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GENERALIZED SPLIT FEASIBILITY PROBLEMS AND WEAK CONVERGENCE THEOREMS IN HILBERT SPACES

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Abstract. In this paper, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems and then obtain weak convergence theorems which are related to the problems. We first obtain some fundamental properties for inverse strongly monotone mappings and resolvents of maximal monotone operators in Hilbert spaces. Then using these properties, we establish two weak convergence theorems which generalize established weak convergence theorems. As applications, we get well-known and new weak convergence theorems which are connected with generalized split feasibility problems and equilibrium problems.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\kappa > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \kappa \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called κ-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be non-empty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [4] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [3] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [3] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^m A_i^{-1}0\big) \cap \big(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0)\big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where A^* is the adjoint operator of A and

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P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is non-empty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility peoblem and generalized split feasibility peoblems including the split common null point problem; see, for instance, [3, 5, 17, 36]. In the study, they used established results for solving the problems. In particular, established convergence theorems have been used for finding solutions of the problems. On the other hand, we know many existence and convergence theorems for inverse strongly monotone mappings in Hilbert spaces; see, for instance, [7, 11, 16, 19, 24, 25, 30].

In this paper, motivated by the ideas of these problems and results, we consider generalized split feasibility problems and then obtain weak convergence theorems which are related to the problems. We first obtain some fundamental properties for inverse strongly monotone mappings and resolvents of maximal monotone operators in Hilbert spaces. For example, we extend the result of (1.1) from metric projections to more general mappings. Then using these properties, we establish two weak convergence theorems for finding solutions of the generalized split feasibility peoblems. The results are generalizations of weak convergence theorems which have already been obtained. As applications, we get well-known and new weak convergence theorems which are connected with generalized split feasibility problems and equilibrium problems.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. From [28] we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$ we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

A Hilbert space satisfies Opial's condition [20], that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [20]. Let C be a non-empty, closed and convex subset of H and let $T\colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points of T. A mapping $T : C \to H$ is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called firmly nonexpansive if $\|Tx-Ty\|^2 \leq \langle Tx-Ty, x-y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [28]. For a non-empty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$.

Such a mapping P_C is also called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \le 0$ holds for all $x \in H$ and $y \in C$; see, for instance, [26]. Let B be a set-valued mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be monotone on H if $\langle x - y, u - v\rangle \ge 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone mapping B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to D(B)$, which is called the resolvent of B for $r > 0$. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$. The following lemma is crucial in order to prove the main theorems.

Lemma 2.1 ([25])**.** *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_s x - J_t x\| \le (|s - t| / s)\, \|x - J_s x\| \tag{2.3}$$

for all $s, t > 0$ and $x \in H$; see also [8, 26].

Lemma 2.2 ([23])**.** *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \le \alpha_n \le b < 1$ for all $n \in \mathbb{N}$ and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that for some c, $\limsup_{n\to\infty} \|v_n\| \le c$, $\limsup_{n\to\infty} \|w_n\| \le c$ and $\limsup_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n) w_n\| = c$. Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.*

Lemma 2.3 ([31])**.** *Let H be a Hilbert space and let E be a non-empty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \le \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in E$, then $\{P_E x_n\}$ converges strongly to some $z \in E$, where P_E is the metric projection on H onto E.*

Using Opial's theorem [20], we have the following lemma; see, for instance, [28].

Lemma 2.4. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a non-empty subset $E \subset H$ satisfying (i) and (ii):*

(i) *For every $x^* \in E$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists:*

(ii) *if a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to x^*, then $x^* \in E$.*

Then there exists $x_0 \in E$ such that $x_n \rightharpoonup x_0$.

