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FIXED POINT THEOREMS FOR NEW NONLINEAR MAPPINGS SATISFYING CONDITION (CC)

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ABSTRACT. We introduce a new condition whose name is Condition (CC), which is weaker than those of Chatterjea mapping and hybrid mapping. We prove fixed point theorems for mappings satisfying Condition (CC) without the convexity of the domain. We also prove convergence theorems.

1. Introduction

In 2010, Takahashi [16] introduced the concept of hybrid mapping. In the case where the whole space is Hilbertian, this concept can be described as follows: A mapping on a nonempty subset C of a Hilbert space E is called *hybrid* iff

$$3\,\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$$

for any $x, y \in C$.

Very recently, inspired by the concept of nonspreading mapping [11], the second author introduced the concept of Chatterjea mapping in Banach spaces in [15]. Let T be a mapping on a subset C of a Banach space E and let η be a continuous, strictly increasing function from $[0, \infty)$ into itself with $\eta(0) = 0$. Then T is call a *Chatterjea mapping* with η if

$$2\,\eta\big(\|Tx - Ty\|\big) \le \eta\big(\|Tx - y\|\big) + \eta\big(\|x - Ty\|\big)$$

for any $x, y \in C$. We note that not every contraction is a Chatterjea mapping; see Example 3.5 below.

In this paper, motivated by the above, we introduce a new condition whose name is Condition (CC). '(CC)' is named after 'Contraction' and 'Chatterjea'; see [4]. We prove fixed point theorems for mappings which satisfy Condition (CC) without the convexity of the domain in Banach spaces. We also prove convergence theorems to a fixed point.

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2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For $n \in \mathbb{N} \cup \{0\}$, we define $n!$ by $0! = 1$ and $(n+1)! = n!\,(n+1)$, that is, $n!$ is the *factorial* of n. For $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$, we define $C(n,k) = n!/\big(k!\,(n-k)!\big)$, that is, $C(n,k)$ is the *binomial coefficient* of $(n;k)$.

A Banach space E is said to be *smooth* if the limit $\lim_{t \to 0}(\|x + t\,y\| - \|x\|)/t$ exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. The *normalized duality mapping* J from E into E^* is defined by $\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2$ for all $x \in E$.

Let E be a Banach space. E is said to be *strictly convex* if $\|x + y\| < 2$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that E is said to be *uniformly convex in every direction* (*UCED*, for short) if for $\varepsilon \in (0, 2]$ and $z \in E$ with $\|z\| = 1$, there exists $\delta > 0$ such that

$$\|x + y\| \leq 2\,(1 - \delta)$$

for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x - y \in \big\{t\,z : t \in [-2, -\varepsilon] \cup [+\varepsilon, +2]\big\}$. It is obvious that UCED implies strictly convexity. We know that every separable Banach space can be equivalently renormed so that it is UCED. See [7, 13] and others. We know UCED is characterized as follows.

Lemma 2.1 ([14])**.** *For a Banach space E, the following are equivalent*:

(i) *E is UCED.*

(ii) *If $\{u_n\}$ is a bounded sequence in E, then a function g on E defined by*

$$g(x) = \limsup_{n \to \infty} \|u_n - x\| \tag{2.1}$$

is strictly quasiconvex, that is,

$$g\big(\lambda\,x + (1 - \lambda)\,y\big) < \max\big\{\,g(x),\ g(y)\,\big\}$$

for all $\lambda \in (0, 1)$ and $x, y \in E$ with $x \neq y$.

Let C be a subset of a Banach space E. C is said to be *boundedly weakly compact* if its intersection with any closed ball is weakly compact. It is obvious that if E is reflexive, then every closed convex subset is boundedly weakly compact. C is said to have the *Opial property* [12] if for each weakly convergent sequence $\{x_n\}$ in C with weak limit $z \in C$,

$$\liminf_{n \to \infty} \|x_n - z\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for $y \in C$ with $y \neq z$. We remark that we may replace 'lim inf' by 'lim sup'. All nonempty compact subsets have the Opial property. Also, all Hilbert spaces, $\ell^p (1 \leq p < \infty)$ and finite dimensional Banach spaces have the Opial property. A Banach space with a duality mapping which is weakly sequentially continuous also has the Opial property [8]. We know that every separable Banach space can be equivalently renormed so that it has the Opial property [6].

Let C be a subset of a Banach space E and let f be a function from C into $\mathbb{R}$. f is said to be *nonincreasing* with respect to a mapping T on C if $f(Tx) \leq f(x)$ for all $x \in C$. Also, from now on, in the case where C is bounded, we consider every function f to satisfy (2.2) below.

In our proof, we use the following.

Lemma 2.2 ([15])**.** *Let C be a boundedly weakly compact subset of a Banach space E and let f be a function from C into $\mathbb{R}$ which is lower semicontinuous in the weak topology. Assume that either C is bounded or f satisfies*

$$\inf\{f(x) : x \in C\} < \lim_{r\to\infty} \inf\big\{f(x) : \ x \in C, \ \|x\| \geq r\big\}. \tag{2.2}$$

Then $\min f(C)$ *exists.*

Lemma 2.3 ([15])**.** *Let η be a continuous, strictly increasing function from $[0,\infty)$ into itself. Then the following hold*:

(i) *$s \leq t$ if and only if $\eta(s) \leq \eta(t)$.*

(ii) *If $\limsup_n t_n \in \mathbb{R}$, then $\eta(\limsup_n t_n) = \limsup_n \eta(t_n)$.*

3. Condition (CC)

In this section, we introduce a new condition whose name is Condition (CC).

Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy *Condition (CC)* iff there exist a continuous, strictly increasing function η from $[0,\infty)$ into itself with $\eta(0) = 0$ and $r, s \in [0,1)$ such that

$$\begin{aligned} &r + 2\,s = 1 \qquad \text{and} \\ &\eta(\|Tx - Ty\|) \leq r\,\eta(\|x - y\|) + s\,\eta(\|x - Ty\|) + s\,\eta(\|Tx - y\|) \end{aligned} \tag{3.1}$$

for all $x, y \in C$.

Remark. We note that $0 < s \leq 1/2$.

From the definition, we can obtain the following propositions.

Proposition 3.1. *Let T be a mapping on a subset C of a Banach space E and let η be a continuous, strictly increasing function from $[0,\infty)$ into itself with $\eta(0) = 0$. Assume that there exist $r, s, t \in [0,1]$ such that $r < 1$, $r + s + t = 1$ and*

$$\eta\big(\|Tx - Ty\|\big) \leq r\,\eta\big(\|x - y\|\big) + s\,\eta\big(\|x - Ty\|\big) + t\,\eta\big(\|Tx - y\|\big)$$

for all $x, y \in C$. Then T satisfies Condition (CC)*.*

Proof. For $x, y \in C$, we have

$$\eta\big(\|Tx - Ty\|\big) \leq r\,\eta\big(\|x - y\|\big) + s\,\eta\big(\|x - Ty\|\big) + t\,\eta\big(\|Tx - y\|\big)$$

and

$$\eta\big(\|Ty - Tx\|\big) \leq r\,\eta\big(\|y - x\|\big) + s\,\eta\big(\|y - Tx\|\big) + t\,\eta\big(\|Ty - x\|\big).$$

Adding the both inequalities, we obtain

$$\eta\big(\|Tx - Ty\|\big) \leq r\,\eta\big(\|x - y\|\big) + \frac{s+t}{2}\,\eta\big(\|x - Ty\|\big) + \frac{s+t}{2}\,\eta\big(\|Tx - y\|\big).$$

Since $r + 2\,(s+t)/2 = 1$, T satisfies Condition (CC) with r, $(s+t)/2$ and η. □

Proposition 3.2. *Let p and q be positive real numbers with $p < q$. Let $r, s \in [0,1)$ satisfy $r + 2\,s = 1$. Let T be a mapping on a subset C of a Banach space E. Assume T satisfies Condition* (CC) *with r, s and $t \mapsto t^p$. Then T also satisfies Condition* (CC) *with r, s and $t \mapsto t^q$.*

Proof. Using the convexity of $t \mapsto t^{q/p}$, we have

$$\begin{aligned}\|Tx - Ty\|^q &= \left(\|Tx - Ty\|^p\right)^{q/p} \\ &\le \left(r\,\|x - y\|^p + s\|x - Ty\|^p + s\,\|Tx - y\|^p\right)^{q/p} \\ &\le r\,\|x - y\|^q + s\,\|x - Ty\|^q + s\,\|Tx - y\|^q\end{aligned}$$

for $x, y \in C$. □

Using the following, we can easily make an example of mapping T which satisfies Condition (CC) but which is not hybrid.

Example 3.3. Let $q \in (1, \infty)$ and let $r, s \in [0, 1)$ satisfy $r + 2\,s = 1$. Put

$$\begin{aligned} c_1 &:= (r/s)^{1/(q-1)} \ge 0 \qquad \text{and} \\ c_2 &:= \frac{1}{(1 + c_1)^q}\left(r + s\,{c_1}^q\right) \ge 0. \end{aligned}$$

Let τ be the unique real number satisfying

$$0 < \tau < 1 \quad \text{and} \quad c_2\,(1 - \tau)^q + s - \tau^q = 0. \tag{3.2}$$

Let $E = \mathbb{R}$ and define a mapping T on E by

$$Tx = \begin{cases} 0 & \text{if } x \ne 1 \\ \tau & \text{if } x = 1. \end{cases}$$

Then T satisfies Condition (CC) with r, s and $t \mapsto t^q$, however, T does not satisfy Condition (CC) with r, s and $t \mapsto t^p$ for any $p \in \mathbb{R}$ with $0 < p < q$.

Proof. Define a function f from $[0, 1]$ by

$$f(t) = c_2\,(1 - t)^q + s - t^q.$$

Then f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Since

$$f'(t) = -\,c_2\,q\,(1 - t)^{q-1} - q\,t^{q-1} < 0$$

for any $t \in (0, 1)$, f is strictly decreasing. Since

$$f(0) = c_2 + s > 0 \quad \text{and} \quad f(1) = s - 1 < 0,$$

we note that there exists a unique τ satisfying (3.2). We next define a function g from $[\tau, 1]$ into $\mathbb{R}$ by

$$g(u) = r\,(1 - u)^q + s + s\,(u - \tau)^q.$$

Put

$$\upsilon := \frac{c_1 + \tau}{c_1 + 1} = \frac{1}{c_1 + 1}\,\tau + \frac{c_1}{c_1 + 1}\,1 \in [\tau, 1).$$

Then we have

$$\begin{aligned} &g'(u) < 0 &&\text{if } \tau < u < \upsilon &&\text{and} \\ &g'(u) > 0 &&\text{if } \upsilon < u < 1. \end{aligned}$$

