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STRICT CONVEXITY AND DE MARR'S THEOREM IN METRIC SPACES

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Abstract. The paper is devoted to a study of fixed points theorems in strictly convex metric spaces and their applications. A variant of De Marr's theorem for the family of Banach operator pairs is obtained. We also give an application in best approximation theory.

1. Introduction

Let (X, d) be a metric space and let $T : X \to X$. T is called a contraction if $d(Tx, Ty) \leq \alpha d(x, y)$ for some $\alpha \in (0, 1)$ and all $x, y \in X$, and nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. Nonexpansive maps can be viewed as natural extensions of contractions. Nevertheless fixed point theory for nonexpansive maps differs sharply from that of contractions in the sense that additional assumptions on the structure of X and/or restrictions on T are needed to guarantee the existence of at least one fixed point.

The systematic study of fixed points of nonexpansive maps was initiated in 1965. In [3], Browder proved that every nonexpansive map T of a closed bounded convex subset K of a Hilbert space X into K has a fixed point. Browder [4], Göhde [14], and Kirk [15] observed that this result could be improved assuming the weaker condition: X is a uniformly convex space or X is a reflexive Banach space with normal structure. These results are significant for the "geometric" conditions which X is required to satisfy. From this point of departure, an extensive theory has been developed which aims to explore more general conditions on the subset K and the space X which still guarantee the existence of a fixed point of the nonexpansive map T. Two important conditions are the convexity of K and strict convexity of X.

A subset K of a normed space X is said to be convex if for all $x, y \in K$, the closed segment $[x, y] = \{z : z = \lambda x+(1-\lambda)y, \lambda \in [0, 1]\} \subseteq K$. It is evident that closed balls are convex and intersection of convex sets is again a convex set. A Banach space X is called strictly convex [10, 12, 13] if any of the following equivalent conditions holds:

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(1) The boundary $\{x \in X : \|x\| = 1\}$ of the unit ball contains no closed segments.
(2) For all $x, y \in X$, with $x \neq y$, if $\|x\| = \|y\| = 1$, then $\|x + y\| < 2$.
(3) For all $x, y \in X$ and $t \in [0, 1]$, there exists a unique $z \in X$ such that $\|x - z\| = t\|x - y\|$ and $\|z - y\| = (1 - t)\|x - y\|$.

In 1963, De Marr [11] obtained the following fixed point theorem:

If K is a nonempty compact convex subset of a Banach space X, and $\mathfrak{A}$ is a nonempty family of commuting nonexpansive self-maps of K, then the family $\mathfrak{A}$ has a common fixed point in K.

In [9], Chen and Li extended De Marr's fixed point theorem to a noncommuting family of nonexpansive maps.

Convexity in metric spaces was first introduced by Takahashi in [17]. In 1999, Bula [5] extended the notion of strict convexity to metric spaces using condition (3). A metric space (X, d) is called convex [1, 5, 6] if for each $x, y \in X$ and for each $t \in [0, 1]$, there exists a $z \in X$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$. If this point z is unique for all possible combinations of x, y and t, then the space X is called strictly convex (see also [2]). In strictly convex metric spaces, the intersection of convex sets is convex, however closed balls in these spaces need not be convex (see [7]). To overcome this difficulty, Bula imposed an additional condition in the notion of strict convexity, namely the convex round balls condition: for all $w \in X$, $d(w, z) < \max\{d(w, x), d(w, y)\}$. Among other things, Bula extended the Browder-Göhde-Kirk fixed point theorem and De Marr's theorem to strictly convex metric spaces.

This paper deals with a study of fixed points theorems in strictly convex metric spaces and their applications. A variant of De Marr's theorem for the family of Banach operator pairs is given. We also derive an application in best approximation theory.

2. Preliminaries

Let (X, d) be a metric space.

