators. For each $i = 1, \ldots, m$, suppose that $r_i > 0$ and J_{r_i} is the corresponding resolvent of T_i. Assume that $\mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m) \neq \emptyset$. Let the given operators $A_n^{(i)} : \mathbb{B} \to \mathbb{B}$, $i = 1, \ldots, m$; $n \in \mathbb{N}$, satisfy for all $x \in \mathbb{B}$, the inequalities*

$$\rho(A_n^{(i)} x, J_{r_i} x) \leq \gamma_n \rho(0, x) \tag{5.13}$$

for some positive numbers γ_n with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then for each $x \in \mathbb{B}$, there exists a point $\overline{x} = \overline{x}(x) \in \mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m)$ such that

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} C(A_j^{(1)}, \ldots, A_j^{(m)}; a_1, \ldots, a_m) \right) x = \overline{x} \quad \textit{weakly,}$$

where $a_1, \ldots, a_m \in (0, 1)$ are real numbers such that $a_1 + a_2 + \cdots + a_m = 1$.

Proof. Fix $z \in \mathrm{Fix}\,(T_1) \cap \cdots \cap \mathrm{Fix}\,(T_m)$. For each $i = 1, \ldots, m$, consider the operators $\widetilde{T}_i w := \mathcal{M}_{-z}(T_i x)$, $\widetilde{J}_{r_i} w := \mathcal{M}_{-z}(J_{r_i} x)$ and $\widetilde{A}_n^{(i)} w := \mathcal{M}_{-z}(A_n^{(i)} x)$; where $x, w \in \mathbb{B}$ are such that $w = \mathcal{M}_{-z}(x)$. Note that $\mathrm{Fix}\,(\widetilde{T}_i) = \mathcal{M}_{-z}(\mathrm{Fix}\,(T_i))$. Since each J_{r_i} is ρ-nonexpansive, so is $\widetilde{J}_{r_i}$. By definition, we know that $z = J_{r_i}((1+r_i)z \ominus r_i y)$ for $y \in T_i z$, and since $z \in T_i z$, it is clear that $z = J_{r_i}((1+r_i)z \ominus r_i z) = J_{r_i} z$ for each $i = 1, \ldots, m$. Recall that $0 = \mathcal{M}_{-z}(z)$. Consequently, $\widetilde{J}_{r_i} 0 = \mathcal{M}_{-z}(J_{r_i} z) = \mathcal{M}_{-z}(z) = 0$. So, since $\widetilde{J}_{r_i}$ is ρ-nonexpansive, we obtain

$$\rho(0, \widetilde{J}_{r_i} w) = \rho(\widetilde{J}_{r_i} 0, \widetilde{J}_{r_i} w) \leq \rho(0, w). \tag{5.14}$$

Take $x \in \mathbb{B}$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = x$ and $x_{n+1} = C(A_n^{(1)}, \ldots, A_n^{(m)}; a_1, \ldots, a_m)x_n$ for each $n \in \mathbb{N}$. We also consider the sequence $(w_n)_{n\in\mathbb{N}}$ defined by $w_1 = w = \mathcal{M}_{-z}(x)$ and $w_{n+1} = C(\widetilde{A}_n^{(1)}, \ldots, \widetilde{A}_n^{(m)}; a_1, \ldots, a_m)w_n$ for each $n \in \mathbb{N}$. Using (2.6), we see that $w_n = \mathcal{M}_{-z}(x_n)$ for each $n \in \mathbb{N}$. By (5.13) and the triangle inequality, we obtain

$$\rho(\widetilde{A}_n^{(i)}w, \widetilde{J}_{r_i}w) \le \gamma_n\rho(0, w) + \gamma_n\rho(0, -z). \tag{5.15}$$

It follows from the triangle inequality, (5.14) and (5.15) that

$$\rho(0, \widetilde{A}_n^{(i)}w) \le (1 + \gamma_n)\rho(0, w) + \gamma_n\rho(0, -z). \tag{5.16}$$

In the rest of the proof we are going to use two claims. Both of them can be proved by induction over k, using (2.7) and (5.16).

