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EXTREMAL STRUCTURE OF ABSOLUTE NORMS AND THE SKEWNESS

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Abstract. The skewness of Banach spaces was introduced by Fitzpatrick and Reznick. In this paper, we consider the skewness for absolute normalized norms on $\mathbb{R}^2$. In particular, we calculate the skewness of $(\mathbb{R}^2, \|\cdot\|)$ when $\|\cdot\|$ is an extreme point of AN_2, where AN_2 is the set of all absolute normalized norms on $\mathbb{R}^2$.

1. Introduction and Preliminaries

The notion of the skewness of a real Banach space X was introduced and investigated by Fitzpatrick and Reznick [3], as follows:

$$s(X) = \sup\left\{ \lim_{t\to+0} \frac{\|x+ty\| - \|y+tx\|}{t} : x, y \in S_X \right\}$$

where $S_X = \{x \in X : \|x\| = 1\}$. We set

$$\langle x, y\rangle = \|x\| \cdot \lim_{t\to+0} \frac{\|x+ty\| - \|x\|}{t}$$

for $x, y \in X$. As in [3] we note that $\langle \cdot,\cdot\rangle$ is the generalized inner product by Ritt [14], and that

$$s(X) = \sup\{\langle x, y\rangle - \langle y, x\rangle : x, y \in S_X\},$$

(cf. [1, 12]). From [3], we have $0 \le s(X) \le 2$ for any Banach space X, X is a Hilbert space if and only if $s(X) = 0$, and $s(X) = s(X^*)$ for any real Banach space X (X^* is the dual space of X). Also, Fitzpatrick and Reznick calculated the skewness for L_p spaces where $1 \le p \le \infty$. Namely, they showed that

$$s(L_p) = \max_{t>0} \frac{2(t - t^{p-1})}{1+t^p}$$

for $2 < p < \infty$, $s(L_2) = 0$, and $s(L_1) = s(L_\infty) = 2$. Note that the computation of skewness is not known for other Banach spaces.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $x, y \in \mathbb{R}$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p (1 \le p \le \infty)$ are basic

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examples,

$$\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ \max(|x|, |y|), & \text{if} \quad p = \infty. \end{cases}$$

Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the set of all continuous convex functions on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ for $t \in [0,1]$. According to Bonsall and Duncan [2], AN_2 and Ψ_2 are in a one-to-one correspondence with $\psi(t) = \|(1-t,\, t)\|$ for $t \in [0,1]$ and

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right), & \text{if} \quad (x,y) \ne (0,0), \\ 0, & \text{if} \quad (x,y) = (0,0) \end{cases}$$

For ℓ_p-norm $\|\cdot\|_p$, the corresponding convex function ψ_p is

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ \max(1-t, t), & \text{if} \quad p = \infty. \end{cases}$$

From this we can consider many non-ℓ_p-type norms easily (see [13]).

A norm $\|\cdot\| \in AN_2$ is an extreme point of AN_2 if

$$\|\cdot\| = \frac{1}{2}(\|\cdot\|' + \|\cdot\|''),\ \|\cdot\|', \|\cdot\|'' \in AN_2 \Rightarrow \|\cdot\|' = \|\cdot\|''.$$

The definition of extreme point of Ψ_2 is similar to that of AN_2. In [6], the authors determined the family of all extreme points of the set AN_2 by considering Ψ_2. For $0 \le \alpha \le \frac{1}{2} \le \beta \le 1$ and for the case $(\alpha, \beta) \ne (\frac{1}{2}, \frac{1}{2})$ we define

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t & \text{if } 0 \le t \le \alpha, \\ \frac{\alpha+\beta-1}{\beta-\alpha}t + \frac{\beta-2\alpha\beta}{\beta-\alpha} & \text{if } \alpha \le t \le \beta, \\ t & \text{if } \beta \le t \le 1. \end{cases}$$

For the case $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$ we put $\psi_{\frac{1}{2},\frac{1}{2}} = \psi_\infty$. Put $E = \{\psi_{\alpha,\beta} : 0 \le \alpha \le 1/2 \le \beta \le 1\}$. Then, for all $\psi \in \Psi_2$, $\|\cdot\|_\psi$ is an extreme point of AN_2 (resp. Ψ_2) if and only if $\psi \in E$(see also [4]). Computation of von Neumann-Jordan constant and James constant for such norms can be found in [5, 6, 7, 8, 9, 10].