Definition 2.1 ([5, Definition 2.5])**.** A set $K \subseteq X$ is said to be *convex* if for each $x, y \in K$ and for each $t \in [0, 1]$, there exists $z \in K$ such that

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

Definition 2.2 ([5, Definition 2.6])**.** A metric space X is said to be *strictly convex* if for each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique $z \in X$ such that

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

Lemma 2.3. *Let $\{K_\alpha : \alpha \in I\}$ be a family of convex sets in a strictly convex metric space. Then $\bigcap_{\alpha \in I} K_\alpha$ is also a convex subset of X.*

Lemma 2.3 lets us define the notion of convex hull in a strictly convex metric space.

Definition 2.4 ([2, Definition 2.13])**.** Let X be a strictly convex metric space and $K \subseteq X$. The *convex hull of K* is the set

$$\operatorname{co}(K) = \bigcap\{C \subseteq X : K \subseteq C \text{ and } C \text{ is convex}\}.$$

$\overline{\mathrm{co}}\,(K)$ will denote the closure of the convex hull of K.

Remark 2.5. Let (X, d) be a strictly convex metric space and $K \subseteq X$. Then:

(i) $\mathrm{co}\,(K)$ is convex and $K \subseteq \mathrm{co}\,(K)$;
(ii) $\mathrm{co}\,(K) = K$ if and only if K is convex;
(iii) $\overline{\mathrm{co}}\,(K) = K$ if and only if K is closed and convex.

In a strictly convex metric space, the intersection of convex sets is a convex set. However, closed balls in strictly convex metric space are not necessarily convex sets (see [7]). So, we require the following definition in addition.

Definition 2.6 ([5, Definition 3.1])**.** A strictly convex metric space (X, d) is said to be a *strictly convex metric space with convex round balls* if for all $x, y, w \in X$ $(x \neq y)$ and for all $t \in (0, 1)$, there exists $z \in X$ such that

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y)$$
$$d(w, z) < \max\{d(w, x), d(w, y)\}.$$

The above strict inequality shows that if x and y belong to

$$S(w, r) = \{a \in X : d(a, w) = r\}, \quad r > 0,$$

then z does not belong to $S(w, r)$, that is, $S(w, r)$ does not contain straight lines.

Lemma 2.7 ([5, Lemma 3.1])**.** *Let (X, d) be a strictly convex metric space with convex round balls. Then the closed ball $B(a, r) = \{y \in X : d(a, y) \leqslant r\}$ is a convex set for every $r > 0$ and every $a \in X$.*

Remark 2.8 ([5, page 8])**.** The condition:
For all $x, y, w \in X$ $(x \neq y)$ and for all $t \in (0, 1)$, there exists $z \in X$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$ and $d(w, z) \leq \max\{d(w, x), d(w, y)\}$ is equivalent with the condition of convexity of closed balls.

Example 2.9 ([5, page 8])**.** The set $\mathbb{R}$ with the metric $d(x, y) = |x - y|$ and the set $\mathbb{R}^2$ with the metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ are both strictly convex metric spaces with convex round balls.

Example 2.10 ([5, page 8])**.** The set $\mathbb{R}^2$ with the metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$ is not a strictly convex metric space.

Example 2.11 ([5, page 8])**.** A trivial example of a strictly convex metric space that is not a strictly convex metric space with convex round balls is $X = \{x\}$ with $d(x, x) = 0$. For a nontrivial example see [7, Section 3].

Lemma 2.12 ([5, Lemma 3.2])**.** *Let (X, d) be a strictly convex metric space with convex round balls, let $K \subseteq X$ be a compact and convex set, and $y \in X$. Then there exists a unique $z \in K$ such that*

$$d(y, z) = \inf\{d(x, y) : x \in K\}.$$

Definition 2.13 ([5, Definition 3.2])**.** A convex set K in a metric space (X, d) is said to have *normal structure* if for each bounded and convex subset $C \subseteq K$ that contains more than one point, there is some point $y \in C$ such that

$$r_y(C) = \sup\{d(x, y) : x \in C\} < \delta(C) = \sup\{d(x, y) : x, y \in C\}.$$

Lemma 2.14 ([5, Lemma 3.3])**.** *Every convex and compact set in a strictly convex metric space* (X,d) *with convex round balls has normal structure.*

In 2007, Chen and Li [8] introduced the class of Banach operator pairs.

