Linear and Nonlinear Analysis
Volume 1, Number 1, 2015, 53–65

Yokohama Publishers
ISSN 2188-8167

STRONG CONVERGENCE FOR A COMMON FIXED POINT OF TWO DIFFERENT GENERALIZATIONS OF CUTTER OPERATORS

YASUNORI KIMURA AND SATIT SAEJUNG*

Abstract. We propose two iterative methods for finding a common fixed point of two different generalizations of cutter mappings in Banach spaces. The results obtained in this paper extend the recent results announced by Kimura et al.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We say that $T : H \to H$ is a *cutter operator* if $\mathrm{Fix}(T) := \{x \in H : x = Tx\} \neq \varnothing$ and

$$\langle Tx - z, Tx - x \rangle \leq 0 \quad \text{for all } x \in H \text{ and } z \in \mathrm{Fix}(T).$$

This type of operators was studied by Bauschke and Combettes [5] and Combettes [9]. The term cutter operator was proposed by Cegielski and Censor [7]. These operators play an important and interesting role in various nonlinear problems. The purpose of this paper is to continue the study of these operators in Banach space setting.

Let E be a real Banach space with the norm $\| \cdot \|$. We say that E is

- *smooth* if the limit $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$;
- *Fréchet smooth* if the limit above does not only exists but is also attained uniformly for all $\|y\| = 1$ whenever x is fixed and $\|x\| = 1$;
- *uniformly smooth* if the limit above does not only exists but is also attained uniformly for all $x, y \in E$ with $\|x\| = \|y\| = 1$.

For more details on the geometry of Banach spaces we refer the reader to [18].

Throughout the paper, we denote by E^* the dual space of E and denote by $\langle \cdot, \cdot \rangle$ the dual pairing acting from $E \times E^*$ into $\mathbb{R}$, that is, whenever $x \in E$ and $x^* \in E^*$, $\langle x, x^* \rangle$ denote the value of x^* at x. We use the notions $\to$ and $\rightharpoonup$ for strong and

2010 *Mathematics Subject Classification.* 47H10, 47H09.
Key words and phrases. Strong convergence theorem, generalized cutter operator, common fixed point, Halpern type iteration, shrinking projection method.
*Corresponding author. The second author was supported by the Thailand Research Fund and Khon Kaen University under Grant Number RSA5680002.

weak convergences, respectively. For a bounded sequence $\{x_n\}$, let

$$\omega_w\{x_n\} = \{z : \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup z \text{ as } k \to \infty\}.$$

The *duality mapping* $J : E \to 2^{E^*}$ is the point-to-set mapping defined by

$$x \mapsto Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The following facts are known and referred in the paper.

- If E is smooth, then Jx is a singleton for all $x \in E$, and hence we treat J as a single-valued mapping from E into E^*.
- If E is Fréchet smooth, then $J : E \to E^*$ is norm-to-norm continuous.
- If E is uniformly smooth, then $J : E \to E^*$ is uniformly norm-to-norm continuous on bounded subsets of E.
- If E^* is Fréchet smooth and $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

In a similar way, we consider the duality mapping $J^* : E^* \to 2^{E^{**}}$. It is not hard to see that if E and E^* are smooth and E is reflexive, then $J : E \to E^*$ is bijective and $J^* = J^{-1}$. We refer the readers to [8] and its review [21] for further information on duality mappings.

Let C be a closed and convex subset of a smooth Banach space E. The following mappings are two different generalizations of cutter operators in Banach space setting. A mapping $T : C \to E$ is said to be

- *cutter mapping of type (P)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, J(Tx - x) \rangle \leq 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$;
- *cutter mapping of type (Q)* if $\mathrm{Fix}(T) \neq \varnothing$ and $\langle Tx - z, JTx - Jx \rangle \leq 0$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$.

The notations (P) and (Q) are from the recent paper of Aoyama et al. (see [3]). This definition of mappings is a particular case of the quasi-Bregman firmly nonexpansive mappings which was introduced first in 2003 by Bauschke, Borwein and Combettes in [4]. This class and several more class of operators with respect to Bregman distances were studied intensively during the last ten years (see, for instance, [4, 17, 24]).