Claim 1: For each $k = 2, \ldots, m$, the following inequality holds:

$$\rho(0, C(\widetilde{A}_n^{(1)}, \ldots, \widetilde{A}_n^{(k)}; \beta_1, \ldots, \beta_k)w) \le \sum_{j=1}^{k} \beta_j\, \rho(0, \widetilde{A}_n^{(j)}w), \tag{5.17}$$

where $\beta_1, \ldots, \beta_k \in (0, 1)$ are such that $\beta_1 + \beta_2 + \cdots + \beta_k = 1$.

By (5.14), (5.17) and since $a_1 + \cdots + a_m = 1$, the following inequality holds:

$$\rho(0, C(\widetilde{A}_n^{(1)}, \ldots, \widetilde{A}_n^{(m)}; a_1, \ldots, a_m)w) \le (1 + \gamma_n)\rho(0, w) + \gamma_n\rho(0, -z).$$

Consequently, for each $n \ge 2$, we obtain the following inequality:

$$\begin{aligned}\rho(0, w_n) \;\le\; & \prod_{j=1}^{n-1}(1 + \gamma_j)\rho(0, w) + \Bigg[\gamma_1 \prod_{j=2}^{n-1}(1 + \gamma_j) + \gamma_2 \prod_{j=3}^{n-1}(1 + \gamma_j) \\ & + \ldots + \gamma_{n-3} \prod_{j=n-2}^{n-1}(1 + \gamma_j) + \gamma_{n-2}(1 + \gamma_{n-1}) + \gamma_{n-1}\Bigg]\rho(0, -z).\end{aligned} \tag{5.18}$$

Since $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, we know that $\prod_{j\in\mathbb{N}}(1 + \gamma_j) < \infty$. Hence there exist numbers $M_1, M_2 > 0$ such that $\prod_{j=p}^{n-1}(1 + \gamma_j) \le M_1$ for all $n \ge p + 1$ and $\sum_{j=1}^{n-1} \gamma_j \le M_2$ for all $n \ge 2$. Combining these inequalities with (5.18), we find a number $M > 0$ such that

$$\rho(0, w_n) \le M[\rho(0, w) + \rho(0, -z)]. \tag{5.19}$$

Claim 2: For each $k = 2, \ldots, m$, we have

$$\begin{aligned}&\rho(C(\widetilde{A}_n^{(1)}, \ldots, \widetilde{A}_n^{(k)}; \beta_1, \ldots, \beta_k)w_n, C(\widetilde{J}_{r_1}, \ldots, \widetilde{J}_{r_k}; \beta_1, \ldots, \beta_k)w_n) \\ &\le \sum_{j=1}^{k} \beta_j\rho(\widetilde{A}_n^{(j)}w_n, \widetilde{J}_{r_j}w_n),\end{aligned} \tag{5.20}$$

where $\beta_1, \beta_2, \ldots, \beta_k \in (0, 1)$ are such that $\beta_1 + \beta_2 + \cdots + \beta_k = 1$.

Set $\widetilde{T} := C(\widetilde{J}_{r_1}, \widetilde{J}_{r_2}, \ldots, \widetilde{J}_{r_m}; a_1, a_2, \ldots, a_m)$. By using inequalities (5.15), (5.19), (5.20) and the fact that $a_1 + \cdots + a_m = 1$, we see that

$$\rho(w_{n+1}, \widetilde{T}w_n) \le \gamma_n[M(\rho(0, w) + \rho(0, -z)) + \rho(0, -z)],$$

which proves that the sequence $(w_n)_{n\in\mathbb{N}}$ is an inexact orbit of $\widetilde{T}$ with summable errors. By (5.12), we know that all exact orbits of $\widetilde{T}$ converge weakly to fixed points of $\widetilde{T}$. Therefore by Theorem 3.5, there exists a point $\overline{w} \in \operatorname{Fix}(\widetilde{T})$ such that $\lim_{n\to\infty} w_n = \overline{w}$ weakly. In addition, by (2.10) we see that $\overline{w} \in \operatorname{Fix}(\widetilde{T}_1) \cap \operatorname{Fix}(\widetilde{T}_2) \cap \cdots \cap \operatorname{Fix}(\widetilde{T}_m)$. Since $(w_n)_{n\in\mathbb{N}} = (\mathcal{M}_{-z}(x_n))_{n\in\mathbb{N}}$ and $\mathcal{M}_z$ is weakly continuous, we conclude that there is a point $\overline{x} = \mathcal{M}_z(\overline{w}) \in \operatorname{Fix}(T_1) \cap \cdots \cap \operatorname{Fix}(T_m)$ such that

$$\overline{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\prod_{j=1}^{n} C(A_j^{(1)}, \ldots, A_j^{(m)}; a_1, \ldots, a_m) \right) x \quad \text{weakly.}$$

□

Note that continuity of the operators $A_n^{(k)}$ is not required in the proofs of all the results in this section. Thus both Theorem 5.2 and Theorem 5.3 hold even for discontinuous operators $A_n^{(k)}$.

