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MAXIMALITY OF THE SUM OF THE SUBDIFFERENTIAL OPERATOR AND A MAXIMALLY MONOTONE OPERATOR

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ABSTRACT. The most important open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximally monotone operators provided that the classical Rockafellar's constraint qualification holds, which is called the "sum problem".

In this paper, we establish the maximal monotonicity of A + B provided that A and B are maximally monotone operators such that dom $A \cap$ int dom $B \neq \emptyset$, and $A + N_{\overline{\text{dom }B}}$ is of type (FPV). This generalizes various current results and also gives an affirmative answer to a problem posed by Borwein and Yao. Moreover, we present an equivalent description of the sum problem.

1. INTRODUCTION

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A: X \rightrightarrows X^*$ be a *set-valued operator* (also known as point-to-set mapping or multifunction) from X to X^* , i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$ be the graph of A, and dom $A := \{x \in X \mid Ax \neq \emptyset\}$ be the domain of A. Recall that A is monotone if

$$\langle x - y, x^* - y^* \rangle \ge 0, \quad \forall (x, x^*) \in \operatorname{gra} A \; \forall (y, y^*) \in \operatorname{gra} A.$$

We say A is maximally monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say (x, x^*) is monotonically related to gra A if

$$\langle x - y, x^* - y^* \rangle \ge 0, \quad \forall (y, y^*) \in \operatorname{gra} A.$$

Let $A: X \rightrightarrows X^*$ be maximally monotone. We say A is of type (FPV) [19,32] if for every open convex set $U \subseteq X$ such that $U \cap \text{dom } A \neq \emptyset$, the implication

 $x \in U$ and (x, x^*) is monotonically related to $\operatorname{gra} A \cap (U \times X^*) \Longrightarrow (x, x^*) \in \operatorname{gra} A$ holds.

Monotone operators have proven important in modern Optimization and Analysis; see, e.g., the books [4, 11, 16, 17, 22, 27, 29, 30, 40–42] and the references therein.

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We adopt standard notation used in these books. Given a subset C of X, int C is the *interior* of C, \overline{C} is the *norm closure* of C, and conv C is the *convex hull* of C. The *indicator function* of C, written as ι_C , is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

If $C, D \subseteq X$, we set $C-D := \{x-y \mid x \in C, y \in D\}$. For every $x \in X$, the normal cone operator of C at x is defined by $N_C(x) := \{x^* \in X^* \mid \sup_{c \in C} \langle c-x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) := \emptyset$, if $x \notin C$. For $x, y \in X$, we set $[x, y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\}$.

Given $f: X \to]-\infty, +\infty]$, we set dom $f := f^{-1}(\mathbb{R})$. We say f is proper if dom $f \neq \emptyset$. Let f be proper. Then $\partial f: X \rightrightarrows X^*: x \mapsto \{x^* \in X^* \mid (\forall y \in X) \langle y - x, x^* \rangle + f(x) \leq f(y)\}$ is the subdifferential operator of f. Thus $N_C = \partial \iota_C$. We also set $P_X: X \times X^* \to X: (x, x^*) \mapsto x$. The open unit ball in X is denoted by $U_X := \{x \in X \mid ||x|| < 1\}$, the closed unit ball in X is denoted by $B_X := \{x \in X \mid ||x|| \leq 1\}$, and $\mathbb{N} := \{1, 2, 3, \ldots\}$. We denote by \longrightarrow and \neg_w^* the norm convergence and weak^{*} convergence of nets, respectively.

Let A and B be maximally monotone operators from X to X^{*}. Clearly, the sum operator $A + B: X \rightrightarrows X^*: x \mapsto Ax + Bx := \{a^* + b^* \mid a^* \in Ax \text{ and } b^* \in Bx\}$ is monotone. Rockafellar established the following significant result in 1970.

Theorem 1.1 (Rockafellar's sum theorem (See [26, Theorem 1] or [11])). Suppose that X is reflexive. Let $A, B : X \Rightarrow X^*$ be maximally monotone. Assume that A and B satisfy the classical constraint qualification:

dom $A \cap$ int dom $B \neq \emptyset$.

Then A + B is maximally monotone.

The generalization of Rockafellar's sum theorem in the setting of a reflexive space can be found in [1,3,11,30,31].

The most famous open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximally monotone operators satisfying Rockafellar's constraint qualification in general Banach spaces; this is called the "sum problem". Some recent developments on the sum problem can be found in Simons' monograph [30] and [5–9,11,13–15,20,34–39], and also see [2] for the subdifferential operators.

In this paper, we focus on the case when A, B are maximally monotone with dom $A \cap$ int dom $B \neq \emptyset$, and $A + N_{\overline{\text{dom }B}}$ is of type (FPV) (see Theorem 3.2).

Corollary 3.4 provides an affirmative answer to the following problem posed by Borwein and Yao in [14, Open problem 4.5].

Let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function, and let $B : X \rightrightarrows X^*$ be maximally monotone with dom $\partial f \cap$ int dom $B \neq \emptyset$. Is $\partial f + B$ necessarily maximally monotone?