We recall the concept of the distance-like function in a smooth Banach space E. Let $\varphi : E \times E \to \mathbb{R}$ be defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

It is clear that $(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. If E is a Hilbert space, then $\varphi(x, y) = \|x - y\|^2$. It is also known that if E and E^* are smooth spaces, then

$$\varphi(x, y) = 0 \iff x = y.$$

Due to this function φ, Alber [1] introduced the following type of projection. Suppose that E is a reflexive Banach space such that E and E^* are smooth, and C is a nonempty, closed and convex subset of E. It is known that for each $x \in E$ there exists a unique element z in C, denoted by $\Pi_C x$, such that

$$\varphi(\Pi_C x, x) = \inf\{\varphi(y, x) : y \in C\}.$$

Moreover, the relation above can be characterized by the following inequalities: for $z \in C$,

$$\begin{aligned} z = \Pi_C x &\iff \langle y - z, Jx - Jz \rangle \leq 0 \quad \text{for all } y \in C \\ &\iff \varphi(y, z) + \varphi(z, x) \leq \varphi(y, x) \quad \text{for all } y \in C. \end{aligned}$$

It is not hard to see that $\Pi_C : E \to C$ is a cutter mapping of type (Q).

In this paper, we also deal with the metric projection. For a closed and convex subset C and for $x \in E$, there exists a unique element z in C, denoted by $P_C x$, such that

$$\|P_C x - x\| = \inf\{\|y - x\| : y \in C\}.$$

It is also not hard to see that $P_C : E \to C$ is a cutter mapping of type (P) (for example, see [28]).

The following result shows a relation between convergences in the sense of φ and of the norm.

Lemma 1.1 (Kamimura and Takahashi [12]). *Suppose that E is a smooth Banach space and E^* is uniformly smooth. If $\{x_n\}$ and $\{y_n\}$ are sequences in E such that one of them is bounded and $\varphi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

We also need the following lemma proved by Maingé.

Lemma 1.2 ([16]). *Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \geq 1$. Then there exists a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty} m_k = \infty$ and the following two inequalities:*

$$\gamma_{m_k} \leq \gamma_{m_k+1} \quad \text{and} \quad \gamma_k \leq \gamma_{m_k+1}$$

hold for all (sufficiently large) numbers k. In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

2. Main results

2.1. **Strong convergence via a new averaged projection method of Halpern type.** Recall that a mapping $U : C \to E$ is *closed at zero* if whenever $\{x_n\}$ is a sequence in C such that $x_n \to p \in C$ and $Ux_n \to 0$ it follows that $Up = 0$.

Theorem 2.1. *Suppose that E and E^* are uniformly smooth spaces. Let C be a closed and convex subset of E. Suppose that $S, T : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is a cutter mapping of type (Q);*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C \text{ is arbitrarily chosen};\\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \le 0\};\\ B_n = \{z \in C : \langle Tx_n - z, JTx_n - Jx_n\rangle \le 0\};\\ C_n = A_n \cap B_n;\\ y_n^* = \alpha_n J\widehat{x} + (1-\alpha_n)\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right);\\ x_{n+1} = \Pi_C J^{-1} y_n^*; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n^k\}_{n,k}$ are sequences in $(0,1)$ such that

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(2) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all n;*
(3) $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0,1)$ *for all k and* $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Then the sequence $\{x_n\}$ converges to $\Pi_F \widehat{x}$.

Remark 2.2. It follows from the assumptions of the theorem that $\sum_{k=1}^{\infty} \beta^k = 1$.

We split the proof of Theorem 2.1 into the following six lemmas.

Lemma 2.3. *If the element x_n is defined, then C_n is a closed and convex subset containing F.*

Denote $z := \Pi_{\bigcap_{n=1}^{\infty} C_n} \widehat{x}$ and $U_n := J^{-1}\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k}\right)$.

Lemma 2.4. *For each $n \ge 1$, the following inequalities hold:*

$$\begin{aligned} \varphi(z, x_{n+1}) &\le \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\varphi(z, U_n x_n)\\ &\le \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\left(\varphi(z, x_n) - \sum_{k=1}^{n} \beta_n^k \varphi(\Pi_{C_k} x_n, x_n)\right)\\ &\le \alpha_n \varphi(z, \widehat{x}) + (1-\alpha_n)\varphi(z, x_n). \end{aligned}$$

In particular, the sequence $\{x_n\}$ is bounded.

Lemma 2.5. *For each $n \ge 1$, the following inequality holds:*

$$\varphi(z, x_{n+1}) \le (1-\alpha_n)\varphi(z, x_n) + 2\alpha_n \langle J^{-1} y_n^* - z, J\widehat{x} - Jz\rangle.$$

Proof. We first observe the following inequality

$$\varphi(u, J^{-1}(\gamma Jv + (1-\gamma)Jw)) \le (1-\gamma)\varphi(u, w) + 2\gamma\langle J^{-1}(\gamma Jv + (1-\gamma)Jw) - u, Jv - Ju\rangle$$

whenever $u, v, w \in E$ and $\gamma \in (0,1)$. In fact, it follows from the subdifferential inequality of $\|\cdot\|^2$ on E^*. Consequently,

$$\begin{aligned} &\varphi(z, J^{-1} y_n^*)\\ &= \varphi\left(z, J^{-1}\left(\alpha_n J\widehat{x} + (1-\alpha_n)\left(\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right)\right)\right) \end{aligned}$$

$$\leq (1-\alpha_n)\varphi\left(z, J^{-1}\left(\sum_{k=1}^{n}\beta_n^k J\Pi_{C_k}x_n\right)\right) + 2\alpha_n\langle J^{-1}y_n^* - z, J\widehat{x} - Jz\rangle.$$