In this paper, we consider the skewness of absolute normalized norms on $\mathbb{R}^2$ by means of the corresponding continuous convex functions in Ψ_2. In particular, we calculate the skewness of $(\mathbb{R}^2, \|\cdot\|_\psi)$ when $\|\cdot\|$ is an extreme point of AN_2. To do this we will use the fact

$$s(X) = \sup\{x^*(y) - y^*(x) : x^* \in \nabla_x, y^* \in \nabla_y\},$$

where ∇_x is the set of all norming (supporting) functionals at x on X ([3]).

2. Norming functionals

In this section we study norming functionals on $\mathbb{R}^2$ by considering Ψ_2 (cf. [11]). An element $x^* \in S_{X^*}$ is said to be a norming functional of $x \in X$ with $x \ne 0$ if $x^*(x) = \|x\|$. We denote by ∇_x the set of all norming functionals of x. The Hahn-Banach theorem yields that, for every $x \in X$ with $x \ne 0$, there exists at least one norming functional of X. For each $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\|_\psi = 1$, we put $t = |x_2|/(|x_1| + |x_2|)$ and

$$x(t) = \frac{1}{\psi(t)}(1 - t, t).$$

For each $t \in (0, 1]$ (resp. $t \in [0, 1)$ we denote by $\psi'_L(t)$ (resp. $\psi'_R(t)$) is the left derivative (resp. the right derivative) of ψ at t. We define a mapping G from $[0, 1]$ into the set of subintervals of $[0, 1]$ as

$$G(t) = \begin{cases} [-1, \psi'_R(0)], \text{ if } t = 0, \\ [\psi'_L(t), \psi'_R(t)], \text{ if } 0 < t < 1, \\ [\psi'_L(1), 1], \text{ if } t = 1. \end{cases}$$

Bonsall and Duncan [2] characterized the family of norming functionals of $x(t)$ on $\mathbb{C}^2$ by considering Ψ_2 (see also [11]). For the case $\mathbb{R}^2$ we similarly have,

Lemma 2.1 (cf. [2, 11])**.** *Let t with $0 \le t \le 1$. Then*

$$\nabla_{x(t)} = \begin{cases} \{(1, c(1 + a)) : a \in G(0), c = \pm 1\}, \text{ if } t = 0, \\ \{(\psi(t) - at, \psi(t) + a(1 - t)) : a \in G(t)\}, \text{ if } 0 < t < 1, \\ \{(c(1 - a), 1) : a \in G(1), c = \pm 1\}, \text{ if } t = 1. \end{cases}$$

As a direct consequence of Lemma 2.1 we have the following.

Lemma 2.2 (cf. [2, 11])**.** *Let $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\|_\psi = 1$ and put $t = |x_2|/(|x_1| + |x_2|)$. Then we can write*

$$x = \frac{1}{\psi(t)}(\xi_1(1 - t), \xi_2 t),$$

where $\xi_1, \xi_2 = \pm 1$. Moreover,

$$\nabla_x = \begin{cases} \{(\xi_1, c(1 + a)) : a \in G(0), c = \pm 1\}, \text{ if } t = 0, \\ \{(\xi_1(\psi(t) - at), \xi_2(\psi(t) + a(1 - t))) : a \in G(t)\}, \text{ if } 0 < t < 1, \\ \{(c(1 - a), \xi_2) : a \in G(1), c = \pm 1\}, \text{ if } t = 1. \end{cases}$$

3. Skewness

In this section, we estimate the skewness of absolute normalized norms on $\mathbb{R}^2$. Fitzpatrick and Reznick [3] proved the following useful lemma.

Lemma 3.1 ([3])**.** *For any Banach space X,*

$$s(X) = \sup\{x^*(y) - y^*(x) : x, y \in S_X, x^* \in \nabla_x, y^* \in \nabla_y\}.$$

For a Banach space X, we put $B_X = \{x \in X : \|x\| \le 1\}$. Let $\mathrm{ext}(B_X)$ be the set of all extreme points of B_X. In 2-dimensional spaces we can improve Lemma 3.1 by using Krein-Milman theorem.