Definition 2.15 ([8, Definition 2.1])**.** The pair (I,T) of two self-maps I and T in a metric space (X,d) is called a *Banach operator pair* if the set $F(T)$ of fixed points of T is I-invariant, namely $I(F(T)) \subseteq F(T)$.

Note that if (I,T) is a Banach operator pair, (T,I) need not be such a pair (see [8, Example 1]).

Definition 2.16 ([9, Definition 3.2])**.** Let T and I be two self-maps of a metric space (X,d). The pair (I,T) is called a *symmetric Banach operator pair* if both (T,I) and (I,T) are Banach operator pairs, i.e., $T(F(I)) \subseteq F(I)$ and $I(F(T)) \subseteq F(T)$.

Definition 2.17 ([9, Definition 3.4])**.** A nonempty family $\mathfrak{A}$ of self-maps of a metric space X is called a Banach operator family if for all $S,T \in \mathfrak{A}$, (S,T) is a symmetrical Banach operator pair.

It is easy to see that the pair (I,T) is a symmetric Banach operator pair if and only if T and I are commuting on $F(T) \cup F(I)$.

3. Fixed point results

Motivated by the results of the paper [5], we prove first the following theorem.

Theorem 3.1. *Let* (X,d) *be a strictly convex metric space with convex round balls. Let* $K \subseteq X$ *be a closed convex set. If* $T : K \to K$ *is a nonexpansive map and* $\overline{\text{co}}\,(T(K))$ *is compact, then* T *has a fixed point in* K*.*

Proof. Let $A = \overline{\text{co}}\,(T(K))$. Since $T(K) \subseteq K$, we have

$$A = \overline{\text{co}}\,(T(K)) \subseteq \overline{\text{co}}(K) = K.$$

Thus A is a compact convex subset of K and

$$T(A) \subseteq T(K) \subseteq \overline{\text{co}}\,(T(K)) = A.$$

So the restriction $T : A \to A$ has a fixed point by [5, Theorem 4.1]. □

In the next result, we consider two nonexpansive mappings which form a Banach operator pair.

Theorem 3.2. *Let* (X,d) *be a strictly convex metric space with convex round balls. Let* $K \subseteq X$ *be a closed convex set and let* $S,T : K \to K$ *be two nonexpansive maps. If* (S,T) *is a Banach operator pair and* $\overline{\text{co}}\,(T(K))$ *is compact, then* $F(S,T) \neq \emptyset$*.*

Proof. By Theorem 3.1, $Fix(T) \neq \emptyset$. Also $Fix(T)$ is closed and convex by [5, Lemma 4.1]. Notice that

$$Fix(T) \subseteq \overline{\text{co}}\,(T(K))$$

and so $Fix(T)$ is compact. Since $S(Fix(T)) \subseteq Fix(T)$, S has a fixed point in $Fix(T)$ by Theorem 3.1. As a result, $F(S,T) = Fix(S) \cap Fix(T) \neq \emptyset$. □

Corollary 3.3. *Let (X, d) be a strictly convex metric space with convex round balls. Let $K \subseteq X$ be a closed convex set and $S, T : K \to K$ such that (T, S) is a Banach operator pair on K. If S is continuous on K, T is nonexpansive, $Fix(S)$ is nonempty and convex and $\overline{\text{co}}\,(T(K))$ is compact, then $F(S, T) \neq \emptyset$.*

Proof. We have $T(Fix(S)) \subseteq Fix(S)$. Also, $Fix(S)$ is closed and convex. It is clear that T is a selfmap of $Fix(S)$. Since $\overline{\text{co}}\,(T(Fix(S))) \subseteq \overline{\text{co}}\,(T(K))$, it follows that $\overline{\text{co}}\,(T(Fix(S)))$ is compact. By Theorem 3.1, $F(S, T) \neq \emptyset$. □

The following result extends [9, Proposition 3.3] from normed to strictly convex metric spaces.