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. In Section 3, our main result (Theorem 3.2) is presented. We also show that Problem 3.8 is equivalent to the sum problem.

2. AUXILIARY RESULTS

We first introduce the well known Banach-Alaoglu Theorem and the two of Rockafellar's results.

Fact 2.1 (The Banach-Alaoglu Theorem). (See [28, Theorem 3.15] or [21, Theorem 2.6.18].) The closed unit ball in X^* , B_{X^*} , is weakly^{*} compact.

Fact 2.2 (Rockafellar). (See [23, Theorem 3], [30, Theorem 18.1], or [40, Theorem 2.8.7(iii)].) Let $f, g: X \to]-\infty, +\infty]$ be proper convex functions. Assume that there exists a point $x_0 \in \text{dom } f \cap \text{dom } g$ such that g is continuous at x_0 . Then $\partial(f+g) = \partial f + \partial g$.

Fact 2.3 (Rockafellar). (See [25, Theorem 1] or [30, Theorem 27.1 and Theorem 27.3].) Let $A: X \rightrightarrows X^*$ be maximally monotone with int dom $A \neq \emptyset$. Then int dom $A = \operatorname{int} \operatorname{dom} A$ and int dom A and $\operatorname{dom} A$ are both convex.

The Fitzpatrick function defined below is an important tool in Monotone Operator Theory.

Fact 2.4 (Fitzpatrick). *(See [18, Corollary 3.9].)* Let $A: X \rightrightarrows X^*$ be monotone, and set

$$F_A \colon X \times X^* \to]-\infty, +\infty] \colon (x, x^*) \mapsto \sup_{(a, a^*) \in \operatorname{gra} A} \left(\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle \right),$$

the *Fitzpatrick function* associated with A. Suppose also A is maximally monotone. Then for every $(x, x^*) \in X \times X^*$, the inequality $\langle x, x^* \rangle \leq F_A(x, x^*)$ is true, and the equality holds if and only if $(x, x^*) \in \operatorname{gra} A$.

The next result is the key to our arguments.

Fact 2.5. (See [35, Theorem 3.4 and Corollary 5.6], or [30, Theorem 24.1(b)].) Let $A, B: X \rightrightarrows X^*$ be maximally monotone operators. Assume $\bigcup_{\lambda>0} \lambda [P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)]$ is a closed subspace. If

$$F_{A+B} \ge \langle \cdot, \cdot \rangle$$
 on $X \times X^*$,

then A + B is maximally monotone.

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Applying Fact 2.6, we can avoid computing the domain of the Fitzpatrick functions in Fact 2.5 (see Corollary 2.8 below).

Fact 2.6. (See [13, Theorem 3.6] or [14].) Let $A : X \rightrightarrows X^*$ be a maximally monotone operator. Then

$$\overline{\operatorname{conv}\left[\operatorname{dom} A\right]} = \overline{P_X\left[\operatorname{dom} F_A\right]}.$$

Lemma 2.7. Let $A, B: X \implies X^*$ be maximally monotone, and suppose that $\bigcup_{\lambda>0} \lambda [\operatorname{dom} A - \operatorname{dom} B]$ is a closed convex subset of X. Then

$$\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B \right] = \bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B \right].$$

Proof. By Fact 2.4 and Fact 2.6, we have

$$\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B \right] \subseteq \bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B \right]$$
$$\subseteq \bigcup_{\lambda>0} \lambda \left[\operatorname{conv} \operatorname{dom} A - \overline{\operatorname{conv}} \operatorname{dom} B \right]$$
$$\subseteq \bigcup_{\lambda>0} \lambda \left[\operatorname{conv} \operatorname{dom} A - \operatorname{conv} \operatorname{dom} B \right]$$
$$= \bigcup_{\lambda>0} \lambda \left[\overline{\operatorname{conv}} \left[\operatorname{dom} A - \operatorname{dom} B \right] \right]$$
$$\subseteq \overline{\bigcup_{\lambda>0} \lambda \operatorname{conv}} \left[\operatorname{dom} A - \operatorname{dom} B \right]$$
$$= \bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B \right]$$
(by the assumption).

Hence $\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B \right] = \bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B \right].$

Corollary 2.8. Let $A, B : X \rightrightarrows X^*$ be maximally monotone operators. Assume that $\bigcup_{\lambda>0} \lambda [\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace. If

$$F_{A+B} \ge \langle \cdot, \cdot \rangle \ on \quad X \times X^*,$$

then A + B is maximally monotone.

Proof. Apply Fact 2.5 and Lemma 2.7 directly.

Now we cite some results on operators of type (FPV).

Fact 2.9 (Fitzpatrick-Phelps and Verona-Verona). (See [19, Corollary 3.4], [33, Theorem 3] or [30, Theorem 48.4(d)].) Let $f: X \to]-\infty, +\infty$] be proper, lower semicontinuous and convex. Then ∂f is of type (FPV).