Kocourek, Takahashi and Yao [13] defined a broad class of nonlinear mappings in a Hilbert space. Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. A mapping $T : C \to H$ is called generalized hybrid [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$. We call such a mapping (α, β)-generalized hybrid. Notice that the class covers several well-known mappings. For example, a (1, 0)-generalized hybrid mapping is nonexpansive. It is nonspreading [14, 15] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [29] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous. We can give the following example [10] of nonspreading mappings. Let H be a Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx = \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D, \end{cases}$$

where P_E is the metric projection of H onto E. Then S is a nonspreading mapping which is not continuous. This implies that the class of nonexpansive mappings does not contain nonspreading mappings. Kawasaki and Takahashi [12] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping S from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx - Sy\|^2 + \beta\|x - Sy\|^2 + \gamma\|Sx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Sx\|^2 + \zeta\|y - Sy\|^2 + \eta\|(x - Sx) - (y - Sy)\|^2 \leq 0 \end{aligned} \tag{2.5}$$

for all $x, y \in C$. Such a mapping S is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [13] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. We know the following theorem from Kawasaki and Takahashi [12].

Theorem 2.5 ([12]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then S has a fixed point if and only if there exists $z \in C$ such that $\{S^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of S is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

The following lemmas for widely more generalized hybrid mappings are essencial for proving our main theorem.

Lemma 2.6 ([12]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(S) \neq \emptyset$ and it satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\zeta+\eta \geq 0$ *and* $\alpha+\beta>0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\varepsilon+\eta \geq 0$ *and* $\alpha+\gamma>0$.

Then S *is quasi-nonexpansive.*

Lemma 2.7 ([9]). *Let* H *be a Hilbert space and let* C *be a non-empty, closed and convex subset of* H. *Let* $S : C \to H$ *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping. Suppose that it satisfies the following conditions* (1) *or* (2):

(1) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta>0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\beta+\zeta+\eta>0$.

If $x_n \rightharpoonup z$ *and* $x_n - Sx_n \to 0$, *then* $z \in F(S)$.

3. Lemmas

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x-Sx, Sx-y\rangle \geq 0 \tag{3.1}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$

$$\begin{aligned}\langle x-Sx, Sx-y\rangle &= \langle x-y+y-Sx, Sx-y\rangle \\ &= \langle x-y, Sx-y\rangle + \langle y-Sx, Sx-y\rangle \\ &\geq \|Sx-y\|^2 - \|Sx-y\|^2 \\ &= 0.\end{aligned}$$

We have the following lemma from Alsulami and Takahashi [1].

Lemma 3.1 ([1]). *Let* H_1 *and* H_2 *be Hilbert spaces and let* $\kappa>0$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$. *Let* $U : H_2 \to H_2$ *be a* κ*-inverse strongly monotone mapping. Then a mapping* $A^*UA : H_1 \to H_1$ *is* $\frac{\kappa}{\|A\|^2}$*-inverse strongly monotone.*

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Since $I-T$ is $\frac{1}{2}$-inverse strongly monotone, we have the following result from Lemma 3.1.

Lemma 3.2. *Let* H_1 *and* H_2 *be Hilbert spaces. Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping. Then a mapping* $A^*(I-T)A : H_1 \to H_1$ *is* $\frac{1}{2\|A\|^2}$*-inverse strongly monotone.*

The following lemma was proved in Takahashi, Xu and Yao [32].

Lemma 3.3 ([32]). *Let* H_1 *and* H_2 *be Hilbert spaces. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping and let* $J_\lambda = (I+\lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda>0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping and let* $A : H_1 \to H_2$ *be a bounded linear operator. Suppose that* $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. *Let* $\lambda, r>0$ *and* $z \in H_1$. *Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I-T)A)z$;
(ii) $0 \in A^*(I-T)Az + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Using Lemma 3.3, we can prove the following lemma.

Lemma 3.4. *Let H_1 and H_2 be Hilbert spaces and let $\kappa > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be a κ-inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*UA)z$;
(ii) $0 \in A^*UAz + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$.

Proof. Since $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$, there exists $z_0 \in B^{-1}0$ such that $Az_0 \in U^{-1}0$. Furthermore, since $U : H_2 \to H_2$ is κ-inverse strongly monotone, $I - 2\kappa U$ is nonexpansive; see [28].