6. CAT(0) SPACES

In this section we consider CAT(0) spaces and establish a result similar to those obtained in Sections 4 and 5. Suppose X is a complete CAT(0) space. Let P_{B_1} and P_{B_2} be the nearest point projections of X onto convex and closed subsets $B_1, B_2 \subset X$, respectively. If $B_1 \cap B_2 \neq \emptyset$, then for each $x \in X$,

$$\lim_{n\to\infty} (P_{B_2} P_{B_1})^n x = \overline{x} \quad \text{weakly,} \tag{6.1}$$

where $\overline{x} \in B_1 \cap B_2$. If, in addition, B_1 and B_2 are boundedly regular, this convergence is in the metric sense. For a proof of these facts we refer the reader to [3, Theorem 4.1].

Remark 6.1. Recall that two subsets $A, B \subset X$ such that $A \cap B \neq \emptyset$ are called *boundedly regular* if for any bounded set $S \subset X$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in S$ and $\max\{d(x, A), d(x, B)\} < \delta$, then $d(x, A \cap B) < \varepsilon$.

Extending the results we have already obtained for Banach spaces and the Hilbert ball, we now study the convergence of infinite products of approximations to these nearest point projections.

A metric space (X, d) is called *metrically homogeneous* if for any $x, y \in X$, there exists an isometry $\mathcal{M}$ of X onto X such that $\mathcal{M}(x) = y$. The Hilbert ball is an example of a metrically homogeneous metric space with the Möbius transformations playing the role of $\mathcal{M}$.

Lemma 6.2. *Suppose (X, d) is a metrically homogeneous CAT(0) space. If $\mathcal{M}$ is an isometry of X onto X, then $\mathcal{M}$ has the following properties:*

(i) *$\mathcal{M}$ is continuous;*
(ii) *$P_{\mathcal{M}(\gamma)}\mathcal{M} = \mathcal{M}P_\gamma$ for all geodesic segments $\gamma \subset X$;*
(iii) *$\mathcal{M}$ is weakly continuous.*

Proof. Point (i) is obvious and point (ii) is not difficult to prove. To prove (iii), consider a sequence $(u_n)_{n\in\mathbb{N}} \subset X$, which converges weakly to $u \in X$. Let σ be a geodesic segment through $\mathcal{M}(u)$. Since $\mathcal{M}$ is an isometry, there is a geodesic segment γ through u such that $\sigma = \mathcal{M}(\gamma)$. So by (ii), we obtain $d(\mathcal{M}(u), P_\sigma \mathcal{M}(u_n)) = d(u, P_\gamma u_n)$; but $\lim_{n\to\infty} d(u, P_\gamma u_n) = 0$ by Proposition 2.9. Thus it is clear that

$$\lim_{n\to\infty} d(\mathcal{M}(u), P_\sigma \mathcal{M}(u_n)) = 0. \tag{6.2}$$

From Proposition 2.9 we see that $\mathcal{M}(u)$ is the weak limit of $(\mathcal{M}(u_n))_{n\in\mathbb{N}}$, because (6.2) holds for any geodesic segment σ through $\mathcal{M}(u)$. This shows that $\mathcal{M}$ is indeed weakly continuous, as asserted. □

Theorem 6.3. *Suppose (X, d) is a complete and metrically homogeneous CAT(0) space. Let P_{B_1} and P_{B_2} be the nearest point projections of X onto convex and closed subsets $B_1, B_2 \subset X$, respectively. Assume that $B_1 \cap B_2 \neq \emptyset$. Let the given operators $A_n^{(i)} : X \to X$, $i = 1, 2$; $n \in \mathbb{N}$, and the point $x_* \in X$ satisfy for all $x \in X$ the inequalities*

$$d(A_n^{(i)} x, P_{B_i} x) \leq \gamma_n d(x_*, x), \tag{6.3}$$

where γ_n are certain positive numbers with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$. Then, for each $x \in X$, there exists a point $\overline{x} = \overline{x}(x) \in B_1 \cap B_2$ such that

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} A_j^{(2)} A_j^{(1)} \right) x = \overline{x} \quad \textit{weakly.}$$

If, in addition, B_1 and B_2 are boundedly regular, the convergence is in the metric sense.