Fact 2.10 (Simons). *(See [30, Theorem 44.2].)* Let $A : X \rightrightarrows X^*$ be of type (FPV). Then

$$\overline{\operatorname{dom} A} = \operatorname{conv} \left(\operatorname{dom} A \right) = P_X \left(\operatorname{dom} F_A \right).$$

The following result presents a sufficient condition for a maximally monotone operator to be of type (FPV).

Fact 2.11 (Simons and Verona-Verona). (See [30, Theorem 44.1], [33] or [8].) Let $A: X \rightrightarrows X^*$ be maximally monotone. Suppose that for every closed convex subset C of X with dom $A \cap \operatorname{int} C \neq \emptyset$, the operator $A + N_C$ is maximally monotone. Then A is of type (FPV).

Fact 2.12 (Boundedness below). (See [12, Fact 4.1].) Let $A : X \Rightarrow X^*$ be monotone and $x \in \operatorname{int} \operatorname{dom} A$. Then there exist $\delta > 0$ and M > 0 such that $x + \delta B_X \subseteq \operatorname{dom} A$ and $\sup_{a \in x + \delta B_X} ||Aa|| \leq M$. Assume that (z, z^*) is monotonically related to gra A. Then

$$\langle z - x, z^* \rangle \ge \delta ||z^*|| - (||z - x|| + \delta)M.$$

We need the following bunch of useful tools from [15].

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Fact 2.13. (See [15, Proposition 3.1].) Let $A : X \rightrightarrows X^*$ be of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that dom $A \cap \operatorname{int} \operatorname{dom} B \neq \emptyset$. Let $(z, z^*) \in X \times X^*$ with $z \in \operatorname{dom} B$. Then

$$F_{A+B}(z, z^*) \ge \langle z, z^* \rangle.$$

Fact 2.14. (See [15, Lemma 2.10].) Let $A : X \rightrightarrows X^*$ be monotone, and let $B: X \rightrightarrows X^*$ be maximally monotone. Let $(z, z^*) \in X \times X^*$. Suppose $x_0 \in \text{dom } A \cap$ int dom B and that there exists a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra } A \cap (\text{dom } B \times X^*)$ such that $(a_n)_{n \in \mathbb{N}}$ converges to a point in $[x_0, z]$, and

$$\langle z - a_n, a_n^* \rangle \longrightarrow +\infty.$$

Then $F_{A+B}(z, z^*) = +\infty$.

Fact 2.15. (See [15, Lemma 2.12].) Let $A : X \rightrightarrows X^*$ be of type (FPV). Suppose $x_0 \in \text{dom } A$ but that $z \notin \overline{\text{dom } A}$. Then there exists a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in gra A such that $(a_n)_{n \in \mathbb{N}}$ converges to a point in $[x_0, z]$ and

$$\langle z - a_n, a_n^* \rangle \longrightarrow +\infty.$$

The proof of Fact 2.16 and Fact 2.17 is mainly extracted from the part of the proof of [15, Proposition 3.2].

Fact 2.16. Let $A : X \rightrightarrows X^*$ be maximally monotone and $z \in \overline{\text{dom } A} \setminus \text{dom } A$. Then for every sequence $(z_n)_{n \in \mathbb{N}}$ in dom A such that $z_n \longrightarrow z$, we have $\lim_{n \to \infty} \inf ||A(z_n)|| = +\infty$.

Proof. Suppose to the contrary that there exists a sequence $z_{n_k}^* \in A(z_{n_k})$ and L > 0 such that $\sup_{k \in \mathbb{N}} ||z_{n_k}^*|| \leq L$. By Fact 2.1, there exists a weak* convergent subnet, $(z_{\beta}^*)_{\beta \in J}$ of $(z_{n_k}^*)_{k \in \mathbb{N}}$ such that $z_{\beta}^* \neg_{w^*} z_{\infty}^* \in X^*$. [12, Fact 3.5] or [10, Section 2, page 539] shows that $(z, z_{\infty}^*) \in \text{gra } A$, which contradicts our assumption that $z \notin \text{dom } A$. Hence we have our result holds.

Fact 2.17. Let $A, B : X \Rightarrow X^*$ be monotone. Let $(z, z^*) \in X \times X^*$. Suppose that $x_0 \in \text{dom } A \cap \text{int dom } B$ and that there exist a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra } A \cap (\text{dom } B \times X^*)$ and a sequence $(K_n)_{n \in \mathbb{N}}$ in \mathbb{R} such that $(a_n)_{n \in \mathbb{N}}$ converges to a point in $[x_0, z[$, and that

$$(2.1) \qquad \langle z - a_n, a_n^* \rangle \ge K_n,$$

Assume that there exists a sequence $b_n^* \in Ba_n$ such that $\frac{K_n}{\|b_n^*\|} \longrightarrow 0$ and $\|b_n^*\| \longrightarrow +\infty$. Then $F_{A+B}(z, z^*) = +\infty$.