(i) $\Leftrightarrow$ (iii). Suppose $z = J_\lambda(I - rA^*UA)z$. We have that

$$\begin{aligned} z &= J_\lambda(I - rA^*UA)z \\ &= J_\lambda(I - \frac{1}{2\kappa}r \cdot 2\kappa A^*UA)z \\ &= J_\lambda(I - \frac{1}{2\kappa}rA^*2\kappa UA)z \\ &= J_\lambda(I - \frac{1}{2\kappa}rA^*(I - (I - 2\kappa U))A)z. \end{aligned}$$

From Lemma 3.3, this equality is equivalent to

$$z \in B^{-1}0 \cap A^{-1}F(I - 2\kappa U) = B^{-1}0 \cap A^{-1}(U^{-1}0).$$

(ii) $\Rightarrow$ (iii). From $0 \in A^*UAz + Bz$, we have $-A^*UAz \in Bz$. Since

$$\begin{aligned} -A^*UAz &= -\frac{1}{2\kappa}2\kappa A^*UAz \\ &= -\frac{1}{2\kappa}A^*2\kappa UAz \\ &= -\frac{1}{2\kappa}A^*(I - (I - 2\kappa U))Az \\ &= -\frac{1}{\kappa}A^*\big(\frac{1}{2}I - \frac{1}{2}(I - 2\kappa U)\big)Az \\ &= -\frac{1}{\kappa}A^*(I - \big(\frac{1}{2}I + \frac{1}{2}(I - 2\kappa U)\big))Az, \end{aligned}$$

we have $-\frac{1}{\kappa}A^*(I - (\frac{1}{2}I + \frac{1}{2}(I - 2\kappa U)))Az \in Bz$. Put $S = \frac{1}{2}I + \frac{1}{2}(I - 2\kappa U)$. Since B is monotone, we have from $0 \in Bz_0$ that

$$\langle -\frac{1}{\kappa}A^*(I - S)Az, z - z_0\rangle \geq 0.$$

Then we have that $\langle A^*(I - S)Az, z - z_0\rangle \leq 0$ and hence

$$\langle Az - SAz, Az - Az_0\rangle \leq 0. \tag{3.2}$$

On the other hand, since S is firmly nonexpansive, we have from (3.1) that

$$\langle Az - SAz, SAz - Az_0\rangle \geq 0. \tag{3.3}$$

From (3.2) and (3.3) we have that

$$\|Az - SAz\|^2 = \langle Az - SAz, Az - SAz\rangle \leq 0 \tag{3.4}$$

and hence $Az = SAz$. This implies that $Az \in F(S) = F(I - 2\kappa U) = U^{-1}0$. Using this, we have $0 \in A^*UAz + Bz = Bz$. Therefore, $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$.

(iii) $\Rightarrow$ (ii). From $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$, we have that $UAz = 0$ and $0 \in Bz$. Thus we have $0 \in A^*(I - T)Az + Bz$. The proof is complete. □

Using Lemmas 3.1 and 3.4, we also have the following lemma.

Lemma 3.5. *Let H_1 and H_2 be Hilbert spaces and let $\kappa > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be a κ-inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $0 \in A^*UAu + Bu$ and $0 \in A^*UAv + Bv$. Then $A^*UAu = A^*UAv$. Furthermore, if $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$, then $(A^*UA + B)^{-1}0$ is closed and convex.*

Proof. If $0 \in A^*UAu + Bu$ and $0 \in A^*UAv + Bv$, then we have that $-A^*UAu \in Bu$ and $-A^*UAv \in Bv$. Since B is monotone, we have

$$\langle u - v, -A^*UAu - (-A^*UAv)\rangle \geq 0$$

and hence

$$\langle u - v, A^*UAu - A^*UAv\rangle \leq 0. \tag{3.5}$$

On the other hand, since A^*UA is $\frac{\kappa}{\|A\|^2}$-inverse strongly monotone from Lemma 3.1, we have

$$\langle u - v, A^*UAu - A^*UAv\rangle \geq \frac{\kappa}{\|A\|^2}\|A^*UAu - A^*UAv\|^2.$$

We have from (3.5) that $\|A^*UAu - A^*UAv\|^2 = 0$ and hence

$$A^*UAu = A^*UAv.$$

Since U is α-inverse strongly monotone, $U^{-1}0$ is closed and convex. Then $A^{-1}(U^{-1}0)$ is closed and convex because A is linear and continuous. We also have that since B is maximal monotone, $B^{-1}0$ is closed and convex. Hence $B^{-1}0 \cap A^{-1}(U^{-1}0)$ is closed and convex. Using Lemma 3.4, we have that $(A^*UA + B)^{-1}0$ is closed and convex. This completes the proof. □

4. Main results

Now we can prove the main theorems.