Note that $z \in \bigcap_{k=1}^{\infty} C_k$. Hence

$$\varphi(z, \Pi_{C_k}x_n) \leq \varphi(z, \Pi_{C_k}x_n) + \varphi(\Pi_{C_k}x_n, x_n) \leq \varphi(z, x_n).$$

It follows then that

$$\varphi\left(z, J^{-1}\left(\sum_{k=1}^{n}\beta_n^k J\Pi_{C_k}x_n\right)\right) \leq \sum_{k=1}^{n}\beta_n^k\varphi(z, \Pi_{C_k}x_n) \leq \varphi(z, x_n).$$

Therefore, since $z \in C$, we have

$$\varphi(z, x_{n+1}) \leq \varphi(z, J^{-1}y_n^*) \leq (1-\alpha_n)\varphi(z, x_n) + 2\alpha_n\langle J^{-1}y_n^* - z, J\widehat{x} - Jz\rangle. \qquad \square$$

The following result can be easily obtained by the recent result of Nilsrakoo and Saejung [20].

Lemma 2.6. *Suppose that*

$$U = J^{-1}\left(\sum_{k=1}^{\infty}\beta^k J\Pi_{C_k}\right)$$

and that $\{z_m\}$ is a bounded sequence in C. Then the following are equivalent:

- $z_m - \Pi_{C_n}z_m \to 0$ *as* $m \to \infty$ *for all* $n \in \mathbb{N}$;
- $z_m - Uz_m \to 0$.

In particular, $\mathrm{Fix}(U) = \bigcap_{n=1}^{\infty} C_n$. *Moreover,* $JU_n \to JU$ *uniformly on bounded sets.*

Proof. We prove only the last assertion. Let B be a bounded set and let M be a number such that $\|x\| \leq M$ for all $x \in B$. It follows from $z \in \bigcap_{k=1}^{\infty} C_k$ that $(\|z\| - \|\Pi_{C_k}x\|)^2 \leq \varphi(z, \Pi_{C_k}x) \leq \varphi(z, x) \leq (\|z\| + \|x\|)^2 \leq (\|z\| + M)^2$ for all $x \in B$ and $k \in \mathbb{N}$. Hence $\|\Pi_{C_k}x\| \leq 2\|z\| + M$ for all $x \in B$ and $k \in \mathbb{N}$. Consequently, for $x \in B$, we get

$$\begin{aligned}\|JU_nx - JUx\| &= \left\|\sum_{k=1}^{n}(\beta_n^k - \beta^k)J\Pi_{C_k}x + \sum_{k=n+1}^{\infty}\beta^k J\Pi_{C_k}x\right\| \\ &\leq \sum_{k=1}^{n}|\beta_n^k - \beta^k|\|J\Pi_{C_k}x\| + \sum_{k=n+1}^{\infty}\beta^k\|J\Pi_{C_k}x\| \\ &\leq \left(\sum_{k=1}^{n}|\beta_n^k - \beta^k| + \sum_{k=n+1}^{\infty}\beta^k\right)(2\|z\| + M).\end{aligned}$$

It follows that $\lim_{n\to\infty}\sup\{\|JU_nx - JUx\| : x \in B\} = 0$. $\square$

Lemma 2.7. *If there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that*

$$\liminf_{j\to\infty}(\varphi(z, x_{m_j+1}) - \varphi(z, x_{m_j})) \geq 0,$$

then $\omega_w\{x_{m_j}\}_{j=1}^{\infty} \subset \bigcap_{n=1}^{\infty} C_n$. Moreover, $\limsup_{j\to\infty}\langle J^{-1}y_{m_j}^* - z, J\widehat{x} - Jz\rangle \leq 0$.

Proof. It follows from Lemma 2.4 and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{j\to\infty}\sum_{k=1}^{m_j}\beta_{m_j}^k\varphi(\Pi_{C_k}x_{m_j},x_{m_j}) = 0.$$

In particular, for each k, we have

$$\beta^k\lim_{j\to\infty}\varphi(\Pi_{C_k}x_{m_j},x_{m_j}) = \lim_{j\to\infty}\beta_{m_j}^k\varphi(\Pi_{C_k}x_{m_j},x_{m_j}) = 0.$$

This implies that $x_{m_j} - \Pi_{C_k}x_{m_j} \to 0$ as $j \to \infty$ because E^* is uniformly smooth. Consequently, $\omega_w\{x_{m_j}\} \subset C_k$. Since the last inclusion holds for all $k \in \mathbb{N}$, we have $\omega_w\{x_{m_j}\} \subset \bigcap_{k=1}^\infty C_k$.