Proof. By the assumption, there exists $0 \le \delta < 1$ such that

(2.2)
$$a_n \longrightarrow x_0 + \delta(z - x_0).$$

Suppose to the contrary that

(2.3)
$$F_{A+B}(z, z^*) < +\infty.$$

By Fact 2.1, there exists a weak^{*} convergent subnet, $(\frac{b_i^*}{\|b_i^*\|})_{i \in I}$ of $\frac{b_n^*}{\|b_n^*\|}$ such that

(2.4)
$$\frac{b_i^*}{\|b_i^*\|} \to_{\mathbf{w}^*} b_\infty^* \in X^*.$$

By (2.1), we have

$$K_n + \left\langle z - a_n, b_n^* \right\rangle + \left\langle z^*, a_n \right\rangle \le \left\langle z - a_n, a_n^* \right\rangle + \left\langle z - a_n, b_n^* \right\rangle + \left\langle z^*, a_n \right\rangle$$
$$\le F_{A+B}(z, z^*)$$

Thus

(2.5)
$$\frac{K_n}{\|b_n^*\|} + \left\langle z - a_n, \frac{b_n^*}{\|b_n^*\|} \right\rangle + \frac{1}{\|b_n^*\|} \left\langle z^*, a_n \right\rangle \le \frac{F_{A+B}(z, z^*)}{\|b_n^*\|}.$$

By the assumption that $\frac{K_n}{\|b_n^*\|} \longrightarrow 0$ and $\|b_n^*\| \longrightarrow +\infty$, (2.2), (2.3) and (2.4), we take the limit along the subnet in (2.5) to obtain

$$\left\langle z - x_0 - \delta(z - x_0), b_\infty^* \right\rangle \le 0.$$

Since $\delta < 1$,

(2.6)
$$\left\langle z - x_0, b_\infty^* \right\rangle \le 0.$$

On the other hand, since $x_0 \in \text{int dom } B$ and $(a_n, b_n^*) \in \text{gra } B$, Fact 2.12 implies that there exist $\eta > 0$ and M > 0 such that

$$\langle a_n - x_0, b_n^* \rangle \ge \eta \|b_n^*\| - (\|a_n - x_0\| + \eta)M$$

Thus

$$\langle a_n - x_0, \frac{b_n^*}{\|b_n^*\|} \rangle \ge \eta - \frac{(\|a_n - x_0\| + \eta)M}{\|b_n^*\|}$$

Since $||b_n^*|| \to +\infty$, by (2.2) and (2.4), we take the limit along the subnet in the above inequality to obtain

$$\left\langle x_0 + \delta(z - x_0) - x_0, b_\infty^* \right\rangle \ge \eta$$

Hence

$$\left\langle z - x_0, b_\infty^* \right\rangle \ge \frac{\eta}{\delta} > 0,$$

which contradicts (2.6). Hence $F_{A+B}(z, z^*) = +\infty$.

3. Our main result

The following result is the key technical tool for our main result (: Theorem 3.2). The proof of Proposition 3.1 follows in part that of [15, Proposition 3.2].

Proposition 3.1. Let $A : X \rightrightarrows X^*$ be of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose $x_0 \in \text{dom } A \cap \text{int dom } B$ and $(z, z^*) \in X \times X^*$. Assume that there exist a sequence $(a_n)_{n \in \mathbb{N}}$ in $\text{dom } A \cap [\overline{\text{dom } B} \setminus \text{dom } B]$ and $\delta \in$ [0,1] such that $a_n \longrightarrow \delta z + (1-\delta)x_0$. Then $F_{A+B}(z, z^*) \ge \langle z, z^* \rangle$.

Proof. Suppose to the contrary that

(3.1)
$$F_{A+B}(z, z^*) < \langle z, z^* \rangle.$$

By the assumption, we have $\delta z + (1 - \delta)x_0 \in \overline{\text{dom }B}$. Since $a_n \notin \text{dom }B$ and $x_0 \in \text{int dom }B$, Fact 2.13 and (3.1) imply that

(3.2)
$$0 < \delta < 1 \quad \text{and} \quad \delta z + (1 - \delta) x_0 \neq x_0.$$

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We set

(3.3)
$$y_0 := \delta z + (1 - \delta) x_0.$$

Since $a_n \in \operatorname{dom} A$, we let

$$(3.4) (a_n, a_n^*) \in \operatorname{gra} A, \quad \forall n \in \mathbb{N}.$$

Since $x_0 \in \text{dom } A \cap \text{int dom } B$, there exist $x_0^*, y_0^* \in X^*$ such that $(x_0, x_0^*) \in \text{gra } A$ and $(x_0, y_0^*) \in \text{gra } B$. By $x_0 \in \text{int dom } B$, there exists $0 < \rho_0 \leq ||y_0 - x_0||$ by (3.2) such that

$$(3.5) x_0 + \rho_0 U_X \subseteq \operatorname{dom} B$$

Now we show that there exists $\delta \leq t_n \in \left[1 - \frac{1}{n}, 1\right]$ such that that

(3.6)
$$H_n \subseteq \operatorname{dom} B \text{ and } \inf \left\| B(H_n) \right\| \ge 4K_0^2(\|a_n^*\| + 1)n,$$

where

(3.7)
$$H_n := t_n a_n + (1 - t_n) x_0 + (1 - t_n) \rho_0 U_X$$
$$K_0 := \max \left\{ 3 \|z\| + 2 + 3 |x_0\|, \frac{1}{\delta} \left(\frac{2 \|y_0 - x_0\|}{\rho_0} + 1 \right) \left(\|x_0^*\| + 1 \right) \right\}.$$