Proof. Fix $z \in B_1 \cap B_2$. Since X is metrically homogeneous, there exists an isometry $\mathcal{M} : X \to X$ such that $\mathcal{M}(z) = x_*$. Consider the subsets $\widetilde{B}_1 := \mathcal{M}(B_1)$ and $\widetilde{B}_2 = \mathcal{M}(B_2)$, and define the operators $P_{\widetilde{B}_i} w := \mathcal{M}(P_{B_i} x)$ and $\widetilde{A}_n^{(i)} w := \mathcal{M}(A_n^{(i)} x)$, where $x, w \in X$ are such that $w = \mathcal{M}(x)$, $i = 1, 2$. Note that $x_* = \mathcal{M}(z) \in \widetilde{B}_i$ because $z \in B_i$.

For all $x, w \in X$ such that $w = \mathcal{M}(x)$, it follows from the definition that $\widetilde{A}_n^{(i)} w = \mathcal{M}(A_n^{(i)} x) = \mathcal{M}(A_n^{(i)} \mathcal{M}^{-1}(w))$ for $i = 1, 2$; therefore

$$\widetilde{A}_n^{(2)} \widetilde{A}_n^{(1)} w = \mathcal{M}(A_n^{(2)} A_n^{(1)} x). \tag{6.4}$$

Since P_{B_i} is nonexpansive, so is $P_{\widetilde{B}_i}$. Moreover, $P_{\widetilde{B}_i}$ is the nearest point projection of X onto $\widetilde{B}_i$. It is clear that $P_{\widetilde{B}_i} x_* = x_*$ because $x_* \in \widetilde{B}_i$ and $P_{\widetilde{B}_i}$ is the nearest point projection of X onto $\widetilde{B}_i$. Hence

$$d(x_*, P_{\widetilde{B}_i} w) = d(P_{\widetilde{B}_i} x_*, P_{\widetilde{B}_i} w) \leq d(x_*, w) \quad \text{for all } w \in X. \tag{6.5}$$

In addition, by (6.3) and the triangle inequality we obtain

$$d(\widetilde{A}_n^{(i)} w, P_{\widetilde{B}_i} w) \leq \gamma_n d(x_*, z) + \gamma_n d(x_*, w) \tag{6.6}$$

for $w \in X$. Given $x \in X$, consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = x$ and $x_{n+1} = A_n^{(2)} A_n^{(1)} x_n$ for all $n \in \mathbb{N}$. Define inductively the sequence $(w_n)_{n\in\mathbb{N}}$ by

$$w_1 = w = \mathcal{M}(x) \quad \text{and} \quad w_{n+1} = \widetilde{A}_n^{(2)} \widetilde{A}_n^{(1)} w_n \quad \text{for all } n \in \mathbb{N}.$$

From (6.4) it is clear that $w_n = \mathcal{M}(x_n)$ for all $n \in \mathbb{N}$. Using the triangle inequality, (6.5) and (6.6), we see that

$$d(x_*, \widetilde{A}_n^{(i)} w) \leq (1 + \gamma_n) d(x_*, w) + \gamma_n d(x_*, z). \tag{6.7}$$

Consequently, we get

$$d(x_*, \widetilde{A}_n^{(2)} \widetilde{A}_n^{(1)} w) \leq (1 + \gamma_n)^2 d(x_*, w) + \gamma_n [1 + (1 + \gamma_n)] d(x_*, z). \tag{6.8}$$

Using induction over $n \geq 2$ and (6.8), we see that

$$\begin{aligned} d(x_*, w_n) \leq & \prod_{j=1}^{n-1} (1 + \gamma_j)^2 d(x_*, w) + d(x_*, z) \gamma_{n-1} [1 + (1 + \gamma_{n-1})] \\ & + d(x_*, z) \sum_{\ell=1}^{n-2} [1 + (1 + \gamma_\ell)] \prod_{j=\ell+1}^{n-1} (1 + \gamma_j)^2. \end{aligned} \tag{6.9}$$