Lemma 3.2. *Let $X = (\mathbb{R}^2, \|\cdot\|)$. Then*

$$s(X) = \sup\{x^*(y) - y^*(x) : x, y \in \mathrm{ext}(B_X), x^* \in \nabla_x, y^* \in \nabla_y\}.$$

Proof. It is enough to show

$$s(X) = \sup\{x^*(y) - y^*(x) : x \in \mathrm{ext}(B_X), y \in S_X, x^* \in \nabla_x, y^* \in \nabla_y\}. \tag{3.1}$$

Let $x, y \in S_X, x^* \in \nabla_x$ and $y^* \in \nabla_y$. Assume that $x = (1-\lambda)x_1 + \lambda x_2$, where $0 < \lambda < 1$ and $x_1, x_2 \in \mathrm{ext}(B_X)$. By $x^* \in \nabla_x$, we have

$$1 = (1-\lambda)x^*(x_1) + \lambda x^*(x_2) \le (1-\lambda)\|x^*\|\|x_1\| + \lambda\|x^*\|\|x_2\| = 1,$$

which implies $x^* \in \nabla_{x_1} \cap \nabla_{x_2}$. Together with

$$x^*(y) - y^*(x) = (1-\lambda)(x^*(y) - y^*(x_1)) + \lambda(x^*(y) - y^*(x_2)),$$

we obtain (3.1). □

Moreover,

Lemma 3.3. *Let* $X = (\mathbb{R}^2, \|\cdot\|)$. *Then*

$$s(X) = \sup\{s(X, x, y) : x, y \in \mathrm{ext}(B_X) \cap (\mathbb{R} \times \mathbb{R}^+)\}, \tag{3.2}$$

where

$$s(X, x, y) = \sup\{|x^*(y) - y^*(x)| : x^* \in \nabla_x, y^* \in \nabla_y\},$$

Proof. Let $x, y \in \mathrm{ext}(B_X)$, $x^* \in \nabla_x$ and $y^* \in \nabla_y$. We consider the case when x is in the first quadrant and y is in the third quadrant. Then $-y \in \mathrm{ext}(B_X)$ and $-y^* \in \nabla_{-y}$, and hence

$$x^*(y) - y^*(x) \le |x^*(-y) - (-y^*)(x)| \le s(X, x, -y).$$

The rest cases are similar. Together with Lemma 3.2, we have (3.2). □

We consider the case $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$. When $\alpha = 1/2$, $\beta = 1/2$ or $(\alpha, \beta) = (0, 1)$, we have $s(X) = 2$. It is clear that $\mathrm{ext}(B_X) \cap (\mathbb{R} \times \mathbb{R}^+) = \{x_1, x_2, x_3, x_4\}$, where $x_1 = (1, \frac{\alpha}{1-\alpha}), x_2 = (\frac{1-\beta}{\beta}, 1), x_3 = (-\frac{1-\beta}{\beta}, 1), x_4 = (-1, \frac{\alpha}{1-\alpha})$. We first calculate norming functional of each x_i.

Lemma 3.4. *Let* $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ *for* α, β. *Let* $\alpha > 0$. *Then*

$$\nabla_{x_1} = \Big\{(-\alpha a + 1 - \alpha, (1-\alpha)a + 1 - \alpha) : a \in \Big[-1, \frac{\alpha+\beta-1}{\beta-\alpha}\Big]\Big\} \tag{3.3}$$

and

$$\nabla_{x_4} = \Big\{(\alpha a - 1 + \alpha, (1-\alpha)a + 1 - \alpha) : a \in \Big[-1, \frac{\alpha+\beta-1}{\beta-\alpha}\Big]\Big\}.$$

If $\alpha = 0$, *then*

$$\nabla_{x_1} = -\nabla_{x_4} = \left\{(1, c(1+a)) : a \in \left[-1, \frac{\beta-1}{\beta}\right], c = \pm 1\right\}.$$

Let $\beta < 1$. *Then*

$$\nabla_{x_2} = \Big\{(-\beta a + \beta, (1-\beta)a + \beta) : a \in \Big[\frac{\alpha+\beta-1}{\beta-\alpha}, 1\Big]\Big\}$$

and

$$\nabla_{x_3} = \Big\{(\beta a - \beta, (1-\beta)a + \beta) : a \in \Big[\frac{\alpha+\beta-1}{\beta-\alpha}, 1\Big]\Big\}.$$

If $\beta = 1$, then

$$\nabla_{x_2} = \nabla_{x_3} = \left\{ (c(1-a), 1) : a \in \left[\frac{\alpha}{1-\alpha}, 1 \right], c = \pm 1 \right\}.$$

Proof. Let $\alpha > 0$. From Lemma 2.1,

$$\nabla_{x_1} = \nabla_{x(\alpha)} = \big\{ (\psi_{\alpha,\beta}(\alpha) - a\alpha, \psi_{\alpha,\beta}(\alpha) + a(1-\alpha)) : a \in G(\alpha) \big\}.$$