For every $s \in [0, 1[$, since $a_n \in \overline{\text{dom } B}$, (3.5) and Fact 2.3 imply that

$$sa_n + (1-s)x_0 + (1-s)\rho_0 U_X = sa_n + (1-s)[x_0 + \rho_0 U_X] \subseteq \overline{\operatorname{dom} B}.$$

By Fact 2.3 again, $sa_n + (1-s)x_0 + (1-s)\rho_0 U_X \subseteq \operatorname{int} \overline{\operatorname{dom} B} = \operatorname{int} \operatorname{dom} B$.

It directly follows from Fact 2.16 and $a_n \in \overline{\text{dom }B} \setminus \text{dom }B$ that the second part of (3.6) holds.

 Set

(3.9)

(3.8)
$$r_n := \frac{\frac{1}{2}(1-t_n)\rho_0}{t_n \|y_0 - a_n\| + (1-t_n)\|y_0 - x_0\|}.$$

Since $\rho_0 \leq ||y_0 - x_0||$, we have $r_n \leq \frac{1}{2}$. Now we show that

$$v_n := r_n y_0 + (1 - r_n) [t_n a_n + (1 - t_n) x_0]$$

= $r_n \delta z + (1 - r_n) t_n a_n + s_n x_0 \in H_n,$

where $s_n := [1 - t_n + r_n(t_n - \delta)].$

Indeed, we have

$$\begin{aligned} \left\| v_n - t_n a_n - (1 - t_n) x_0 \right\| &= \left\| r_n y_0 + (1 - r_n) \left[t_n a_n + (1 - t_n) x_0 \right] - t_n a_n - (1 - t_n) x_0 \right\| \\ &= \left\| r_n y_0 - r_n \left[t_n a_n + (1 - t_n) x_0 \right] \right\| \\ &= r_n \left\| t_n y_0 + (1 - t_n) y_0 - \left[t_n a_n + (1 - t_n) x_0 \right] \right\| \\ &= r_n \left\| t_n (y_0 - a_n) + (1 - t_n) (y_0 - x_0) \right\| \\ &\leq r_n \left(t_n \| y_0 - a_n \| + (1 - t_n) \| y_0 - x_0 \| \right) \end{aligned}$$

$$=\frac{1}{2}(1-t_n)\rho_0$$
 (by (3.8)).

Hence $v_n \in H_n$ and thus (3.9) holds by (3.3).

Since $a_n \longrightarrow y_0$ and $v_n \in H_n$ by (3.9), $v_n \longrightarrow y_0$. Then we can and do suppose that

$$(3.10) ||v_n|| \le ||y_0|| + 1 \le ||z|| + ||x_0|| + 1, \quad \forall n \in \mathbb{N} \quad (by (3.3)).$$

Since $a_n \longrightarrow y_0$ and $||y_0 - x_0|| > 0$ by (3.2), we can suppose that

 $||y_0 - a_n|| \le ||y_0 - x_0||, \quad \forall n \in \mathbb{N}.$

Then by (3.8),

(3.11)
$$\frac{1-t_n}{r_n} \le \frac{2\|y_0 - x_0\|}{\rho_0}, \quad \forall n \in \mathbb{N}.$$

Since $s_n = [1 - t_n + r_n(t_n - \delta)]$, by (3.11) and $\delta \le t_n < 1$, we have

(3.12)
$$\frac{s_n}{r_n} = \frac{1 - t_n}{r_n} + t_n - \delta \le \frac{2\|y_0 - x_0\|}{\rho_0} + 1, \quad \forall n \in \mathbb{N}$$

Now we show there exists $(\widetilde{a_n}, \widetilde{a_n}^*)_{n \in \mathbb{N}}$ in gra $A \cap (H_n \times X^*)$ such that

(3.13)
$$\left\langle z - \widetilde{a_n}, \widetilde{a_n}^* \right\rangle \ge -4K_0^2(\|a_n^*\| + 1)$$

We consider two cases.

 $Case \ 1: \ (v_n, (2 - t_n)a_n^*) \in \text{gra } A.$ Set $(\widetilde{a_n}, \widetilde{a_n}^*) := (v_n, (2 - t_n)a_n^*).$ Then we have $\langle z - \widetilde{a_n}, \widetilde{a_n}^* \rangle = \langle z - v_n, (2 - t_n)a_n^* \rangle$ $\geq -2 \|z - v_n\| \cdot \|a_n^*\|$ $\geq -2(2\|z\| + \|x_0\| + 1) \cdot \|a_n^*\|$ $\geq -4K_0^2(\|a_n^*\| + 1) \quad (by \ (3.10) \text{ and } (3.7)).$

Hence (3.13) holds since $v_n \in H_n$ by (3.9). Case 2: $(v_n, (2 - t_n)a_n^*) \notin \operatorname{gra} A$.

