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COMMUTATIVE SEMIGROUP OPERATIONS ON $\mathbf{R}^2$ COMPATIBLE WITH THE ORDINARY ADDITIVE OPERATION

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Abstract. We give a commutative semigroup operation $*_\varphi$ on the 2 dimensional Euclid space $\mathbf{R}^2$ associated with a bijection φ from $\mathbf{R}$ to $\mathbf{R}_+$, the set of all positive numbers. We show that $(A *_\varphi B) + C = (A + C) *_\varphi (B + C)$ holds for all $A, B, C \in \mathbf{R}^2$ if and only if $\varphi(x)\varphi(y) = \varphi(0)\varphi(x + y)$ holds for all $x, y \in \mathbf{R}$.

1. Introduction

Semigroup operations on the real numbers $\mathbf{R}$ have a long history (cf. [1, 2, 3]). However, the history of semigroup operations on the 2 dimensional Euclid space $\mathbf{R}^2$ is not the case. In this note, we reveal a certain commutative semigroup operation on $\mathbf{R}^2$ from the standpoint of distribution law.

Denote by $+$ the ordinary additive operation on $\mathbf{R}^2$ defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

for each $a, b, c, d \in \mathbf{R}$. Then there are several commutative semigroup operations $\times$ on $\mathbf{R}^2$ satisfying the following distribution law :

$$(A + B) \times C = A \times C + B \times C \quad (A, B, C \in \mathbf{R}^2). \tag{1.1}$$

Of course we can consider its converse distribution law :

$$(A \times B) + C = (A + C) \times (B + C) \quad (A, B, C \in \mathbf{R}^2). \tag{1.2}$$

The distribution law (1.1) is incompatible with (1.2). Actually, assume that both (1.1) and (1.2) hold. Taking $A = B = C = 0$ in (1.1), we have $0 \times 0 = 0$. Then, taking $A = B = 0$ in (1.2), we have $C = C \times C$. Therefore, taking $A = B = C$ in (1.2), we have $2C = (2C) \times (2C) = 4C$. So we obtain that $C = 0$ for all $C \in \mathbf{R}^2$, a contradiction.

Therefore it is raised a natural question whether there exists a commutative semigroup operation on $\mathbf{R}^2$ which is compatible with the ordinary additive operation $+$ on $\mathbf{R}^2$.

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The purpose of this note is to give a commutative semigroup operation $*_\varphi$ on $\mathbf{R}^2$ associated with a bijection φ from $\mathbf{R}$ to $\mathbf{R}_+$, the set of all positive numbers and to characterize a bijection φ satisfying the distribution law :

$$(A *_\varphi B) + C = (A + C) *_\varphi (B + C) \quad (A, B, C \in \mathbf{R}^2). \tag{1.3}$$

Indeed, such a φ can be characterized as a bijection such that $\varphi(x)\varphi(y) = \varphi(0)\varphi(x+y)$ for all $x, y \in \mathbf{R}$. In particular if φ is continuous, then $\varphi(x) = \beta\alpha^x$ $(x \in \mathbf{R})$ for some positive numbers α and β.

2. Main results

For each $A = (a, b) \in \mathbf{R}^2$, we put

$$A^+ = \frac{a+b}{2} \quad \text{and} \quad A^- = \frac{a-b}{2}.$$

Then it is obvious that

$$A = (A^+ + A^-, A^+ - A^-)$$

holds.

Throughout the remainder of the note, let φ be a bijection from $\mathbf{R}$ to $\mathbf{R}_+$. For each $A, B \in \mathbf{R}^2$, take an element $C \in \mathbf{R}^2$ such that

$$C^+ = \varphi^{-1}(\varphi(A^+) + \varphi(B^+)) \quad \text{and} \quad C^- = \frac{\varphi(A^+)A^- + \varphi(B^+)B^-}{\varphi(A^+) + \varphi(B^+)}.$$

Such an element C is unique and is denoted by $A *_\varphi B$. In this case we have the following

Theorem 2.1. *The binary operation $*_\varphi$ is a commutative semigroup operation on $\mathbf{R}^2$.*

Moreover we have the following interesting characterization.

Theorem 2.2. *The following three conditions are equivalent:*

(i) *The distribution law* (1.3) *holds.*
(ii) $((A *_\varphi B) + C)^- = ((A + C) *_\varphi (B + C))^-$ *holds for all* $A, B, C \in \mathbf{R}^2$.
(iii) $\varphi(x)\varphi(y) = \varphi(0)\varphi(x + y)$ *holds for all* $x, y \in \mathbf{R}$.