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be a κ-inverse strong monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)J_{\lambda_n}(I - \lambda_n A^*UA)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the following:*

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty, \quad 0 < a \le \lambda_n \le \frac{2\kappa}{\|A\|^2} \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then $x_n \rightharpoonup z_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$, *where* $z_0 = \lim_{n\to\infty} P_{B^{-1}0\cap A^{-1}(U^{-1}0)} x_n$.

Proof. Let $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Then we have that $z = J_{\lambda_n} z$ and $UAz = 0$. Put $y_n = J_{\lambda_n}(I - \lambda_n A^*UA)x_n$ for all $n \in \mathbb{N}$. Since J_{λ_n} is nonexpansive and U is κ-inverse strongly monotone, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A^*UA)x_n - J_{\lambda_n} z\|^2 \\
&\le \|x_n - \lambda_n A^*UAx_n - z\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, A^*UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, UAx_n\rangle + (\lambda_n)^2 \|A^*UAx_n\|^2 \\
&\le \|x_n - z\|^2 - 2\kappa\lambda_n \|UAx_n\|^2 + (\lambda_n)^2 \|A\|^2 \|UAx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\kappa) \|UAx_n\|^2 .
\end{aligned} \tag{4.1}$$

From $0 < a \le \lambda_n \le \frac{2\kappa}{\|A\|^2}$ we have that $\|y_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$ and hence

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n x_n + (1-\beta_n) y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n) \|y_n - z\| \\
&\le \beta_n \|x_n - z\| + (1-\beta_n) \|x_n - z\| \\
&\le \|x_n - z\| .
\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus $\{x_n\}$, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Since

$$\kappa\|UAx_n\|^2 \le \langle UAx_n, Ax_n - Az\rangle \le \|UAx_n\|\|Ax_n - Az\|,$$

$\{UAx_n\}$ is bounded. Then $\{A^*UAx_n\}$ is bounded. Using the equality (2.1), we have that for $n \in \mathbb{N}$ and $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n) y_n - z\|^2 \\
&= \beta_n \|x_n - z\|^2 + (1-\beta_n) \|y_n - z\|^2 - \beta_n(1-\beta_n) \|x_n - y_n\|^2 \\
&\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \|x_n - z\|^2 - \beta_n(1-\beta_n) \|x_n - y_n\|^2 \\
&= \|x_n - z\|^2 - \beta_n(1-\beta_n) \|x_n - y_n\|^2
\end{aligned}$$

and hence

$$\beta_n(1-\beta_n) \|x_n - y_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 .$$

Thus we have

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) \|x_n - y_n\|^2 \le \|x_1 - z\|^2 - \lim_{n\to\infty} \|x_n - z\|^2 .$$

Since $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, we have from [28, p. 114] that

$$\liminf_{n\to\infty} \|x_n - y_n\|^2 = 0. \tag{4.2}$$

Put $u_n = (I - \lambda_n A^* UA)x_n$. Since $J_{\lambda_n}(I - \lambda_n A^* UA)$ is nonexpansive, we have from Lemma 2.1 that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \left\|J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_{n+1} - J_{\lambda_n}(I - \lambda_n A^* UA)x_n\right\| \\
&\leq \left\|J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_{n+1} - J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_n\right\| \\
&\quad + \left\|J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_n - J_{\lambda_n} u_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \left\|J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_n - J_{\lambda_n} u_n\right\| \\
&\leq \left\|J_{\lambda_{n+1}}(I - \lambda_{n+1} A^* UA)x_n - J_{\lambda_{n+1}}(I - \lambda_n A^* UA)x_n\right\| \\
&\quad + \left\|J_{\lambda_{n+1}} u_n - J_{\lambda_n} u_n\right\| + \|x_{n+1} - x_n\| \\
&\leq \|(I - \lambda_{n+1} A^* UA)x_n - (I - \lambda_n A^* UA)x_n\| \\
&\quad + \left\|J_{\lambda_{n+1}} u_n - J_{\lambda_n} u_n\right\| + \|x_{n+1} - x_n\| \\
&\leq |\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| \\
&\quad + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\| + \|x_{n+1} - x_n\| .
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &= \|\beta_n x_n + (1 - \beta_n) y_n - y_{n+1}\| \\
&\leq \beta_n \|x_n - y_{n+1}\| + (1 - \beta_n) \|y_n - y_{n+1}\| \\
&\leq \beta_n \|x_n - x_{n+1} + x_{n+1} - y_{n+1}\| + (1 - \beta_n)\{\|x_{n+1} - x_n\| \\
&\quad + |\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\|\} \\
&\leq \beta_n \|x_n - x_{n+1}\| + \beta_n \|x_{n+1} - y_{n+1}\| \\
&\quad + (1 - \beta_n)\{\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| \\
&\quad + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\|\} \\
&= \|x_n - x_{n+1}\| + \beta_n \|x_{n+1} - y_{n+1}\| \\
&\quad + (1 - \beta_n)\{|\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| \\
&\quad + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\|\}
\end{aligned}$$