Finally, to prove the "Moreover" part, we claim that $J^{-1}y_{m_j}^* - x_{m_j} \to 0$ as $j \to \infty$. If this is so, then it follows from $\omega_w\{x_{m_j}\}_{j=1}^\infty \subset \bigcap_{k=1}^\infty C_k$ that

$$\begin{aligned}&\limsup_{j\to\infty}\langle J^{-1}y_{m_j}^* - z, J\widehat{x} - Jz\rangle\\ &= \limsup_{j\to\infty}\langle x_{m_j} - \Pi_{\bigcap_{n=1}^\infty C_n}\widehat{x}, J\widehat{x} - J\Pi_{\bigcap_{n=1}^\infty C_n}\widehat{x}\rangle \le 0.\end{aligned}$$

To prove the last claim, let us note from the first part that $x_{m_j} - \Pi_{C_k}x_{m_j} \to 0$ as $j \to \infty$ for all $k = 1, 2, \dots$. In virtue of Lemma 2.6, we have $x_{m_j} - Ux_{m_j} \to 0$ as $j \to \infty$ and hence $Jx_{m_j} - JUx_{m_j} \to 0$ as $j \to \infty$. Note that $JU_n \to JU$ uniformly on bounded sets. It follows then that $Jx_{m_j} - y_{m_j}^* = Jx_{m_j} - JU_{m_j}x_{m_j} \to 0$ as $j \to \infty$, that is, $J^{-1}y_{m_j}^* - x_{m_j} \to 0$ as $j \to \infty$. □

The following lemma also plays an important role in this subsection. However, its proof given there is not quite accurate.

Lemma 2.8 (Saejung and Yotkaew [26]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^\infty \alpha_n = \infty$, and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n \quad \text{for all } n \ge 1.$$

If $\limsup_{j\to\infty} t_{m_j} \le 0$ for every subsequence $\{s_{m_j}\}$ of $\{s_n\}$ satisfying

$$\liminf_{j\to\infty}(s_{m_j+1} - s_{m_j}) \ge 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Proof. The proof is split into two cases.

Case 1: There exists an $n_0 \in \mathbb{N}$ such that $s_{n+1} \le s_n$ for all $n \ge n_0$. It follows then that $\lim_{n\to\infty} s_n = s$ for some $s \ge 0$. In particular, $\liminf_{n\to\infty}(s_{n+1} - s_n) = 0$ and hence $\limsup_{n\to\infty} t_n \le 0$. On the other hand, for $n \ge n_0$, we have

$$\alpha_n(s_n - t_n) \le s_n - s_{n+1}.$$

Let $\varepsilon > 0$ be given. Then there exists an integer $n_1 \ge n_0$ such that $s_n \ge s - \varepsilon$ and $t_n \le \varepsilon$ for all $n \ge n_1$. For any $n \ge n_1$, we have

$$\alpha_n(s - 2\varepsilon) \le \alpha_n(s_n - t_n) \le s_n - s_{n+1}.$$

In particular,

$$(s-2\varepsilon)\sum_{n=n_1}^{\infty}\alpha_n \le s_{n_1}-s<\infty.$$

It follows from $\sum_{n=1}^{\infty}\alpha_n=\infty$ that $s\le 2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we conclude that $s=0$.

Case 2: There exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j}<s_{n_j+1}$ for all $j\in\mathbb{N}$. In this case, we can apply Lemma 1.2 to find a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty}m_k=\infty$ and the following two inequalities:

$$s_{m_k}\le s_{m_k+1} \quad \text{and} \quad s_k\le s_{m_k+1}$$

hold for all (sufficiently large) numbers k. Note that $\{s_{m_k}\}$ is not necessarily a subsequence of $\{s_n\}$. Let $\{p_j\}$ be the subsequence of $\{m_k\}$ such that $\{p_j\}$ is strictly increasing and each term in $\{m_k\}$ belongs to $\{p_j\}$. Now $\{s_{p_j}\}$ is a subsequence of $\{s_n\}$. It follows from the first inequality that $\liminf_{j\to\infty}(s_{p_j+1}-s_{p_j})\ge 0$ and hence $\limsup_{j\to\infty}t_{p_j}\le 0$. Moreover, by the first inequality again, we have

$$s_{p_j+1}\le(1-\alpha_{p_j})s_{p_j}+\alpha_{p_j}t_{p_j}\le(1-\alpha_{p_j})s_{p_j+1}+\alpha_{p_j}t_{p_j}.$$

In particular, since each $\alpha_{p_j}>0$, we have $s_{p_j+1}\le t_{p_j}$. Finally, it follows from the second inequality that

$$\limsup_{k\to\infty}s_k\le\limsup_{k\to\infty}s_{m_k+1}=\limsup_{j\to\infty}s_{p_j+1}\le\limsup_{j\to\infty}t_{p_j}\le 0.$$

Hence $\lim_{k\to\infty}s_k=0$.