Hence

$$\min\{g(u) :\ \tau \le u \le 1\} = g(\upsilon) = c_2\,(1 - \tau)^q + s.$$

Therefore we obtain

$$0 = f(\tau) = g(\upsilon) - \tau^q = \min\big\{r\,(1-u)^q + s + s\,(u-\tau)^q : \ \tau \le u \le 1\big\} - \tau^q. \tag{3.3}$$

We shall show that T satisfies Condition (CC) with r, s and $t \mapsto t^q$. In the case where $x < \tau$, we have by (3.3)

$$\begin{aligned}\|T1 - Tx\|^q = \tau^q = g(\upsilon) \le g(\tau) &\le r\,(1-x)^q + s + s\,(\tau - x)^q \\ &= r\,\|1-x\|^q + s\,\|1-Tx\|^q + s\,\|T1-x\|^q.\end{aligned}$$

In the case where $\tau \le x < 1$, we have (3.3)

$$\begin{aligned}\|T1 - Tx\|^q = \tau^q = g(\upsilon) &\le r\,(1-x)^q + s + s\,(x-\tau)^q \\ &= r\,\|1-x\|^q + s\,\|1-Tx\|^q + s\,\|T1-x\|^q.\end{aligned}$$

In the case where $1 < x$, we have by (3.3)

$$\begin{aligned}\|T1 - Tx\|^q = \tau^q = g(\upsilon) \le g(1) &\le r\,(x-1)^q + s + s\,(x-\tau)^q \\ &= r\,\|1-x\|^q + s\,\|1-Tx\|^q + s\,\|T1-x\|^q.\end{aligned}$$

Therefore T satisfies Condition (CC) with r, s and $t \mapsto t^q$. Let $p \in \mathbb{R}$ satisfy $0 < p < q$. Then using (3.3), $s > 0$, $1 \ne \upsilon - \tau$ and the strict convexity of $t \mapsto t^{q/p}$, we have

$$\begin{aligned}&r\,\|1-\upsilon\|^p + s\,\|1-T\upsilon\|^p + s\,\|T1-\upsilon\|^p \\ &= \big(r\,(1-\upsilon)^p + s\,(1)^p + s\,(\upsilon-\tau)^p\big)^{(q/p)\,(p/q)} \\ &< \big(r\,(1-\upsilon)^q + s\,(1)^q + s\,(\upsilon-\tau)^q\big)^{p/q} \\ &= (\tau^q)^{p/q} = \tau^p = \|T1 - T\upsilon\|^p.\end{aligned}$$

Therefore T does not satisfy Condition (CC) with r, s and $t \mapsto t^p$. □

The following two inform that there exists an example of mapping T which satisfies Condition (CC), but which is not a Chatterjea mapping.

Example 3.4. Let T be a contraction on a subset C of a Banach space E, that is, there exists $r \in [0,1)$ such that $\|Tx - Ty\| \le r\,\|x-y\|$ for any $x, y \in C$. Then T satisfies Condition (CC).

Proof. Obvious. □

Example 3.5. Let $r \in (1/3, 1)$. Let E be a Banach space and define a mapping T on E by $Tx = -r\,x$ for any $x \in E$. Then T is not a Chatterjea mapping.

Proof. Arguing by contradiction, we assume that T is a Chatterjea mapping. Then there exists η such that

$$2\,\eta(\|Tx - Ty\|) \le \eta(\|x - Ty\|) + \eta(\|Tx - y\|) \tag{3.4}$$

for any $x, y \in E$. Fix $w \in E \setminus \{0\}$. Put $x = w$ and $y = -w$. Then (3.4) becomes

$$2\,\eta(2\,r\|w\|) \le 2\,\eta\big((1-r)\,\|w\|\big).$$

This is a contradiction because $2\,r > 1 - r$. □

4. Basic properties

In this section, we prove some basic properties of a mapping T which satisfies Condition (CC).

A mapping T on a subset C of a Banach space E is said to be *quasinonexpansive* [5] if

$$\|Tx - z\| \le \|x - z\| \tag{4.1}$$

for all $x \in C$ and $z \in F(T)$.

Proposition 4.1. *Assume that a mapping T on a subset C of a Banach space E satisfies Condition* (CC). *Assume also that T has a fixed point. Then T is a quasinonexpansive mapping.*

Proof. Let r, s and η satisfy (3.1). For $x \in C$ and $z \in F(T)$, we have

$$\eta\big(\|Tx - z\|\big) \le r\,\eta\big(\|x - z\|\big) + s\,\eta\big(\|x - Tz\|\big) + s\,\eta\big(\|Tx - z\|\big).$$

So,

$$(1 - s)\,\eta\big(\|Tx - z\|\big) \le (r + s)\,\eta\big(\|x - z\|\big) = (1 - s)\,\eta\big(\|x - z\|\big)$$

holds. Using this and the strict increasingness of η, we obtain (4.1). □

From Proposition 4.1, we obtain the following.

Lemma 4.2. *Assume that a mapping T on a subset C of a Banach space E satisfies Condition* (CC). *Assume also that T has a fixed point. Then $\{T^n u\}$ is bounded for all $u \in C$.*

Proposition 4.3. *Let T be a mapping on a closed subset C of a Banach space E which satisfies Condition* (CC). *Then $F(T)$ is closed. Moreover, if E is strictly convex and C is convex, then $F(T)$ is also convex.*

The following lemma plays a very important role in this paper.