Proposition 3.4. *Let (X, d) be a strictly convex metric space. Let $K \subseteq X$ be a convex set, $S : K \to K$ and $\alpha : K \to [0, 1]$ be a map such that the set $\{x \in X : \alpha(x) = 0\}$ is S-invariant, that is, $\alpha(Sx) = 0$, for all $x \in \{x \in X : \alpha(x) = 0\}$. Let $T_\alpha : K \to K$ satisfy*

$$d(x, T_\alpha x) = \alpha(x)d(x, Sx) \text{ and } d(Sx, T_\alpha x) = (1 - \alpha(x))d(x, Sx).$$

Then (S, T_α) is a symmetric Banach operator pair.

Proof. If $x \in Fix(S)$, then $d(x, Sx) = 0$ and so $d(x, T_\alpha x) = 0$ and $d(Sx, T_\alpha x) = 0$, which imply that $x = T_\alpha x$ and $Sx = T_\alpha x$. As a result, $T_\alpha x = Sx = x \in Fix(S)$ and hence $T_\alpha(Fix(S)) \subseteq Fix(S)$.

Now let $x \in Fix(T_\alpha)$. Then $x = T_\alpha x$ and so $\alpha(x)d(x, Sx) = 0$. We consider two cases. If $\alpha(x) \neq 0$, then $Sx = x \in Fix(T_\alpha)$. If $\alpha(x) = 0$, then $\alpha(Sx) = 0$ and so $d(Sx, T_\alpha(Sx)) = \alpha(Sx)d(Sx, S(Sx)) = 0$, which implies that $T_\alpha(Sx) = Sx$, that is, $Sx \in Fix(T_\alpha)$. Consequently, $S(Fix(T_\alpha)) \subseteq Fix(T_\alpha)$. Hence, (S, T_α) is a symmetric Banach operator pair. □

The following theorem extends [9, Lemma 2.2] from normed to strictly convex metric spaces.

Theorem 3.5. *Let (X, d) be a strictly convex metric space with convex round balls. Let $K \subseteq X$ be a closed convex set and let $T : K \to K$ be a nonexpansive map such that there exists a nonempty compact convex set $C \subseteq K$ satisfying $T(C) = C$ and the last set does not reduce to a point. Then there exists a closed convex set K_1 such that:*

(i) *$K_1 \subseteq K$ and $T(K_1) \subseteq K_1$,*
(ii) *$C \cap (K_1)^c \neq \emptyset$.*

Proof. By Lemma 2.14, there is $u \in C$ such that

$$p = r_u(C) = \sup\{d(x, u) : x \in C\} < \delta(C)$$

where $\delta(C)$ is the diameter of C. Since C is not reduced to a point, $\delta(C) > 0$.

Define, for each $x \in C$,

$$U(x) = \{y : d(y, x) \leqslant p\}.$$

Since $u \in U(x)$ for each $x \in C$, we have

$$K_1 = \bigcap_{x \in C} U(x) \neq \emptyset.$$

Note that K_1 is closed and convex. For any $x \in K_1 \cap K$ and any $z \in C$ we have $x \in U(z)$, that is, $d(x,z) \leqslant p$. Since $T(C) = C$, there exists $y \in C$ such that $z = Ty$. Since T is nonexpansive,

$$d(Tx,z) = d(Tx,Ty) \leqslant d(x,y) \leqslant p$$

and so $Tx \in U(z)$. Since this holds for any $z \in C$, we have

$$Tx \in \bigcap_{z\in C} U(z) = K_1,$$

which implies that $Tx \in K_1 \cap K$. Thus $Tx \in K_1 \cap K$ for all $x \in K_1 \cap K$. Since C is compact, there exist $x_0, x_1 \in C$ such that

$$d(x_0,x_1) = \delta(C) > p.$$

Note also that $x_1 \notin U(x_0) \supseteq K_1$ and hence $x_1 \in C \cap (K_1)^c$, that is, $C \cap (K_1)^c \neq \emptyset$. □

We conclude this section with the following result which generalizes De Marr's theorem, that is, main result of [9] (Theorem 3.5) for a family of nonexpansive mappings, from normed to strictly convex spaces.