By $\psi'_L(\alpha) = -1$ and $\psi'_R(\alpha) = \frac{\alpha+\beta-1}{\beta-\alpha}$, we obtain $G(\alpha) = [-1, \frac{\alpha+\beta-1}{\beta-\alpha}]$ and hence (3.3) holds. The rest cases are similar. □

For each $x, y \in \mathrm{ext}(B_X)$, we define

$$M(X, x, y) = \sup\{x^*(y) - y^*(x) : x^* \in \nabla_x, y^* \in \nabla_y\}$$

and

$$m(X, x, y) = \inf\{x^*(y) - y^*(x) : x^* \in \nabla_x, y^* \in \nabla_y\}.$$

We shall calculate $s(X, x_i, x_j)$. Note that $s(X, x_i, x_i) = 0$ for all i.

Lemma 3.5. *Let $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ for α, β. If $\alpha > 0$ and $\beta < 1$, then*

$$s(X, x_1, x_2) = \max\left\{ \frac{1-2\alpha}{1-\alpha}, \frac{2\beta-1}{\beta} \right\}. \tag{3.4}$$

If $\alpha = 0$ and $\beta < 1$, then

$$s(X, x_1, x_2) = \max\left\{ 1, \frac{2(2\beta-1)}{\beta} \right\}. \tag{3.5}$$

If $\alpha > 0$ and $\beta = 1$, then

$$s(X, x_1, x_2) = \max\left\{ 1, \frac{2(1-2\alpha)}{1-\alpha} \right\}. \tag{3.6}$$

Proof. Let $\alpha > 0$ and $\beta < 1$. Take $x_1^* \in \nabla_{x_1}$ and $x_2^* \in \nabla_{x_2}$. By Lemma 3.4 we can write $x_1^* = (-\alpha a + 1 - \alpha, (1-\alpha)a + 1 - \alpha)$ and $x_2^* = (-\beta b + \beta, (1-\beta)b + \beta)$, where $a \in \big[-1, \frac{\alpha+\beta-1}{\beta-\alpha}\big]$ and $b \in \big[\frac{\alpha+\beta-1}{\beta-\alpha}, 1\big]$. Hence

$$\begin{aligned} x_1^*(x_2) - x_2^*(x_1) &= \Big\langle (-\alpha a + 1 - \alpha, (1-\alpha)a + 1 - \alpha), \big(\frac{1-\beta}{\beta}, 1\big) \Big\rangle \\ &\quad - \Big\langle (-\beta b + \beta, (1-\beta)b + \beta), \big(1, \frac{\alpha}{1-\alpha}\big) \Big\rangle \\ &= \frac{\beta-\alpha}{\beta} a + \frac{\beta-\alpha}{1-\alpha} b + \frac{1-\alpha}{\beta} - \frac{\beta}{1-\alpha}. \end{aligned}$$

By $\frac{\beta-\alpha}{\beta} \geq 0$ and $\frac{\beta-\alpha}{1-\alpha} \geq 0$,

$$\begin{aligned} M(X, x_1, x_2) &= \frac{\beta-\alpha}{\beta} \cdot \frac{\alpha+\beta-1}{\beta-\alpha} + \frac{\beta-\alpha}{1-\alpha} + \frac{1-\alpha}{\beta} - \frac{\beta}{1-\alpha} \\ &= \frac{1-2\alpha}{1-\alpha} \end{aligned}$$

and

$$m(X, x_1, x_2) = \frac{\beta-\alpha}{\beta} \cdot (-1) + \frac{\beta-\alpha}{1-\alpha} \cdot \frac{\alpha+\beta-1}{\beta-\alpha} + \frac{1-\alpha}{\beta} - \frac{\beta}{1-\alpha}$$

$$= -\frac{2\beta - 1}{\beta}.$$

Thus we obtain (3.4). If $\alpha = 0$ and $\beta < 1$, then we similarly have

$$x_1^*(x_2) - x_2^*(x_1) = \frac{1-\beta}{\beta} + c(1+a) + \beta b - \beta,$$

where $a \in [-1, \frac{\beta-1}{\beta}]$, $c = \pm 1$ and $b \in [\frac{\beta-1}{\beta}, 1]$. Hence $M(X, x_1, x_2) = 1$ and