and hence

$$\begin{aligned}
(1 - \beta_n) \|x_{n+1} - y_{n+1}\| &\leq \|x_n - x_{n+1}\| \\
&\quad + (1 - \beta_n\{|\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\|\} \\
&= (1 - \beta_n) \|x_n - y_n\| \\
&\quad + (1 - \beta_n)\{|\lambda_{n+1} - \lambda_n| \, \|A^* UAx_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\|\}.
\end{aligned}$$

From $1 - \beta_n > 0$, we have

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &\leq \|x_n - y_n\| + |\lambda_{n+1} - \lambda_n| \, \|A^*(I - T)Ax_n\| \\
&\quad + \frac{|\lambda_{n+1} - \lambda_n|}{a} \left\|J_{\lambda_{n+1}} u_n - u_n\right\| .
\end{aligned}$$

Using Tan and Xu's lemma [33], we have that $\lim_{n\to\infty}\|x_n - y_n\|$ exists. Hence, we have from (4.2) that

$$\lim_{n\to\infty}\|x_n - y_n\| = \liminf_{n\to\infty}\|x_n - y_n\| = 0.$$

We finally show that $x_n \rightharpoonup z_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$. We first show $x^* \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Since $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty$, $\{\lambda_n\}$ is a Cauchy sequence. Then there exists $\lambda_0 \in [a, \frac{2\kappa}{\|A\|^2}]$ such that $\lambda_n \to \lambda_0$. We have that $x_{n_i} \rightharpoonup x^*$ and $\lambda_{n_i} \to \lambda_0$. For such λ_0, we have that

$$\begin{aligned}
&\|J_{\lambda_0}(I-\lambda_0 A^*UA)x_{n_i} - y_{n_i}\| \\
&\quad = \|J_{\lambda_0}(I-\lambda_0 A^*UA)x_{n_i} - J_{\lambda_{n_i}}(I-\lambda_{n_i}A^*UA)x_{n_i}\| \\
&\quad \le \|J_{\lambda_0}(I-\lambda_0 A^*UA)x_{n_i} - J_{\lambda_0}(I-\lambda_{n_i}A^*UA)x_{n_i}\| \\
&\qquad + \|J_{\lambda_0}(I-\lambda_{n_i}A^*UA)x_{n_i} - J_{\lambda_{n_i}}(I-\lambda_{n_i}A^*UA)x_{n_i}\| \\
&\quad \le \|(I-\lambda_0 A^*UA)x_{n_i} - (I-\lambda_{n_i}A^*UA)x_{n_i}\| \\
&\qquad + \|J_{\lambda_0}u_{n_i} - J_{\lambda_{n_i}}u_{n_i}\| \\
&\quad \le |\lambda_0 - \lambda_{n_i}|\|A^*UAx_{n_i}\| + \frac{|\lambda_0 - \lambda_{n_i}|}{\lambda_0}\|J_{\lambda_0}u_{n_i} - u_{n_i}\| \to 0,
\end{aligned} \tag{4.3}$$

where $u_{n_i} = (I-\lambda_{n_i}A^*UA)x_{n_i}$. We also have that

$$\begin{aligned}
&\|x_{n_i} - J_{\lambda_0}(I-\lambda_0 A^*UA)x_{n_i}\| \\
&\qquad \le \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - J_{\lambda_0}(I-\lambda_0 A^*UA)x_{n_i}\| \to 0.
\end{aligned} \tag{4.4}$$