This completes the proof. □

We now give the proof of the main result.

Proof of Theorem 2.1. Denote $s_n:=\varphi(z,x_n)$ and $t_n:=2\langle J^{-1}y_n^*-z,J\widehat{x}-Jz\rangle$. It follows from Lemma 2.5 that

$$s_{n+1}\le(1-\alpha_n)s_n+\alpha_n t_n \quad \text{for all } n\ge 1.$$

All prerequisites of Lemma 2.8 are satisfied. Then $x_n\to z$.

We are going to make use of the closedness of $I-S$ and $I-T$ at zero. Since $z=\Pi_{\bigcap_{k=1}^{\infty}C_k}\widehat{x}\in\bigcap_{k=1}^{\infty}A_k\subset A_n$ for all n and $Sx_n=P_{A_n}x_n$, we have

$$\|Sx_n-x_n\|\le\|z-x_n\|\to 0.$$

It follows then that $z=Sz$. Similarly, since $z=\Pi_{\bigcap_{n=1}^{\infty}C_n}\widehat{x}\in\bigcap_{n=1}^{\infty}B_n\subset B_n$ for all n and $Tx_n=\Pi_{B_n}x_n$, we have

$$\varphi(Tx_n,x_n)\le\varphi(z,x_n)\to 0.$$

In particular, $x_n-Tx_n\to 0$ by Lemma 1.1 and hence $z=Tz$.

Moreover, it follows from $z=\Pi_{\bigcap_{n=1}^{\infty}C_n}\widehat{x}$ and $F\subset\bigcap_{n=1}^{\infty}C_n$ that $\varphi(z,\widehat{x})\le\varphi(\Pi_F\widehat{x},\widehat{x})$. Because $z\in F$, so $z=\Pi_F\widehat{x}$. The proof is finished. □

Using the same proof (with a slight modification) as the preceding result, we also have the following:

Theorem 2.9. *Suppose that E and E^* are uniformly smooth. Let C be a closed convex subset of E. Suppose that $T : C \to C$ and $S : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is relatively quasi-nonexpansive, that is, $\operatorname{Fix}(T) \neq \varnothing$ and $\varphi(z, Tx) \leq \varphi(z, x)$ for all $x \in C$ and $z \in \operatorname{Fix}(T)$;*
- *$F := \operatorname{Fix}(S) \cap \operatorname{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C \text{ is arbitrarily chosen}; \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \varphi(z, Tx_n) \leq \varphi(z, x_n)\}; \\ C_n = A_n \cap B_n; \\ y_n^* = \alpha_n J\widehat{x} + (1-\alpha_n)\left(\displaystyle\sum_{k=1}^{n} \beta_n^k J\Pi_{C_k} x_n\right); \\ x_{n+1} = \Pi_C J^{-1} y_n^*; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n^k\}_{n,k}$ are sequences in $(0,1)$ such that

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all n;*
(3) $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0,1)$ *for all k and* $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Then the sequence $\{x_n\}$ converges to $\Pi_F \widehat{x}$.

Remark 2.10. Theorem 2.9 can be viewed as an extension of the recent result of Kimura et al. [15]. It is worth mentioning that our assumption on the sequence $\{\beta_n^k\}_{n,k}$ is strictly weaker than that of the aforementioned result. In fact, if $\{\beta_n^k\}_{n,k}$ is a sequence in $(0,1)$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for all n and $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\beta_n^k - \beta_{n+1}^k| < \infty$ and $\lim_{n\to\infty} \beta_n^k = \beta^k \in (0,1)$ for all k, then $\lim_{n\to\infty} \sum_{k=1}^{n} |\beta_n^k - \beta^k| = 0$.

Remark 2.11. Theorem 2.1 itself can be regarded as an extension of Kimura et al. In fact, let $T' : C \to H$ be a quasi-nonexpansive mapping. It is easy to see that

$$\{z \in C : \|z - T'x\| \leq \|z - x\|\} = \{z \in C : \langle Tx - z, Tx - x\rangle \leq 0\}$$

where $T = \frac{1}{2}(I + T')$. Moreover, T is a cutter mapping.