Remark 2.3. There are many bijections ψ from $\mathbf{R}$ to $\mathbf{R}_+$ such that $\psi(x)\psi(y) = \psi(0)\psi(x+y)$ for all $x, y \in \mathbf{R}$. In fact, put $\Psi(x) = \log\psi(x) - b$ for each $x \in \mathbf{R}$, where $b = \log\psi(0)$. Then Ψ is a real-valued function on $\mathbf{R}$ such that $\Psi(x+y) = \Psi(x)+\Psi(y)$ for all $x, y \in \mathbf{R}$. Such a function Ψ can be constructed by using Hamel bases (see [3, Theorem 10 in 2.2]). Hence there exist infinitely many function ψ by defining $\psi(x) = e^b e^{\Psi(x)}$ for each $x \in \mathbf{R}$. If, in addition, a bijection ψ is continuous, then so is Ψ. Therefore $\Psi(x) = cx$ $(x \in \mathbf{R})$ for some $c \in \mathbf{R}$, and hence $\psi(x) = \beta\alpha^x$ $(x \in \mathbf{R})$ for some $\alpha, \beta \in \mathbf{R}_+$.

3. Proofs of main results

Proof of Theorem 2.1. By definition, it is obvious that $*_\varphi$ is a commutative binary operation on $\mathbf{R}^2$. Take $A, B, C \in \mathbf{R}^2$ arbitrarily. Then

$$\begin{aligned}((A *_\varphi B) *_\varphi C)^+ &= \varphi^{-1}(\varphi((A *_\varphi B)^+) + \varphi(C^+)) \\ &= \varphi^{-1}(\varphi(A^+) + \varphi(B^+) + \varphi(C^+))\end{aligned}$$

holds. Therefore $((A *_\varphi B) *_\varphi C)^+ = ((A' *_\varphi B') *_\varphi C')^+$ holds for any permutation $A'B'C'$ of $\{ABC\}$, and hence we have

$$\begin{aligned}((A *_\varphi B) *_\varphi C)^+ &= ((C *_\varphi B) *_\varphi A)^+ \\ &= (A *_\varphi (C *_\varphi B))^+ = (A *_\varphi (B *_\varphi C))^+.\end{aligned} \tag{3.1}$$

Also we have

$$\begin{aligned}((A *_\varphi B) *_\varphi C)^- &= \frac{\varphi((A *_\varphi B)^+)(A *_\varphi B)^- + \varphi(C^+)C^-}{\varphi((A *_\varphi B)^+) + \varphi(C^+)} \\ &= \frac{\varphi(A^+)A^- + \varphi(B^+)B^- + \varphi(C^+)C^-}{\varphi(A^+) + \varphi(B^+) + \varphi(C^+)}.\end{aligned}$$

Then $((A *_\varphi B) *_\varphi C)^- = ((A' *_\varphi B') *_\varphi C')^-$ holds for any permutation $A'B'C'$ of $\{ABC\}$, and hence we have

$$\begin{aligned}((A *_\varphi B) *_\varphi C)^- &= ((C *_\varphi B) *_\varphi A)^- \\ &= (A *_\varphi (C *_\varphi B))^- = (A *_\varphi (B *_\varphi C))^-.\end{aligned} \tag{3.2}$$

Therefore we have from (3.1) and (3.2) that $*_\varphi$ is associative. Consequently, $*_\varphi$ is a commutative semigroup operation on $\mathbf{R}^2$. □

We need the following lemma to show Theorem 2.2.

Lemma 3.1. *Suppose that $\varphi(x)\varphi(y) = \varphi(0)\varphi(x + y)$ holds for all $x, y \in \mathbf{R}$. Then*

$$\varphi^{-1}(\varphi(a + c) + \varphi(b + c)) = \varphi^{-1}(\varphi(a) + \varphi(b)) + c$$

holds for all $a, b, c \in \mathbf{R}$.

Proof. Put $\lambda = \frac{1}{\varphi(0)}$. Then we have from hypothesis that

$$x + y = \varphi^{-1}(\lambda\varphi(x)\varphi(y)) \tag{3.3}$$

for all $x, y \in \mathbf{R}$. Take $a, b, c \in \mathbf{R}$ arbitrarily. Then we have from hypothesis and (3.3), applied to $x = \varphi^{-1}(\varphi(a) + \varphi(b))$ and $y = c$ that

$$\begin{aligned}\varphi^{-1}(\varphi(a + c) + \varphi(b + c)) &= \varphi^{-1}(\lambda\varphi(a)\varphi(c) + \lambda\varphi(b)\varphi(c)) \\ &= \varphi^{-1}(\lambda(\varphi(a) + \varphi(b))\varphi(c)) \\ &= \varphi^{-1}(\lambda\varphi(\varphi^{-1}(\varphi(a) + \varphi(b)))\varphi(c)) \\ &= \varphi^{-1}(\varphi(a) + \varphi(b)) + c,\end{aligned}$$

which implies the desired equation. □

Proof of Theorem 2.2. The implication (i) $\Rightarrow$ (ii) follows from definition.