Lemma 4.4. *Put $I_0 = \big\{(m, n) : \ m, n \in \mathbb{N} \cup \{0\}, \ m \le n\big\}$ and $I = \big\{(m, n) : m, n \in \mathbb{N}, \ m < n\big\}$. Let $r, s \in [0, 1)$ satisfy $r + 2\,s = 1$. Define a function A from I_0 into $[0, \infty)$ by the following*:

(4.2) $A(0, n) = 1$ *for* $n \in \mathbb{N}$.

(4.3) $A(n, n) = 0$ *for* $n \in \mathbb{N} \cup \{0\}$.

(4.4) $A(m, n) = r\,A(m - 1, n - 1) + s\,A(m - 1, n) + s\,A(m, n - 1)$ *for* $(m, n) \in I$.

Then the following hold:

(i) $A(m, n) \le 1$ *for* $(m, n) \in I_0$.

(ii) $A(m - 1, n - 1) \ge A(m, n)$ *for* $(m, n) \in I$.

(iii) *For* $(m, n) \in I$ *and* $k \in \mathbb{N}$,

$$A(m + k, n + k) \le r^k + \frac{1}{2}\,A(m - 1, n) + \frac{1}{2}\,A(m, n - 1).$$

(iv) *For* $j, n \in \mathbb{N} \cup \{0\}$ *and* $\ell \in \mathbb{N}$,

$$A(j + 2\,n + 2\,n\ell, j + 2\,n + 1 + 2\,n\ell)$$
$$\leq 2\,nr^{\ell} + \frac{1}{2^{2n}} \sum_{k=0}^{n} C(2\,n, k)\,\frac{2\,n - 2\,k + 1}{2\,n - k + 1}\,A(j + k, j + 2\,n + 1 - k).$$

(v) *For* $j, n \in \mathbb{N} \cup \{0\}$ *and* $\ell \in \mathbb{N}$,

$$A(j + 2\,n + 1 + (2\,n + 1)\,\ell, j + 2\,n + 2 + (2\,n + 1)\,\ell)$$
$$\leq (2\,n + 1)r^{\ell} + \frac{1}{2^{2n+1}} \sum_{k=0}^{n} C(2\,n + 1, k)\,\frac{2\,n - 2\,k + 2}{2\,n - k + 2}\,A(j + k, j + 2\,n + 2 - k).$$

(vi) $\lim_{n\to\infty} A(n, n + 1) = 0$.

We use the following in the proof of Lemma 4.4.

Lemma 4.5. *The following hold*:

(i) *For* $n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n, k)\,\frac{2\,n - 2\,k + 1}{2\,n - k + 1} \leq 1.$$

(ii) *For* $n, j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\sum_{k=0}^{n} C(2\,n, k)\,\frac{2\,n - 2\,k + 1}{2\,n - k + 1}\,\big(&A(j + k, j + 2\,n + 2 - k)\\ &+ A(j + k + 1, j + 2\,n + 1 - k)\big)\\ = \sum_{k=0}^{n} C(2\,n + 1, k)\,&\frac{2\,n - 2\,k + 2}{2\,n - k + 2}\,A(j + k, j + 2\,n + 2 - k).\end{aligned}$$

(iii) *For* $n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{2^{2\,n+1}} \sum_{k=0}^{n} C(2\,n + 1, k)\,\frac{2\,n - 2\,k + 2}{2\,n - k + 2} \leq 1.$$

(iv) *For* $n, j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\sum_{k=0}^{n} C(2\,n + 1, k)\,\frac{2\,n - 2\,k + 2}{2\,n - k + 2}\,\Big(&A(j + k, j + 2\,n + 3 - k)\\ &+ A(j + k + 1, j + 2\,n + 2 - k)\Big)\\ = \sum_{k=0}^{n+1} C(2\,n + 2, k)\,&\frac{2\,n - 2\,k + 3}{2\,n - k + 3}\,A(j + k, j + 2\,n + 3 - k).\end{aligned}$$

(v)

$$\lim_{n\to\infty} \frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n, k)\,\frac{2\,n - 2\,k + 1}{2\,n - k + 1} = 0.$$

(vi) *The convergence radius of* $t \mapsto \sum_{n=0}^{\infty} n\,t^{n}$ *is* 1.

Proof. (i)–(v) are proved in the proof of Lemma 11 in [15]. (vi) is obvious. □

Proof of Lemma 4.4. We first define a total order $\leq$ on I by $(m_1, n_1) \leq (m_2, n_2)$ iff either $n_1 < n_2$ or $n_1 = n_2$ and $m_1 \leq m_2$. We will show (i). We note $A(m, n) \leq 1$ for $(m, n) \in I_0 \setminus I$ by (4.2) and (4.3). Fix $(m, n) \in I$ and assume $A(m', n') \leq 1$ for $(m', n') \in I$ with $(m', n') < (m, n)$. Then we have by (4.4)

$$\begin{aligned} A(m,n) &= r\,A(m-1,n-1) + s\,A(m-1,n) + s\,A(m,n-1) \\ &\leq r + s + s = 1. \end{aligned}$$

By induction, we obtain (i). We next show (ii). Since

$$A(1,n) \leq A(0,n-1) = 1$$

for $n \in \mathbb{N}$ with $n \geq 2$, (ii) holds when $m = 1$. Let $(m, n) \in I$ with $m > 1$ and assume $A(m'-1, n'-1) \leq A(m', n')$ for $(m', n') \in I$ with $(m', n') < (m, n)$. Then noting $A(m-1, m-1) = A(m, m)$ if necessary, we have

$$\begin{aligned} A(m,n) &= r\,A(m-1,n-1) + s\,A(m-1,n) + s\,A(m,n-1) \\ &\leq r\,A(m-2,n-2) + s\,A(m-2,n-1) + s\,A(m-1,n-2) \\ &= A(m-1,n-1). \end{aligned}$$