2.2. **Strong convergence via the shrinking projection method.** In this subsection, we present another strong convergence theorem without assuming the uniform smoothness of E and E^*.

Let us recall the concept of Mosco convergence [19] for a sequence of closed and convex sets in a Banach space. Suppose that E is a reflexive Banach space and $\{C_n\}$ is a sequence of nonempty closed and convex subsets of E. We consider the following two sets:

$$x \in \text{s-liminf}_{n\to\infty} C_n \iff \exists \{x_n\} \subset E \text{ such that } x_n \to x \text{ and } x_n \in C_n \text{ for all } n;$$
$$x \in \text{w-limsup}_{n\to\infty} C_n \iff \exists \{n_k\} \subset \{n\}\ \exists \{x_k\} \subset E \text{ such that } x_k \rightharpoonup x$$

$$\text{and } x_k \in C_{n_k} \text{ for all } k.$$

If there exists a subset $C_0 \subset E$ such that $C_0 = \text{s-liminf}_{n\to\infty} C_n = \text{w-limsup}_{n\to\infty} C_n$, then we say that $\{C_n\}$ *converges to* C_0 *in the sense of Mosco* and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. The proof of the following main result makes use of the so-called Tsukada's Theorem.

Lemma 2.12 (Tsukada [30]). *Suppose that E is a smooth Banach space and E^* is Fréchet smooth. If $\{C_n\}$ is a sequence of nonempty closed and convex subsets of E such that $C_0 := \text{M-lim}_{n\to\infty} C_n \neq \varnothing$, then $P_{C_n}x \to P_C x$ for all $x \in E$.*

We also need the following lemma.

Lemma 2.13. *Suppose that E and E^* are Fréchet smooth. If $\{x_n\}$ and $\{y_n\}$ are two sequences in E such that $\varphi(x_n, y_n) \to 0$ and $y_n \to z \in E$, then $x_n \to z$.*

Proof. Note that $\{x_n\}$ and $\{y_n\}$ are bounded, $\varphi(y_n, z) \to 0$, and $Jy_n \to Jz$. Consequently,

$$\varphi(x_n, z) = \varphi(x_n, y_n) + \varphi(y_n, z) + 2\langle x_n - y_n, Jy_n - Jz\rangle \to 0.$$

Next, we show that $\omega_w\{x_n\} = \{z\}$. Suppose that $x_{n_k} \rightharpoonup z'$ for some $\{x_{n_k}\} \subset \{x_n\}$. It follows then that

$$\varphi(z', z) \leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jz\rangle + \|z\|^2) = \liminf_{k\to\infty} \varphi(x_{n_k}, z) = 0.$$

In particular, $z' = z$. Hence, $x_n \rightharpoonup z$. It follows from $\|x_n\| \to \|z\|$ that $x_n \to z$. $\square$

Theorem 2.14. *Let E be a Banach space such that both E and its dual space E^* are Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $S, T : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is a cutter mapping of type (Q);*
- *$F := \text{Fix}(S) \cap \text{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C_1 := C \text{ is arbitrarily chosen}; \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \leq 0\}; \\ B_n = \{z \in C : \langle Tx_n - z, JTx_n - Jx_n\rangle \leq 0\}; \\ C_{n+1} = A_n \cap B_n \cap C_n; \\ x_{n+1} = P_{C_{n+1}}\widehat{x}. \end{cases}$$

Then the sequence $\{x_n\}$ converges to $P_F\widehat{x}$.

Proof. It is clear from the assumption that $F \subset A_n \cap B_n$ for all n and hence $F \subset \bigcap_{n=1}^{\infty} C_n$. In particular, each C_n is a nonempty closed and convex subset of E. Thus $\{x_n\}$ is well-defined. Note that $C_n \supset C_{n+1}$ for all n. This implies that

$$C_0 := \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n \neq \varnothing.$$

It follows from Lemma 2.12 that $x_n \to P_{C_0}\widehat{x} =: x'$. It is clear from the iteration that

$$Sx_n = P_{A_n}x_n \quad \text{and} \quad Tx_n = \Pi_{B_n}x_n.$$

As $x_{n+1} \in C_{n+1} \subset A_n \cap B_n$, we have

$$\|Sx_n - x_n\| \le \|x_{n+1} - x_n\| \quad \text{and} \quad \varphi(Tx_n, x_n) \le \varphi(x_{n+1}, x_n).$$

We will prove that

(1) $x' \in \mathrm{Fix}(S)$;
(2) $x' \in \mathrm{Fix}(T)$.

To see (1), we will make use of the closedness of $I - S$ at zero. It is clear that $Sx_n \to x'$ and hence (1) holds.