By Fact 2.10 and the assumption that $\{a_n, y_0, x_0\} \subseteq \overline{\operatorname{dom} A}$, (3.9) shows that $v_n \in \overline{\operatorname{dom} A}$. Thus $H_n \cap \operatorname{dom} A \neq \emptyset$ by (3.9) again. Since $(v_n, (2-t_n)a_n^*) \notin \operatorname{gra} A$, $v_n \in H_n$ by (3.9), and A is of type (FPV), there exists $(\widetilde{a_n}, \widetilde{a_n}^*) \in \operatorname{gra} A \cap (H_n \times X^*)$ such that

$$\left\langle v_n - \widetilde{a_n}, \widetilde{a_n}^* - (2 - t_n)a_n^* \right\rangle > 0.$$

Thus by (3.9), we have

$$\left\langle v_n - \widetilde{a_n}, \widetilde{a_n}^* - (2 - t_n) a_n^* \right\rangle > 0$$

$$\Longrightarrow \left\langle r_n \delta z + (1 - r_n) t_n a_n + s_n x_0 - \widetilde{a_n}, \widetilde{a_n}^* - a_n^* - (1 - t_n) a_n^* \right\rangle > 0$$

$$\Longrightarrow \left\langle r_n \delta z + (1 - r_n) t_n a_n + s_n x_0 - \widetilde{a_n}, \widetilde{a_n}^* - a_n^* \right\rangle$$

$$> \left\langle r_n \delta z + (1 - r_n) t_n a_n + s_n x_0 - \widetilde{a_n}, (1 - t_n) a_n^* \right\rangle$$

$$> \left\langle (3.15) \qquad \geq -(1 - t_n) \left(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\| \right) \|a_n^*\|$$

Note that $\widetilde{a_n} = r_n \delta \widetilde{a_n} + (1 - r_n) t_n \widetilde{a_n} + s_n \widetilde{a_n}$. Thus (3.15) implies that

$$\left\langle r_{n}\delta(z-\widetilde{a_{n}})+(1-r_{n})t_{n}(a_{n}-\widetilde{a_{n}})+s_{n}(x_{0}-\widetilde{a_{n}}),\widetilde{a_{n}}^{*}-a_{n}^{*}\right\rangle$$

$$> -(1-t_{n})\left(\|z\|+\|a_{n}\|+\|x_{0}\|+\|\widetilde{a_{n}}\|\right)\|a_{n}^{*}\|$$

$$\Longrightarrow \left\langle r_{n}\delta(z-\widetilde{a_{n}})+s_{n}(x_{0}-\widetilde{a_{n}}),\widetilde{a_{n}}^{*}-a_{n}^{*}\right\rangle$$

$$\ge (1-r_{n})t_{n}\left\langle a_{n}-\widetilde{a_{n}},a_{n}^{*}-\widetilde{a_{n}}^{*}\right\rangle$$

$$-(1-t_{n})\left(\|z\|+\|a_{n}\|+\|x_{0}\|+\|\widetilde{a_{n}}\|\right)\|a_{n}^{*}\|$$

$$\ge -(1-t_{n})\left(\|z\|+\|a_{n}\|+\|x_{0}\|+\|\widetilde{a_{n}}\|\right)\|a_{n}^{*}\|$$

$$\Longrightarrow \left\langle r_{n}\delta(z-\widetilde{a_{n}})+s_{n}(x_{0}-\widetilde{a_{n}}),a_{n}^{*}\right\rangle$$

$$> \left\langle r_{n}\delta(z-\widetilde{a_{n}})+s_{n}(x_{0}-\widetilde{a_{n}}),a_{n}^{*}\right\rangle$$

$$-(1-t_{n})\left(\|z\|+\|a_{n}\|+\|x_{0}\|+\|\widetilde{a_{n}}\|\right)\|a_{n}^{*}\|$$

$$\Longrightarrow r_{n}\delta\left\langle z-\widetilde{a_{n}},\widetilde{a_{n}}^{*}\right\rangle > s_{n}\left\langle \widetilde{a_{n}}-x_{0},\widetilde{a_{n}}^{*}\right\rangle$$

$$+ \left\langle r_{n}\delta(z-\widetilde{a_{n}})+s_{n}(x_{0}-\widetilde{a_{n}}),a_{n}^{*}\right\rangle$$

$$(3.16) -(1-t_{n})\left(\|z\|+\|a_{n}\|+\|x_{0}\|+\|\widetilde{a_{n}}\|\right)\|a_{n}^{*}\|$$

Since $\{(x_0, x_0^*), (\widetilde{a_n}, \widetilde{a_n}^*)\} \subseteq \operatorname{gra} A$, we have $\langle \widetilde{a_n} - x_0, \widetilde{a_n}^* \rangle \geq \langle \widetilde{a_n} - x_0, x_0^* \rangle$ by the monotonicity of A. Thus, by (3.16),

$$r_n \delta \left\langle z - \widetilde{a_n}, \widetilde{a_n}^* \right\rangle > s_n \left\langle \widetilde{a_n} - x_0, x_0^* \right\rangle + \left\langle r_n \delta(z - \widetilde{a_n}) + s_n(x_0 - \widetilde{a_n}), a_n^* \right\rangle$$

$$- (1 - t_n) \left(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\| \right) \|a_n^*\|$$

$$\ge -s_n \|\widetilde{a_n} - x_0\| \cdot \|x_0^*\| - r_n \|z - \widetilde{a_n}\| \cdot \|a_n^*\| - s_n \|x_0 - \widetilde{a_n}\| \cdot \|a_n^*\|$$

$$- (1 - t_n) \left(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\| \right) \|a_n^*\|$$