So, we obtain (ii). Next we prove (iii). Using (4.4) and (ii) several times, (i) and $s/(1-r) = 1/2$, we have

$$\begin{aligned} &A(m+k,n+k) \\ &= r\,A(m+k-1,n+k-1) + s\,A(m+k-1,n+k) + s\,A(m+k,n+k-1) \\ &\leq r^2\,A(m+k-2,n+k-2) \\ &\qquad + r\,s\,A(m+k-2,n+k-1) + r\,s\,A(m+k-1,n+k-2) \\ &\qquad + s\,A(m+k-2,n+k-1) + s\,A(m+k-1,n+k-2) \\ &= r^2\,A(m+k-2,n+k-2) + s\,(1+r)\,A(m+k-2,n+k-1) \\ &\qquad + s\,(1+r)\,A(m+k-1,n+k-2) \\ &\leq r^3\,A(m+k-3,n+k-3) + s\,(1+r+r^2)\,A(m+k-3,n+k-2) \\ &\qquad + s\,(1+r+r^2)\,A(m+k-2,n+k-3) \\ &\ \ \vdots \\ &\leq r^{k+1}\,A(m-1,n-1) + s\,\frac{1-r^{k+1}}{1-r}\,A(m-1,n) + s\,\frac{1-r^{k+1}}{1-r}\,A(m,n-1) \\ &\leq r^{k+1} + \frac{1}{2}\,A(m-1,n) + \frac{1}{2}\,A(m,n-1) \\ &\leq r^{k} + \frac{1}{2}\,A(m-1,n) + \frac{1}{2}\,A(m,n-1) \end{aligned}$$

for $k \in \mathbb{N}$. In order to show (iv) and (v), we also use induction with respect to n. When $n = 0$, (iv) becomes $A(j, j+1) \leq A(j, j+1)$, which clearly holds. We assume

(iv) holds for some $n \in \mathbb{N} \cup \{0\}$. Using (iii) and Lemma 4.5 (i) and (ii), we have

$$
\begin{aligned}
&A(j+2\,n+1+(2\,n+1)\,\ell, j+2\,n+2+(2\,n+1)\,\ell) \\
&= A(j+1+\ell+2\,n+2\,n\ell, j+1+\ell+2\,n+1+2\,n\ell) \\
&\le 2\,nr^{\ell} + \frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n,k)\,\frac{2\,n-2\,k+1}{2\,n-k+1}\,A(j+1+\ell+k, j+1+\ell+2\,n+1-k) \\
&= 2\,nr^{\ell} + \frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n,k)\,\frac{2\,n-2\,k+1}{2\,n-k+1}\,A(j+1+k+\ell, j+2+2\,n-k+\ell) \\
&\le 2\,nr^{\ell} + \frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n,k)\,\frac{2\,n-2\,k+1}{2\,n-k+1}\,\Big(r^{\ell} + \frac{1}{2}\,A(j+k, j+2+2\,n-k) \\
&\qquad\qquad + \frac{1}{2}\,A(j+1+k, j+1+2n-k)\Big) \\
&= 2\,nr^{\ell} + \frac{1}{2^{2\,n}} \sum_{k=0}^{n} C(2\,n,k)\,\frac{2\,n-2\,k+1}{2\,n-k+1}\,r^{\ell} \\
&\qquad\qquad + \frac{1}{2^{2n+1}} \sum_{k=0}^{n} C(2\,n+1,k)\,\frac{2\,n-2\,k+2}{2\,n-k+2}\,A(j+k, j+2\,n+2-k) \\
&\le (2\,n+1)\,r^{\ell} + \frac{1}{2^{2n+1}} \sum_{k=0}^{n} C(2\,n+1,k)\,\frac{2\,n-2\,k+2}{2\,n-k+2}\,A(j+k, j+2\,n+2-k).
\end{aligned}
$$

Hence (v) holds provided (iv) holds. Using (iii) and Lemma 4.5 (iii) and (iv), we also have

$$
\begin{aligned}
&A(j+2\,(n+1)+2\,(n+1)\ell, j+2\,(n+1)+1+2\,(n+1)\ell) \\
&= A(j+1+\ell+2\,n+1+(2\,n+1)\,\ell, j+1+\ell+2\,n+2+(2\,n+1)\,\ell) \\
&\le (2\,n+1)\,r^{\ell} + \frac{1}{2^{2\,n+1}} \sum_{k=0}^{n} C(2\,n+1,k)\,\frac{2\,n-2\,k+2}{2\,n-k+2} \\
&\qquad \times A(j+k+1+\ell, j+2\,n+2-k+1+\ell) \\
&\le (2\,n+1)\,r^{\ell} + \frac{1}{2^{2\,n+1}} \sum_{k=0}^{n} C(2\,n+1,k)\,\frac{2\,n-2\,k+2}{2\,n-k+2} \\
&\qquad \times \Big(r^{\ell} + \frac{1}{2}\,A(j+k, j+2\,n+3-k) + \frac{1}{2}\,A(j+k+1, j+2\,n+2-k)\Big) \\
&= (2\,n+1)\,r^{\ell} + \frac{1}{2^{2\,n+1}} \sum_{k=0}^{n} C(2\,n+1,k)\,\frac{2\,n-2\,k+2}{2\,n-k+2}\,r^{\ell} \\
&\qquad + \frac{1}{2^{2\,n+2}} \sum_{k=0}^{n+1} C(2\,n+2,k)\,\frac{2\,n-2\,k+3}{2\,n-k+3}\,A(j+k, j+2\,n+3-k) \\
&\le 2\,(n+1)\,r^{\ell} + \frac{1}{2^{2\,n+2}} \sum_{k=0}^{n+1} C(2\,n+2,k)\,\frac{2\,n-2\,k+3}{2\,n-k+3}\,A(j+k, j+2\,n+3-k).
\end{aligned}
$$