(ii) $\Rightarrow$ (iii). Suppose that

$$((A *_\varphi B) + C)^- = ((A + C) *_\varphi (B + C))^-$$

holds for all $A, B, C \in \mathbf{R}^2$. Take $x, y \in \mathbf{R}$ arbitrarily and put

$$A = (x + 1, x - 1),\ B = (0, 0) \text{ and } C = (y, y).$$

Then

$$A^+ = x,\ B^+ = 0,\ C^+ = y,\ A^- = 1,\ B^- = 0 \text{ and } C^- = 0. \tag{3.4}$$

By hypothesis, we have

$$(A *_\varphi B)^- + C^- = ((A *_\varphi B) + C)^- = ((A + C) *_\varphi (B + C))^-,$$

and hence

$$\begin{aligned} &\frac{\varphi(A^+)A^- + \varphi(B^+)B^-}{\varphi(A^+) + \varphi(B^+)} + C^- \\ &\qquad = \frac{\varphi(A^+ + C^+)(A^- + C^-) + \varphi(B^+ + C^+)(B^- + C^-)}{\varphi(A^+ + C^+) + \varphi(B^+ + C^+)}. \end{aligned} \tag{3.5}$$

Then we have from (3.4) and (3.5) that

$$\frac{\varphi(x)}{\varphi(x) + \varphi(0)} = \frac{\varphi(x + y)}{\varphi(x + y) + \varphi(y)}. \tag{3.6}$$

By deforming (3.6), we obtain that $\varphi(x)\varphi(y) = \varphi(0)\varphi(x + y)$.

(iii) $\Rightarrow$ (i). Suppose that $\varphi(x)\varphi(y) = \varphi(0)\varphi(x + y)$ holds for all $x, y \in \mathbf{R}$. Take $A, B, C \in \mathbf{R}^2$ arbitrarily. Then we have

$$((A *_\varphi B) + C)^+ = (A *_\varphi B)^+ + C^+ = \varphi^{-1}(\varphi(A^+) + \varphi(B^+)) + C^+ \tag{3.7}$$

and

$$((A *_\varphi B) + C)^- = (A *_\varphi B)^- + C^- = \frac{\varphi(A^+)A^- + \varphi(B^+)B^-}{\varphi(A^+) + \varphi(B^+)} + C^-. \tag{3.8}$$

Also we have from Lemma 3.1 that

$$\begin{aligned} ((A + C) *_\varphi (B + C))^+ &= \varphi^{-1}(\varphi(A^+ + C^+) + \varphi(B^+ + C^+)) \\ &= \varphi^{-1}(\varphi(A^+) + \varphi(B^+)) + C^+, \end{aligned}$$

and hence we obtain from (3.7) that

$$((A *_\varphi B) + C)^+ = ((A + C) *_\varphi (B + C))^+. \tag{3.9}$$

Put $\lambda = \frac{1}{\varphi(0)}$. Then we have from hypothesis that

$$\begin{aligned} ((A + C) *_\varphi (B + C))^- &= \frac{\varphi(A^+ + C^+)(A^- + C^-) + \varphi(B^+ + C^+)(B^- + C^-)}{\varphi(A^+ + C^+) + \varphi(B^+ + C^+)} \\ &= \frac{\lambda\varphi(A^+)\varphi(C^+)(A^- + C^-) + \lambda\varphi(B^+)\varphi(C^+)(B^- + C^-)}{\lambda\varphi(A^+)\varphi(C^+) + \lambda\varphi(B^+)\varphi(C^+)} \\ &= \frac{\varphi(A^+)(A^- + C^-) + \varphi(B^+)(B^- + C^-)}{\varphi(A^+) + \varphi(B^+)} \end{aligned}$$

$$= \frac{\varphi(A^+)A^- + \varphi(B^+)B^-}{\varphi(A^+) + \varphi(B^+)} + C^-$$

and hence we obtain from (3.8) that

$$((A *_\varphi B) + C)^- = ((A + C) *_\varphi (B + C))^-. \tag{3.10}$$

By (3.9) and (3.10), we have $(A *_\varphi B) + C = (A + C) *_\varphi (B + C)$. Thus we see that the distribution law (1.3) holds. □

Remark 3.2. One of the reviewers of this paper obtained a simple proof of the implication (iii) ⇒ (i) in Theorem 2.2 by using the transform $\Psi(x) = \log \psi(x) - \log \psi(0)$ introduced in Remark 2.3.

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