Theorem 3.6. *Let (X,d) be a strictly convex metric space with convex round balls. Let $K \subseteq X$ be a nonempty closed convex set and $\mathfrak{A}$ a nonempty family of nonexpansive maps of K into itself. If $\mathfrak{A}$ is a Banach operator family and there exists a $T \in \mathfrak{A}$ such that $\overline{\mathrm{co}}\,(T(K))$ is compact, then $\mathfrak{A}$ has a common fixed point in K.*

Proof. We shall show that

$$F(T,S_1,S_2) \neq \emptyset$$

for any two maps $S_1, S_2 \in \mathfrak{A}$. Let Γ denote the set of all nonempty closed and convex subsets C of K such that $T(C) \subseteq C, S_1(C) \subseteq C, S_2(C) \subseteq C$ and $\overline{\mathrm{co}}\,(T(C))$ is compact for all $C \in \Gamma$. On the set Γ, define a partial order "$\leq$" as the set inclusion, that is, $C_i \leq C_j$ if and only if $C_i \subseteq C_j$. We can find a minimal set $C_0 \in \Gamma$. If the set C_0 is a singleton, then $F(T,S_1,S_2) \neq \emptyset$. Suppose to the contrary that C_0 contains at least two different points. By Theorem 3.2, T and S_1 have a nonempty compact convex common fixed point set $F = F(T,S_1)$ in C_0 satisfying $T(F) = F$ and $S_1(F) = F$. Since both (S_2,T) and (S_2,S_1) are Banach operator pairs, we have $S_2(F) \subseteq F$. Using Zorn's lemma, there is a minimal nonempty compact convex subset of C_0, say C_1 such that

$$T(C_1) = C_1,\ S_1(C_1) = C_1 \text{ and } S_2(C_1) \subseteq C_1.$$

Next, we show that $S_2(C_1) = C_1$. Indeed, if $S_2(C_1) \neq C_1$, then $S_2(S_2(C_1)) \subseteq S_2(C_1)$ and $S_2(C_1)$ is compact. Hence, $S_2(C_1) \subseteq C_1 \subseteq F$ and $T(S_2(C_1)) = S_2(C_1)$ and $S_1(S_2(C_1)) = S_2(C_1)$. But this contradicts the minimality of C_1. If C_1 has only one point, then

$$F(T,S_1,S_2) \neq \emptyset.$$

So suppose that C_1 has at least two points. Then, by Theorem 3.5, there exists a set $K_1 \in \Gamma$ satisfying $C_1 \cap (K_1)^c \neq \emptyset$, which implies that K_1 is a proper subset of C_0. This contradicts the minimality of C_0. Consequently C_0 is a singleton and so

$$F(T,S_1,S_2) \neq \emptyset.$$

It can be shown by induction that for any finite family of maps $S_j \in \mathfrak{A}$, $j = 1, 2, \ldots, n$, the common fixed point set $F(T, S_1, S_2, \ldots, S_n) \neq \emptyset$. Let

$$\Lambda = \{F(T, S) : S \in \mathfrak{A}\}.$$

Then for any $S \in \mathfrak{A}$, $F(T, S)$ is a nonempty compact set and for any $S_j \in \mathfrak{A}$, $j = 1, 2, \ldots, n$, we have

$$\bigcap_{j=1}^{n} F(T, S_j) = F(T, S_1, S_2, \ldots, S_n) \neq \emptyset.$$

Thus the set family Λ has the finite intersection property and hence

$$\bigcap_{S \in \mathfrak{A}} F(T, S) \neq \emptyset,$$

that is, the family $\mathfrak{A}$ has a fixed point in K. $\square$

4. Best approximations and fixed points

Let (X, d) be a metric space, $K \subseteq X$ and $x \in X$. Recall (see, e.g., [8, 16]) that a point $y \in K$ is called a best approximation of x in K if

$$d(x, y) = \operatorname{dist}(x, K) = \inf\{d(x, z) : z \in K\}.$$

The set of all best approximations of x in K will be denoted by $P_K(x)$. The problem of proving the existence, and possibly finding, best approximations is one of the important ones in applications.