Thus (iv) holds when $n := n+1$. By induction, we obtain (iv) and (v). We finally prove (vi). Putting $j = 0$ and $\ell = 2n$ in (iv), we have

$$\begin{aligned}
&\lim_{n\to\infty} A(2n + 4n^2, 2n + 4n^2 + 1) \\
&\le \lim_{n\to\infty} \Big(2nr^{2n} + \frac{1}{2^{2n}} \sum_{k=0}^{n} C(2n,k)\, \frac{2n - 2k + 1}{2n - k + 1}\, A(k, 2n + 1 - k) \Big) \\
&\le \lim_{n\to\infty} \Big(2nr^{2n} + \frac{1}{2^{2n}} \sum_{k=0}^{n} C(2n,k)\, \frac{2n - 2k + 1}{2n - k + 1} \Big) \\
&= 0
\end{aligned}$$

by Lemma 4.5 (v) and (vi). By (ii), we obtain (vi). $\square$

Lemma 4.6. *Let I_0, I and A be as in Lemma 4.4. Let B be a function from I_0 into $[0,\infty)$ satisfying the following*:

- $B(0,n) \le 1$ *for* $n \in \mathbb{N}$.
- $B(n,n) = 0$ *for* $n \in \mathbb{N} \cup \{0\}$.
- *There exist $r, s \in [0,1)$ such that $r + 2s = 1$ and*
$$B(m,n) \le r\,B(m-1,n-1) + s\,B(m-1,n) + s\,B(m,n-1)$$
for $(m,n) \in I$.

Then $B(m,n) \le A(m,n)$ holds for $(m,n) \in I_0$. Hence $\lim_{n\to\infty} B(n,n+1) = 0$ holds.

Proof. We first note that $B(m,n) \le A(m,n)$ obviously holds for $(m,n) \in I_0 \setminus I$. Define a total order $\le$ on I as in the proof of Lemma 4.4. Fix $(m,n) \in I$ and assume $B(m',n') \le A(m',n')$ for $(m',n') \in I$ with $(m',n') < (m,n)$. Then we have

$$\begin{aligned}
B(m,n) &\le r\,B(m-1,n-1) + s\,B(m-1,n) + s\,B(m,n-1) \\
&\le r\,A(m-1,n-1) + s\,A(m-1,n) + s\,A(m,n-1) = A(m,n).
\end{aligned}$$

By induction, $B(m,n) \le A(m,n)$ holds for $(m,n) \in I$. By Lemma 4.4, $\lim_{n\to\infty} B(n,n+1) = 0$ holds. $\square$

A mapping T on C is said to be *asymptotically regular* at $x \in C$ [3] if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0. \tag{4.5}$$

T is said to be asymptotically regular on C if T is asymptotically regular at all $x \in C$.

Lemma 4.7. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume $\{T^n x\}$ is bounded for some $x \in C$. Then T is asymptotically regular at x.*

Proof. Let r, s and η satisfy (3.1). From the assumption, there exists a positive real number M such that $M > \eta\big(2\,\|T^n x\|\big)$ for $n \in \mathbb{N} \cup \{0\}$. Define a function B by

$$B(m,n) = \frac{1}{M}\,\eta\big(\|T^m x - T^n x\|\big)$$

for $m, n \in \mathbb{N} \cup \{0\}$ with $m \le n$. Then all the assumption of Lemma 4.6 are satisfied. So we obtain

$$\lim_{n\to\infty} \eta\big(\|T^n x - T^{n+1} x\|\big) = M \lim_{n\to\infty} B(n, n+1) = 0.$$

Therefore T is asymptotically regular at x. $\square$

Proposition 4.8. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume $\{T^n u\}$ is bounded for some $u \in C$. Then the following hold*:

(i) *$\{T^n x\}$ is bounded for all $x \in C$.*
(ii) *T is asymptotically regular on C.*

Proof. Let r, s and η satisfy (3.1). Define a continuous function f from C into $[0, \infty)$ by

$$f(x) = \limsup_{n\to\infty} \eta\big(\|T^n u - x\|\big) \tag{4.6}$$

for all $x \in C$. Then f is well defined from the assumption. We have

$$\begin{aligned}
&f(Tx)\\
&= \limsup_{n\to\infty} \eta\big(\|T^n u - Tx\|\big)\\
&\le \limsup_{n\to\infty} \big(r\,\eta\big(\|T^{n-1} u - x\|\big) + s\,\eta\big(\|T^{n-1} u - Tx\|\big) + s\,\eta\big(\|T^n u - x\|\big)\big)\\
&\le r \limsup_{n\to\infty} \eta\big(\|T^{n-1} u - x\|\big) + s \limsup_{n\to\infty} \eta\big(\|T^{n-1} u - Tx\|\big) + s \limsup_{n\to\infty} \eta\big(\|T^n u - x\|\big)\\
&= r\, f(x) + s\, f(Tx) + s\, f(x),
\end{aligned}$$

which implies $f(Tx) \le f(x)$. Thus, f is nonincreasing with respect to T. Hence $f(T^n x) \le f(x)$ for $n \in \mathbb{N}$. This implies that $\{T^n x\}$ is bounded. We have shown (i). By Lemma 4.7, we obtain (ii). $\square$

5. Convergence theorems

In the section, we prove convergence theorems under the assumption that the domain C has the Opial property.