By hypothesis, we know that $\sum_{j \in \mathbb{N}} \gamma_j < \infty$, hence $\prod_{j \in \mathbb{N}} (1 + \gamma_j) < \infty$. Thus there exist numbers $M_1, M_2, M_3 > 0$ such that $\prod_{j=p}^{n-1} (1+\gamma_j)^2 \leq M_1$ for all $n \geq p+1$; $1 + (1 + \gamma_j) \leq M_2$ for each $j \in \mathbb{N}$ and $\sum_{j=1}^{n-1} \gamma_j \leq M_3$ for all $n \geq 2$. These facts, along with (6.9), imply that there exists a number $M > 0$ such that

$$d(x_*, w_n) \leq M [d(x_*, w) + d(x_*, z)]. \tag{6.10}$$

Let $\widetilde{T} = P_{\widetilde{B}_2} P_{\widetilde{B}_1}$. By the triangle inequality, the nonexpansivity of $P_{\widetilde{B}_2}$, (6.6), (6.7) and (6.10), we obtain

$$\begin{aligned} d(w_{n+1}, \widetilde{T} w_n) & \leq d(\widetilde{A}_n^{(2)} \widetilde{A}_n^{(1)} w_n, P_{\widetilde{B}_2} \widetilde{A}_n^{(1)} w_n) + d(P_{\widetilde{B}_2} \widetilde{A}_n^{(1)} w_n, P_{\widetilde{B}_2} P_{\widetilde{B}_1} w_n) \\ & \leq \gamma_n [(1 + (1 + \gamma_n)) d(x_*, w_n) + (1 + \gamma_n) d(x_*, z)] \\ & \leq \gamma_n [M^* M (d(x_*, w) + d(x_*, z)) + M^* d(x_*, z)], \end{aligned} \tag{6.11}$$

where $M^* > 0$ is a number such that $1 + (1 + \gamma_n) \leq M^*$ for all $n \in \mathbb{N}$.

Inequality (6.11) shows that the sequence $(w_n)_{n \in \mathbb{N}}$ is an inexact orbit of $\widetilde{T}$ with summable errors. Note that by (2.21), we have $\widetilde{B}_1 \cap \widetilde{B}_2 = \operatorname{Fix}(P_{\widetilde{B}_2} P_{\widetilde{B}_1})$. According to (6.1), all exact orbits of $\widetilde{T}$ converge weakly to fixed points of $P_{\widetilde{B}_2} P_{\widetilde{B}_1}$. Hence by Theorem 3.6, there exists a point $\overline{w} \in \widetilde{B}_1 \cap \widetilde{B}_2$ such that

$$\overline{w} = \lim_{n \to \infty} w_n = \lim_{n \to \infty} \left(\prod_{j=1}^{n-1} \widetilde{A}_j^{(2)} \widetilde{A}_j^{(1)} \right) w \quad \text{weakly.} \tag{6.12}$$

Consequently, since $\mathcal{M}$ is weakly continuous (see Lemma 6.2) and $w_n = \mathcal{M}(x_n)$, there exists a point $\overline{x} = \mathcal{M}^{-1}(\overline{w}) \in B_1 \cap B_2$ such that

$$\overline{x} = \lim_{n \to \infty} x_n = \lim_{n \to \infty} \left(\prod_{j=1}^{n-1} A_j^{(2)} A_j^{(1)} \right) x \quad \text{weakly.} \tag{6.13}$$

If B_1 and B_2 are boundedly regular, then all exact orbits of $\widetilde{T}$ are convergent in the metric sense. So we can use Theorem 3.2, which gives us the existence of a point $\overline{w} \in \widetilde{B}_1 \cap \widetilde{B}_2$ such that (6.12) is true in the metric sense. Since $\mathcal{M}$ is continuous (by Lemma 6.2), there exists $\overline{x} = \mathcal{M}^{-1}(\overline{w}) \in B_1 \cap B_2$ such that (6.13) holds in the metric sense. □

Since continuity of the operators $A_n^{(i)}$ is not used in its proof, Theorem 6.3 remains true even when these approximations are discontinuous.

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