$$m(X, x_1, x_2) = \frac{2(1-2\beta)}{\beta}.$$

Thus we obtain (3.5). If $\alpha > 0$ and $\beta = 1$, then

$$x_1^*(x_2) - x_2^*(x_1) = (1-\alpha)a + 1 - \alpha - c(1-b) - \frac{\alpha}{1-\alpha},$$

where $a \in [-1, \frac{\alpha}{1-\alpha}]$, $b \in [\frac{\alpha}{1-\alpha}, 1]$ and $c = \pm 1$. Hence

$$M(X, x_1, x_2) = \frac{2(1-2\alpha)}{1-\alpha}$$

and $m(X, x_1, x_2) = -1$. Thus we obtain (3.6). □

In other cases, we similarly have,

Lemma 3.6. *Let $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ for α, β. If $\alpha > 0$ and $\beta < 1$, then*

$$s(X, x_1, x_3) = \max\left\{\frac{2(1-2\alpha)(2\beta-1)}{\beta-\alpha}, \frac{\alpha}{1-\alpha} + \frac{1-\beta}{\beta}\right\}. \tag{3.7}$$

If $\alpha = 0$ and $\beta < 1$, then

$$s(X, x_1, x_3) = \max\left\{1, \frac{2(2\beta-1)}{\beta}\right\}. \tag{3.8}$$

If $\alpha > 0$ and $\beta = 1$, then

$$s(X, x_1, x_3) = \max\left\{1, \frac{2(1-2\alpha)}{1-\alpha}\right\}. \tag{3.9}$$

Proof. Let $\alpha > 0$ and $\beta < 1$. Then

$$\begin{aligned} &x_1^*(x_3) - x_3^*(x_1) \\ &\quad = \frac{\alpha+\beta-2\alpha\beta}{\beta}\cdot a - \frac{\alpha+\beta-2\alpha\beta}{1-\alpha}\cdot b + \frac{(1-\alpha)(2\beta-1)}{\beta} - \frac{(2\alpha-1)\beta}{1-\alpha}, \end{aligned}$$

where $a \in \big[-1, \frac{\alpha+\beta-1}{\beta-\alpha}\big]$ and $b \in \big[\frac{\alpha+\beta-1}{\beta-\alpha}, 1\big]$. Hence we have by $\alpha + \beta - 2\alpha\beta > 0$,

$$\begin{aligned} M(X, x_1, x_3) &= \frac{\alpha+\beta-2\alpha\beta}{\beta}\cdot\frac{\alpha+\beta-1}{\beta-\alpha} - \frac{\alpha+\beta-2\alpha\beta}{1-\alpha}\cdot\frac{\alpha+\beta-1}{\beta-\alpha} \\ &\quad + \frac{(1-\alpha)(2\beta-1)}{\beta} - \frac{(2\alpha-1)\beta}{1-\alpha} \\ &= \frac{2}{\beta-\alpha}(1-2\alpha)(2\beta-1) \end{aligned}$$

and

$$\begin{aligned} m(X, x_1, x_3) &= \frac{\alpha+\beta-2\alpha\beta}{\beta} \cdot (-1) - \frac{\alpha+\beta-2\alpha\beta}{1-\alpha} \\ &\quad + \frac{(1-\alpha)(2\beta-1)}{\beta} - \frac{(2\alpha-1)\beta}{1-\alpha} \\ &= -\frac{\alpha}{1-\alpha} - \frac{1-\beta}{\beta}. \end{aligned}$$

Thus we obtain (3.7). Let $\alpha = 0$ and $\beta < 1$. Then

$$x_1^*(x_3) - x_3^*(x_1) = -\frac{1-\beta}{\beta} + c(1+a) - (\beta b - \beta),$$

where $a \in [-1, \frac{\beta-1}{\beta}], c = \pm 1$ and $b \in [\frac{\beta-1}{\beta}, 1]$. Hence

$$M(X, x_1, x_3) = \frac{2(2\beta-1)}{\beta}$$

and $m(X, x_1, x_3) = -1$. Thus we obtain (3.8). Let $\alpha > 0$ and $\beta = 1$. Then

$$x_1^*(x_3) - x_3^*(x_1) = (1-\alpha)a + 1 - \alpha - c(1-b) - \frac{\alpha}{1-\alpha},$$

where $a \in [-1, \frac{\alpha}{1-\alpha}], b \in [\frac{\alpha}{1-\alpha}, 1]$ and $c = \pm 1$. Hence