Proposition 5.1. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume C has the Opial property. If $\{x_n\}$ converges weakly to z and* $\lim_n \|Tx_n - x_n\| = 0$, *then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Proof. Let r, s and η satisfy (3.1). We note that $\{x_n\}$ is bounded. Since

$$\|Tx_n - y\| - \|Tx_n - x_n\| \le \|x_n - y\| \le \|Tx_n - x_n\| + \|Tx_n - y\|,$$

we have

$$\limsup_{n\to\infty} \|Tx_n - y\| = \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in C$. Using this, we have

$$\begin{aligned}
&\eta\big(\limsup_{n\to\infty}\|x_n - Tz\|\big)\\
&= \eta\big(\limsup_{n\to\infty}\|Tx_n - Tz\|\big)\\
&= \limsup_{n\to\infty}\eta\big(\|Tx_n - Tz\|\big)\\
&\le \limsup_{n\to\infty}\big(r\,\eta(\|x_n - z\|) + s\,\eta(\|x_n - Tz\|) + s\,\eta(\|Tx_n - z\|)\big)\\
&\le \limsup_{n\to\infty} r\,\eta\big(\|x_n - z\|\big) + \limsup_{n\to\infty} s\,\eta\big(\|x_n - Tz\|\big) + \limsup_{n\to\infty} s\,\eta\big(\|Tx_n - z\|\big)\\
&= r\,\eta\big(\limsup_{n\to\infty}\|x_n - z\|\big) + s\,\eta\big(\limsup_{n\to\infty}\|x_n - Tz\|\big) + s\,\eta\big(\limsup_{n\to\infty}\|Tx_n - z\|\big)\\
&= r\,\eta\big(\limsup_{n\to\infty}\|x_n - z\|\big) + s\,\eta\big(\limsup_{n\to\infty}\|x_n - Tz\|\big) + s\,\eta\big(\limsup_{n\to\infty}\|x_n - z\|\big)
\end{aligned}$$

and hence

$$(1-s)\,\eta\big(\limsup_{n\to\infty}\|x_n - Tz\|\big) \le (r+s)\,\eta\big(\limsup_{n\to\infty}\|x_n - z\|\big).$$

So, we have

$$\eta\big(\limsup_{n\to\infty}\|x_n - Tz\|\big) \le \eta\big(\limsup_{n\to\infty}\|x_n - z\|\big).$$

Since C has the Opial property, we obtain $Tz = z$. □

Remark. A function $y \mapsto \limsup_n \eta\big(\|x_n - y\|\big)$ from C into $[0,\infty)$ is also nonincreasing with respect to T.

Theorem 5.2. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume $\{T^n u\}$ is bounded for some $u \in C$; and C is boundedly weakly compact and has the Opial property. Then $\{T^n x\}$ converges weakly to a fixed point of T for all $x \in C$.*

Remark. We do not need the convexity of C.

Proof. Fix $x \in C$. By Proposition 4.8, $\{T^n x\}$ is bounded and $\lim_n \|T^n x - T\circ T^n x\| = 0$. From the assumption, there exist a subsequence $\{T^{n_j} x\}$ of $\{T^n x\}$ and $z \in C$ such that $\{T^{n_j} x\}$ converges weakly to z. By Proposition 5.1, z is a fixed point of T. By Proposition 4.1, we note that $\{\|T^n x - z\|\}$ is a nonincreasing sequence. Arguing by contradiction, assume that $\{T^n x\}$ does not converge to z. Then there exist a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}$ and $w \in C$ such that $\{T^{n_k} x\}$ converges weakly to w and $z \neq w$. We note $Tw = w$. From the Opial property,

$$\begin{aligned}
\lim_{n\to\infty}\|T^n x - z\| &= \lim_{j\to\infty}\|T^{n_j} x - z\| < \lim_{j\to\infty}\|T^{n_j} x - w\| = \lim_{n\to\infty}\|T^n x - w\|\\
&= \lim_{k\to\infty}\|T^{n_k} x - w\| < \lim_{k\to\infty}\|T^{n_k} x - z\| = \lim_{n\to\infty}\|T^n x - z\|.
\end{aligned}$$

This is a contradiction. Therefore $\{T^n x\}$ converges weakly to z. □

As direct consequences of Theorem 5.2, we obtain the following.

Corollary 5.3. *Let T be a mapping on a weakly compact subset C of a Banach space E which satisfies Condition* (CC). *Assume C has the Opial property. Then $\{T^n x\}$ converges weakly to a fixed point of T for all $x \in C$.*

Corollary 5.4. *Let T be a mapping on a compact subset C of a Banach space E which satisfies Condition* (CC). *Then $\{T^n x\}$ converges strongly to a fixed point of T for all $x \in C$.*

It is well known that every metric space (X, d) is isometric to some subset of $B(X)$, where $B(X)$ is the set of all bounded real functions on X with supremum norm. So we can rewrite Corollary 5.4 as follows.