To see (2), let us note from Lemma 2.13 and $\varphi(Tx_n, x_n) \to 0$ that $Tx_n \to x'$. It follows from the closedness of $I - T$ at zero that (2) holds.

Finally, it follows from $F \subset \bigcap_{n=1}^{\infty} C_n$ and $x' \in F$ that $x' = P_F\widehat{x}$. □

Remark 2.15. This type of iterative scheme called the shrinking projection method was first proposed by Takahashi et al. [29]. The technique of the proof using Mosco convergence is due to Kimura and Takahashi [14]; see also [13].

The following result can be obtained with a slight modification of the preceding proof so its proof is omitted.

Theorem 2.16. *Let E be a Banach space such that both E and its dual space E^* are Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $T : C \to C$ and $S : C \to C$ are two mappings such that the following properties are satisfied:*

- *S is a cutter mapping of type (P);*
- *T is relatively quasi-nonexpansive;*
- *$F := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \varnothing$;*
- *$I - S$ and $I - T$ are closed at zero.*

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C := C_1 \text{ is arbitrarily chosen}; \\ A_n = \{z \in C : \langle Sx_n - z, J(Sx_n - x_n)\rangle \le 0\}; \\ B_n = \{z \in C : \varphi(z, Tx_n) \le \varphi(z, x_n)\}; \\ C_{n+1} = A_n \cap B_n \cap C_n; \\ x_{n+1} = P_{C_{n+1}}\widehat{x}. \end{cases}$$

Then the sequence $\{x_n\}$ converges to $P_F\widehat{x}$.

Remark 2.17. Let us note that the metric projection involved in our iterations in the preceding two theorems can be replaced by Alber's generalized projections. To prove this, we just invoke the analogue of Tsukada's Theorem for generalized projections. In fact, in the same setting as Tsukada's theorem, Ibaraki et al. [11] proved that $\Pi_{C_n}x \to \Pi_{C_0}x$ for all $x \in E$.

Finally, we present a related result which is deduced from our Theorem 2.14 where T is the identity mapping.

Theorem 2.18. *Let E be a smooth Banach space such that E^* is Fréchet smooth. Let C be a closed and convex subset of E. Suppose that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:*

- *$f(x,x) = 0$ for all $x \in C$;*
- *$f(x,y) + f(y,x) \le 0$ for all $x, y \in C$;*
- *$f(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$;*
- *for every $x \in C$ and $x^* \in E^*$ the following implication holds:*

$$f(z,x) + \langle x - z, x^* \rangle \le 0 \quad \forall z \in C \implies f(x,y) + \langle y - x, x^* \rangle \ge 0 \quad \forall y \in C.$$

Define an iterative sequence $\{x_n\}$ by the following way:

$$\begin{cases} x_1 = \widehat{x} \in C =: C_1 \text{ is arbitrarily chosen}; \\ C_{n+1} = \{z \in C : \langle F_{r_n} x_n - z, J(F_{r_n} x_n - x_n) \rangle \le 0\} \cap C_n; \\ x_{n+1} = P_{C_{n+1}} \widehat{x}, \end{cases}$$

where $\{r_n\}$ is a sequence of positive real numbers such that $\liminf_{n\to\infty} r_n > 0$. If $\mathrm{EP}(f) \neq \varnothing$, then the sequence $\{x_n\}$ converges to $P_{\mathrm{EP}(f)}\widehat{x}$. Here for each $x \in E$ and $r > 0$, the element $F_r x$ is a unique element in C such that

$$f(F_r x, y) + \frac{1}{r}\langle y - F_r x, J(F_r x - x) \rangle \ge 0 \quad \forall y \in C.$$

Remark 2.19. The preceding theorem is proved in [27, Theorem 3.2] under the assumption that E^* is uniformly smooth. It is noted that F_r is a cutter mapping of type (P) and $\mathrm{Fix}(F_r) = \mathrm{EP}(f)$. Moreover, the proof of Theorem 2.14 does not alter if we can replace a single mapping S with a sequence of mappings $\{S_n\}$ such that $\bigcap_{n=1}^\infty \mathrm{Fix}(S_n) \neq \varnothing$ and the following condition holds:

$$\{z_n\} \subset C,\ z_n \to z,\ S_n z_n \to z \implies z \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n).$$

3. Concluding remarks

We propose a new alternative iterative scheme for approximation of a common fixed point of two different types of generalizations of cutters mappings. This appears as the first theoretical framework dealing with two different types of mappings in just only one scheme. Let us consider the convex feasibility problem, that is, the problem of fining a common element in the intersection of two (or more) closed and convex subsets of a certain Banach space. As already mentioned that there are two types of projections for these two sets, we can choose the easier calculated projection on each set. If these two projections are different, the schemes in this paper will generates an appropriate sequence for the problem.