Hence

$$\left\langle z - \widetilde{a_n}, \widetilde{a_n}^* \right\rangle > -\frac{s_n}{r_n \delta} \|\widetilde{a_n} - x_0\| \cdot \|x_0^*\| - \frac{1}{\delta} \|z - \widetilde{a_n}\| \cdot \|a_n^*\| - \frac{s_n}{r_n \delta} \|x_0 - \widetilde{a_n}\| \cdot \|a_n^*\| - \frac{1 - t_n}{r_n \delta} \left(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\| \right) \|a_n^*\|$$

Then combining (3.11) and (3.12), we have

$$\left\langle z - \widetilde{a_n}, \widetilde{a_n}^* \right\rangle > -\left(\frac{2\|y_0 - x_0\|}{\rho_0} + 1\right) \frac{1}{\delta} \|\widetilde{a_n} - x_0\| \cdot \|x_0^*\| - \frac{1}{\delta} \|z - \widetilde{a_n}\| \cdot \|a_n^*\| - \left(\frac{2\|y_0 - x_0\|}{\rho_0} + 1\right) \frac{1}{\delta} \|x_0 - \widetilde{a_n}\| \cdot \|a_n^*\|$$

$$(3.17) \qquad -\frac{2\|y_0 - x_0\|}{\rho_0 \delta} \Big(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\|\Big) \|a_n^*\| \\ \ge -K_0 \|\widetilde{a_n} - x_0\| - K_0 \|z - \widetilde{a_n}\| \cdot \|a_n^*\| - K_0 \|x_0 - \widetilde{a_n}\| \cdot \|a_n^*\| \\ - K_0 \Big(\|z\| + \|a_n\| + \|x_0\| + \|\widetilde{a_n}\|\Big) \|a_n^*\| \quad (by (3.7))$$

$$(3.17) -K_0 (||z|| + ||a_n|| + ||x_0|| + ||\widetilde{a_n}||) ||a_n^*|| (by (3.7))$$

Since $\widetilde{a_n} \in H_n, t_n \longrightarrow 1^-$ and $a_n \longrightarrow y_0$,

$$(3.18) \qquad \qquad \widetilde{a_n} \longrightarrow y_0.$$

Then we can and do suppose that

(3.19)
$$\max\{\|a_n\|, \|\widetilde{a_n}\|\} \le \|y_0\| + 1 \le \|x_0\| + \|z\| + 1, \quad \forall n \in \mathbb{N}.$$
 (by (3.3))
Then by (3.19), (3.17) and (3.7), we have

$$(3.20) \quad \left\langle z - \widetilde{a_n}, \widetilde{a_n}^* \right\rangle > -K_0^2 - K_0^2 \|a_n^*\| - K_0^2 \|a_n^*\| - K_0^2 \|a_n^*\| \ge -4K_0^2 (\|a_n^*\| + 1).$$

Hence (3.13) holds.

Combining the above two cases, we have (3.13) holds.

Since $\widetilde{a_n} \in H_n$, (3.6) implies that $\widetilde{a_n} \in \text{dom } B$. Then combining (3.18), (3.2), (3.6) and (3.13), Fact 2.17 implies that $F_{A+B}(z, z^*) = +\infty$, which contradicts (3.1). Hence $F_{A+B}(z, z^*) \ge \langle z, z^* \rangle$.

Now we come to our main result.

Theorem 3.2 (Main result). Let $A, B : X \Rightarrow X^*$ be maximally monotone with dom $A \cap$ int dom $B \neq \emptyset$. Assume that $A + N_{\overline{\text{dom }B}}$ is of type (FPV). Then A + B is maximally monotone.

Proof. After translating the graphs if necessary, we can and do assume that $0 \in$ dom $A \cap$ int dom B and that $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} B$. By Corollary 2.8, it suffices to show that

(3.21)
$$F_{A+B}(z, z^*) \ge \langle z, z^* \rangle, \quad \forall (z, z^*) \in X \times X^*.$$

Take $(z, z^*) \in X \times X^*$. Suppose to the contrary that

$$(3.22) F_{A+B}(z, z^*) < \langle z, z^* \rangle$$

Since B is maximally monotone, $B = B + N_{\overline{\text{dom }B}}$. Thus

(3.23)
$$A + B = A + B + N_{\overline{\operatorname{dom} B}} = (A + N_{\overline{\operatorname{dom} B}}) + B.$$

Since $A + N_{\overline{\text{dom }B}}$ is of type (FPV) and $0 \in \text{dom } [A + N_{\overline{\text{dom }B}}] \cap \text{int dom } B$, Fact 2.13 and (3.22) imply that