Lemma 4.1. *In a strictly convex metric space (X, d) with convex round balls, $P_K(x)$ is a singleton if $P_K(x)$ is nonempty and $K \subseteq X$ is convex.*

Proof. Suppose $P_K(x)$ is nonempty. If $y_1, y_2 \in P_K(x)$ with $y_1 \neq y_2$, then

$$d(y_1, x) = \operatorname{dist}(x, K) \text{ and } d(y_2, x) = \operatorname{dist}(x, K).$$

Since K is convex, for fixed $t \in [0, 1]$, there exists $y_0 \in K$ such that

$$d(y_1, y_0) = td(y_1, y_2) \text{ and } d(y_2, y_0) = (1 - t)d(y_1, y_2),$$
$$d(x, y_0) < \max\{d(x, y_1), d(x, y_2)\}.$$

This implies that

$$\operatorname{dist}(x, K) \leq d(x, y_0) < \max\{\operatorname{dist}(x, K), \operatorname{dist}(x, K)\} = \operatorname{dist}(x, K),$$

which is a contradiction. Hence $P_K(x)$ is a singleton. $\square$

The following theorem gives sufficient conditions in order that the set $P_K(z)$ be nonempty, for some specific $z \in X$.

Theorem 4.2. *Let (X, d) be a strictly convex metric space. Let $K \subseteq X$ be a closed convex set and $T : X \to X$ be a nonexpansive map. If T has a fixed point $z \in X$, K is T-invariant and $\overline{\text{co}}\,(T(K))$ is compact, then the set of best approximations $P_K(z)$ is not empty.*

Proof. Since $\overline{\text{co}}\,(T(K))$ is compact, there exists $y \in \overline{\text{co}}\,(T(K))$ such that

$$\text{dist}\,(z, \overline{\text{co}}(T(K))) = d(z, y).$$

Note that $y \in K$, since K is closed and $\overline{\text{co}}\,(T(K)) \subseteq K$. Also,

$$\begin{aligned}\text{dist}\,(z, \overline{\text{co}}(T(K))) &\leqslant \text{dist}\,(z, T(K)) \leqslant d(z, Tw)\\ &= d(Tz, Tw) \leqslant d(z, w)\end{aligned}$$

for all $w \in K$. Now

$$\text{dist}\,(z, K) \leq \text{dist}\,(z, \overline{\text{co}}(T(K))) \leq d(z, w)$$

for all $w \in K$. Thus

$$\text{dist}\,(z, K) = \text{dist}\,(z, \overline{\text{co}}(T(K)) = d(z, y).$$

So $y \in P_K(z)$, which means that $P_K(z) \neq \emptyset$. □

By strengthening the conditions of the previous theorem, we can prove the uniqueness of best approximation.

Theorem 4.3. *Let (X, d) be a strictly convex metric space with convex round balls. Let $K \subseteq X$ be a closed convex set and $T : K \to K$ a nonexpansive map. If T has a fixed point z and $\overline{\text{co}}\,(T(K))$ is compact, then the point z has a unique best approximation y in K which is also a fixed point of T.*

Proof. By Theorem 4.2, $P_K(z) \neq \emptyset$. Let $y \in P_K(z)$. Then $d(z, y) = \text{dist}\,(z, K)$. Notice that

$$\text{dist}\,(z, K) \leq d(z, Ty) = d(Tz, Ty) \leq d(z, y) = \text{dist}\,(z, K).$$

This implies that $Ty \in P_K(z)$ and so $T(P_K(z)) \subseteq P_K(z)$. But since X is a strictly convex metric space with convex round balls, $P_K(z)$ is a singleton by Lemma 4.1 and so $P_K(z) = \{y\}$. Hence $Ty = y$. □

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