The calculation of the projection onto general closed and convex sets is a hard task. However, if C in our theorems is the whole space E, the closed and convex set we are dealing with is a half space. To calculate such a projection, we refer to a formula proposed by Butnariu and Resmerita (see [6, Theorem 4.7] with $p = 2$).

In the recent works of Reich and Sabach (see [22, 23, 24, 25, 17]), they considered the classes of operators containing the cutter mappings of type (Q). It is very interesting to extend our results to these classes.

Acknowledgement. The authors would like to express their sincere thanks to the referee for his/her comments and suggestions on their manuscript.

References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, 1996, pp. 15–50.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: their relations and continuity properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[4] H. H. Bauschke, J. M. Borwein, and P. L. Combettes, *Bregman monotone optimization algorithms*, SIAM J. Control Optim. **42** (2003), 596–636.

[5] H. H. Bauschke and P. L. Combettes, *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces*, Math. Oper. Res. **26** (2001), 248–264.

[6] D. Butnariu and E. Resmerita, *Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal. 2006, Art. ID 84919, 39 pp.

[7] A. Cegielski and Y. Censor, *Opial-type theorem and the common fixed point problems*, In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, H. H. Bauschke et al. (eds), Springer Optimization and its Applications No. 49, Springer, New York, 2011, pp. 155–184.

[8] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Mathematics and its Applications, 62. Kluwer Academic Publishers Group, Dordrecht, 1990.

[9] P. L. Combettes, *Quasi-Fejérian analysis of some optimization algorithms*, In: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000), Stud. Comput. Math., 8, North-Holland, Amsterdam, 2001, pp. 115–152.

[10] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[11] T. Ibaraki, Y. Kimura and W. Takahashi, *Convergence theorems for generalized projections and maximal monotone operators in Banach spaces*, Abstr. Appl. Anal. (2003), 621–629.

[12] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[13] Y. Kimura, K. Nakajo and W. Takahashi, *Strongly convergent iterative schemes for a sequence of nonlinear mappings*, J. Nonlinear Convex Anal. **9** (2008), 407–416.

[14] Y. Kimura and W. Takahashi, *On a hybrid method for a family of relatively nonexpansive mappings in a Banach space*, J. Math. Anal. Appl. **357** (2009), 356–363.

[15] Y. Kimura, W. Takahashi and J. C. Yao, *Strong convergence of an iterative scheme by a new type of projection method for a family of quasinonexpansive mappings*, J. Optim Theory Appl. **149** (2011), 239–253.

[16] P. E. Maingé, *The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces*, Comput. Math. Appl. **59** (2010), 74–79.

[17] V. Martín-Márquez, S. Reich and S. Sabach, *Iterative methods for approximating fixed points of Bregman nonexpansive operators*, Discrete and Continuous Dynamical Systems, accepted for publication.

[18] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, 183. Springer-Verlag, New York, 1998.

[19] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math. **3** (1969), 510–585.

[20] W. Nilsrakoo and S. Saejung, *On the fixed-point set of a family of relatively nonexpansive and generalized nonexpansive mappings*, Fixed Point Theory Appl. 2010, Art. ID 414232, 14 pp.

[21] S. Reich, *Book Review: Geometry of Banach spaces, duality mappings and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 367–370.

[22] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **31** (2010), 22–44.

[23] S. Reich and S. Sabach, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces*, Nonlinear Anal. **73** (2010), 122–135.

[24] S. Reich and S. Sabach, *Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces*, In: Fixed-point algorithms for inverse problems in science and engineering, Springer Optim. Appl., 49, Springer, New York, 2011, pp. 301–316.

[25] S. Reich and S. Sabach, *Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach spaces*, In: Optimization Theory and Related Topics, Contemp. Math., 568, Amer. Math. Soc., Providence, RI, 2012, pp. 225–240.

[26] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

[27] H. Takahashi and W. Takahashi, *Existence theorems and strong convergence theorems by a hybrid method for equilibrium problems in Banach spaces*, In: Fixed Point Theory and its Applications, Yokohama Publ., Yokohama, 2008, pp. 163–174.

[28] W. Takahashi, *Nonlinear Functional Analysis, -Fixed point theory and its applications-*, Yokohama Publishers, Yokohama, 2000.

[29] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[30] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), 301–309.

Manuscript received 28 October 2014
revised 12 December 2014

Yasunori Kimura
Department of Information Science, Faculty of Science, Toho University, Miyama, Funabashi, Chiba 274-8502, Japan
E-mail address: yasunori@is.sci.toho-u.ac.jp

Satit Saejung
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
E-mail address: saejung@kku.ac.th