(3.24)
$$z \notin \overline{\operatorname{dom} B}$$
 and then $z \notin \operatorname{dom} \left[A + N_{\overline{\operatorname{dom} B}}\right]$

Then by Fact 2.15, there exist a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\operatorname{gra}(A + N_{\overline{\operatorname{dom}}B})$ and $\delta \in$ [0,1] such that

(3.25)
$$a_n \longrightarrow \delta z \quad \text{and} \quad \langle z - a_n, a_n^* \rangle \longrightarrow +\infty.$$

Thus $a_n \in \operatorname{dom} \left[A + N_{\overline{\operatorname{dom} B}}\right] \cap \overline{\operatorname{dom} B}, \forall n \in \mathbb{N}.$

Now we consider two cases.

Case 1: There exists a subsequence of $(a_n)_{n \in \mathbb{N}}$ in dom B.

We can and do suppose that $a_n \in \text{dom } B$ for every $n \in \mathbb{N}$. Thus $(a_n)_{n \in \mathbb{N}}$ is in $\text{dom} \left[A + N_{\overline{\text{dom } B}}\right] \cap \text{dom } B$.

Combining Fact 2.14 and (3.25),

$$F_{A+B}(z, z^*) = F_{A+N_{\overline{\mathrm{dom}\,B}}+B}(z, z^*) = +\infty,$$

which contradicts (3.22).

Case 2: There exists $N_1 \in \mathbb{N}$ such that $a_n \notin \text{dom } B, \forall n \ge N_1$.

Then we can and do suppose that $a_n \notin \operatorname{dom} B$ for every $n \in \mathbb{N}$. Thus, $a_n \in \operatorname{dom} \left[A + N_{\overline{\operatorname{dom} B}}\right] \cap \left[\overline{\operatorname{dom} B} \setminus \operatorname{dom} B\right]$. By Proposition 3.1 and (3.25),

$$F_{A+B}(z, z^*) = F_{A+N_{\overline{\operatorname{dom} B}}+B}(z, z^*) \ge \langle z, z^* \rangle,$$

which contradicts (3.22).

Combing all the above cases, we have $F_{A+B}(z, z^*) \ge \langle z, z^* \rangle$ for all $(z, z^*) \in X \times X^*$. Hence A + B is maximally monotone.

Remark 3.3. Theorem 3.2 generalizes the main result in [37] (see [37, Theorem 3.4]).

Corollary 3.4. Let $f: X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function, and let $B: X \rightrightarrows X^*$ be maximally monotone with dom $\partial f \cap \operatorname{int} \operatorname{dom} B \neq \emptyset$. Then $\partial f + B$ is maximally monotone.

Proof. By Fact 2.3 and Fact 2.2 (or [2, Theorem 1.1]), $\partial f + N_{\overline{\text{dom }B}} = \partial (f + \iota_{\overline{\text{dom }B}})$. Then Fact 2.9 shows that $\partial f + N_{\overline{\text{dom }B}}$ is of type (FPV). Applying Theorem 3.2, we have $\partial f + B$ is maximally monotone.

Remark 3.5. Corollary 3.4 provides an affirmative answer to a problem posed by Borwein and Yao in [14, Open problem 4.5].

Given a set-valued operator $A: X \rightrightarrows X^*$, we say A is a *linear relation* if gra A is a linear subspace.

Corollary 3.6 (Linear relation (See [14, Theorem 3.1] or [15, Corollary 4.5]). Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that dom $A \cap$ int dom $B \neq \emptyset$. Then A + B is maximally monotone.

Proof. Apply Fact 2.3, [38, Corollary 3.3] and Theorem 3.2 directly.

Corollary 3.7 (Convex domain (See [15, Corollary 4.3]). Let $A : X \rightrightarrows X^*$ be of type (FPV) with convex domain, and let $B : X \rightrightarrows X^*$ be maximally monotone. Suppose that dom $A \cap$ int dom $B \neq \emptyset$. Then A + B is maximally monotone.

Proof. Apply Fact 2.3, [37, Corollary 2.10] and Theorem 3.2 directly.

Applying Fact 2.11 and Theorem 3.2, we can obtain that the sum problem is equivalent to the following problem:

Open Problem 3.8. Let $A : X \rightrightarrows X^*$ be maximally monotone, and C be a nonempty closed and convex subset of X. Assume that dom $A \cap \operatorname{int} C \neq \emptyset$. Is $A + N_C$ necessarily maximally monotone?

Clearly, Problem 3.8 is a special case of the sum problem. However, if we would have an affirmative answer to Problem 3.8 for every maximally monotone operator A and every nonempty closed and convex set C satisfying Rockafellar's constraint qualification: dom $A \cap \text{int } C \neq \emptyset$, then Fact 2.11 implies that every maximally monotone operator is of type (FPV), and thus $A + N_C$ is of type (FPV) (since $A + N_C$ is maximally monotone by the assumption). Thus applying Theorem 3.2, we have an affirmative answer to the sum problem.

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