Corollary 5.5. *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that there exist a continuous, strictly increasing function η from $[0, \infty)$ into itself with $\eta(0) = 0$ and $r, s \in [0, 1)$ such that*

$$r + 2\,s = 1 \qquad \textit{and}$$
$$\eta\big(d(Tx, Ty)\big) \le r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big)$$

for all $x, y \in X$. Then $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

6. Existence theorems

In this section, we prove the existence of fixed points of mappings T which satisfy Condition (CC). By Lemma 4.2 and Theorem 5.2, we obtain the following.

Theorem 6.1. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume C is boundedly weakly compact and has the Opial property. Then the following are equivalent*:

(i) *$\{T^n u\}$ is bounded for some $u \in C$.*
(ii) *T has a fixed point.*

As direct consequences of Theorem 6.1, we obtain the following.

Corollary 6.2. *Let T be a mapping on a subset C of a Banach space E which satisfies Condition* (CC). *Assume that either of the following holds*:

- *C is compact;*
- *C is weakly compact and has the Opial property.*

Then T has a fixed point.

Remark. It is obvious that Corollary 6.2 also can be proved by Corollaries 5.3 and 5.4.

In order to prove fixed point theorems in UCED Banach spaces, we use the following lemmas.

Lemma 6.3 ([15])**.** *Let C be a boundedly weakly compact and convex subset of a Banach space E. Let T be a mapping on a subset C. Assume that there exists a lower semicontinuous, strictly quasiconvex function f from C into $\mathbb{R}$ such that f is nonincreasing with respect to T and f satisfies* (2.2). *Then T has a fixed point.*

Lemma 6.4 ([15])**.** *Let C be a boundedly weakly compact and convex subset of a Banach space E. Let T_0, T_1, T_2, ..., T_ℓ be commuting mappings on C. Assume that for every $j = 0, 1, 2, \ldots, \ell$, there exists a lower semicontinuous, strictly quasiconvex function f_j from C into $\mathbb{R}$ such that f_j is nonincreasing with respect to T_j and f_j satisfies* (2.2). *Assume also that $F(T_j)$ is closed and convex for $j = 1, 2, \ldots, \ell$. Then $\bigcap_{j=0}^{\ell} F(T_j)$ is nonempty.*

Lemma 6.5 ([15]). *Let C be a weakly compact and convex subset of a Banach space E. Let $S = \{T_0\} \cup S'$ be a family of commuting mappings on C. Assume that for every $T \in S$, there exists a lower semicontinuous, strictly quasiconvex function f_T from C into $\mathbb{R}$ such that f_T is nonincreasing with respect to T. Assume also that $F(T)$ is closed and convex for $T \in S'$. Then S has a common fixed point.*

Lemma 6.6. *Let C be a convex subset of a UCED Banach space E. Let T be a mapping on C which satisfies Condition* (CC)*. Assume that $\{T^n u\}$ is bounded for some $u \in C$. Define a function f from C into $[0, \infty)$ by* (4.6)*. Then f is a continuous, strictly quasiconvex function such that f is nonincreasing with respect to T and f satisfies* (2.2)*.*

Proof. We note that a function g defined by (2.1) is continuous and strictly quasiconvex; and g satisfies (2.2). So f is also continuous and strictly quasiconvex; and f satisfies (2.2). We have shown that f is nonincreasing with respect to T in the proof of Proposition 4.8. □

Using Lemmas 6.3–6.6, we obtain the following.

Theorem 6.7. *Let C be a boundedly weakly compact and convex subset of a UCED Banach space E. Let T be a mapping on C which satisfies Condition* (CC)*. Then the following are equivalent*:

(i) *$\{T^n u\}$ is bounded for some $u \in C$.*
(ii) *T has a fixed point.*

Theorem 6.8. *Let C be a boundedly weakly compact and convex subset of a UCED Banach space E. Let $T_1, T_2, \ldots, T_\ell$ be commuting mappings on C which satisfy Condition* (CC)*. Assume that $\{T_j{}^n u\}$ is bounded for all $u \in C$ and j. Then $\bigcap_{j=1}^{\ell} F(T_j)$ is nonempty.*

Theorem 6.9. *Let C be a weakly compact and convex subset of a UCED Banach space E. Let S be a family of commuting mappings on C which satisfy Condition* (CC)*. Then S has a common fixed point.*

7. A problem

In [2], Aoyama and Kohsaka introduced the concept of α-nonexpansive mappings. Let T be a mapping on a subset C of a Banach space E. Then T is called α-nonexpansive iff there exists $\alpha \in \mathbb{R}$ such that $\alpha < 1$ and

$$\|Tx - Ty\|^2 \le \alpha \|x - Ty\|^2 + \alpha \|Tx - y\|^2 + (1 - 2\alpha) \|x - y\|^2 \tag{7.1}$$

for any $x, y \in C$.

Remark.

(i) In the case where $\alpha < 0$, the identity mapping on C is a unique α-nonexpansive mapping.
(ii) In the case where $\alpha = 0$, every α-nonexpansive mapping is nonexpansive.
(iii) In the case where $0 < \alpha \le 1/2$, every α-nonexpansive mapping satisfies Condition (CC) with $1 - 2\alpha$, α and $t \mapsto t^2$.
(iv) In the case where $1/2 < \alpha < 1$, we do not know the relation between the condition on α-nonexpansive mapping and Condition (CC).

We know more concept of nonlinear mappings; see also [1, 9, 10]. Since we use the mapping $t \mapsto t^2$ in their concepts as in (7.1), the authors believe that Condition (CC) is not stronger than their concepts.

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