$$M(X, x_1, x_3) = \frac{2(1-2\alpha)}{1-\alpha}$$

and $m(X, x_1, x_3) = -1$. Thus we obtain (3.9). □

Lemma 3.7. *Let $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ for α, β. Then*

$$s(X, x_1, x_4) = \frac{2\alpha(2\beta-1)}{\beta-\alpha}. \tag{3.10}$$

Proof. Let $\alpha > 0$ and $\beta < 1$. Then

$$x_1^*(x_4) - x_4^*(x_1) = 2\alpha(a-b),$$

where $a, b \in \big[-1, \frac{\alpha+\beta-1}{\beta-\alpha}\big]$. Hence

$$M(X, x_1, x_4) = \frac{2\alpha(2\beta-1)}{\beta-\alpha}$$

and

$$m(X, x_1, x_4) = -\frac{2\alpha(2\beta-1)}{\beta-\alpha}.$$

Thus we have (3.10). For the rest cases, we similarly have (3.10). □

Lemma 3.8. *Let $X = (\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ for α, β. Then*

$$s(X, x_2, x_3) = \frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha}. \tag{3.11}$$

Proof. Let $\beta < 1$. Then

$$x_2^*(x_3) - x_3^*(x_2) = 2(1-\beta)(a-b),$$

where $a, b \in \left[\frac{\alpha+\beta-1}{\beta-\alpha}, 1\right]$. Hence

$$M(X, x_2, x_3) = \frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha}$$

and

$$m(X, x_2, x_3) = -\frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha}.$$

Thus we have (3.11). If $\beta = 1$, then $s(X, x_2, x_3) = 0$ by $x_2 = x_3$. □

Moreover, we similarly obtain $s(X, x_2, x_4) = s(X, x_1, x_3)$ and $s(X, x_3, x_4) = s(X, x_1, x_2)$.

Remark 3.9. (i) Let $0 \le \alpha \le 1/2 < \beta \le 1$. Then it is easy to prove that

$$\max\left\{\frac{2(1-2\alpha)(2\beta-1)}{\beta-\alpha}, \frac{2\alpha(2\beta-1)}{\beta-\alpha}\right\} \ge \frac{2\beta-1}{\beta}$$

and

$$\max\left\{\frac{2(1-2\alpha)(2\beta-1)}{\beta-\alpha}, \frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha}\right\} \ge \frac{1-2\alpha}{1-\alpha}.$$

(ii) When $\alpha = 0$, note that

$$\frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha} \ge \frac{\alpha}{1-\alpha} + \frac{1-\beta}{\beta}$$

and

$$\max\left\{\frac{2(2\beta-1)}{\beta}, \frac{2(1-\beta)}{\beta}\right\} \ge 1.$$

When $\beta = 1$, note that

$$\frac{2\alpha(2\beta-1)}{\beta-\alpha} \ge \frac{\alpha}{1-\alpha} + \frac{1-\beta}{\beta}$$

and

$$\max\left\{\frac{2(1-2\alpha)}{1-\alpha}, \frac{2\alpha}{1-\alpha}\right\} \ge 1.$$

Thus we obtain the main theorem.

Theorem 3.10. *Let $0 \le \alpha \le 1/2 < \beta \le 1$. Then*

$$s\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})\big) = \max\left\{\frac{2(1-2\alpha)(2\beta-1)}{\beta-\alpha}, \frac{2\alpha(2\beta-1)}{\beta-\alpha}, \frac{2(1-\beta)(1-2\alpha)}{\beta-\alpha}, \frac{\alpha}{1-\alpha} + \frac{1-\beta}{\beta}\right\}.$$

Corollary 3.11. *Let $0 \le \alpha \le 1/2$. Then*

$$s\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,1-\alpha}})\big) = 2\max\left\{1-2\alpha, \frac{\alpha}{1-\alpha}\right\}.$$

Remark 3.12. It follows from Corollary 3.11 that the minimum of $s\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,1-\alpha}})\big)$ over $0 \le \alpha \le 1/2$ attains at $\alpha = 1 - \sqrt{2}/2$, and the value is $2(\sqrt{2}-1)(= 0.8284\cdots)$. This is just the case that the unit ball is a regular octagon. However, it is not always true for general ψ in E. Indeed, for $\alpha_0 = 0.25$ and $\beta_0 = 0.67$, we have $s\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha_0,\beta_0}})\big) = 0.8258\cdots$ by Theorem 3.10.

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