Since $J_{\lambda_0}(I-\lambda_0 A^*UA)$ is nonexpansive, we have from (4.4) and $x_{n_i} \rightharpoonup x^*$ that $x^* = J_{\lambda_0}(I-\lambda_0 A^*UA))x^*$. We have from Lemma 3.3 that $x^* \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. We next show that if $x_{n_i} \rightharpoonup x^*$ and $x_{n_j} \rightharpoonup y^*$, then $x^* = y^*$. We know $x^*, y^* \in B^{-1}0 \cap A^{-1}(U^{-1}0)$ and hence $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty}\|x_n - y^*\|$ exist. Suppose $x^* \neq y^*$. Since H satisfies Opial's condition, we have that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - x^*\| &= \lim_{i\to\infty}\|x_{n_i} - x^*\| < \lim_{i\to\infty}\|x_{n_i} - y^*\| \\
&= \lim_{n\to\infty}\|x_n - y^*\| = \lim_{j\to\infty}\|x_{n_j} - y^*\| \\
&< \lim_{j\to\infty}\|x_{n_j} - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|.
\end{aligned}$$

This is a contradiction. Then we have $x^* = y^*$. Therefore, $x_n \rightharpoonup x^* \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. Moreover, since for any $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$

$$\|x_{n+1} - z\| \le \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

we have from Lemma 2.3 that $P_{B^{-1}0\cap A^{-1}(U^{-1}0)}x_n \to z_0$ for some $z_0 \in B^{-1}0 \cap A^{-1}(U^{-1}0)$. The property of metric projection implies that

$$\langle x^* - P_{B^{-1}0\cap A^{-1}(U^{-1}0)}x_n, x_n - P_{B^{-1}0\cap A^{-1}(U^{-1}0)}x_n\rangle \le 0.$$

Therefore, we have

$$\langle x^* - z_0, x^* - z_0\rangle = \|x^* - z_0\|^2 \le 0.$$

This means that $x^* = z_0$, i.e., $x_n \rightharpoonup z_0$. □

Theorem 4.2. *Let H_1 and H_2 be real Hilbert spaces and let C be a non-empty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that the domain of B is included in C and let $J_\lambda = (I + rB)^{-1}$ be the resolvent of B for $r > 0$. Let $S : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

and let $U : H_2 \to H_2$ be a κ-inverse strong monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. For any $x_1 = x \in C$, define

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(J_{\lambda_n}(I - \lambda_n A^* U A)x_n), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following:

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < \frac{2\kappa}{\|A\|^2}.$$

Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$, where $z_0 = \lim_{n\to\infty} P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)} x_n$.

Proof. Set $E = F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$. Then we have that E is closed and convex. Let $y_n = J_{\lambda_n}(I - \lambda_n A^* U A)x_n$ for all $n \in \mathbb{N}$ and let $z \in E$. Since $z = J_{\lambda_n} z$, $UAz = 0$ and U is κ-inverse strongly monotone, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A^* U A)x_n - J_{\lambda_n} z\|^2 \\
&\leq \|x_n - \lambda_n A^* U A x_n - z\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, A^* U A x_n\rangle + (\lambda_n)^2 \|A^* U A x_n\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n \langle A x_n - Az, U A x_n\rangle + (\lambda_n)^2 \|A^* U A x_n\|^2 \\
&\leq \|x_n - z\|^2 - 2\kappa\lambda_n \|UAx_n\|^2 + (\lambda_n)^2 \|A\|^2 \|UAx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\kappa) \|UAx_n\|^2 .
\end{aligned} \tag{4.5}$$

From (4.5) we have that $\|y_n - z\| \leq \|x_n - z\|$ for all $n \in \mathbb{N}$ and hence

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n x_n + (1 - \beta_n) S y_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1 - \beta_n) \|S y_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1 - \beta_n) \|y_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1 - \beta_n) \|x_n - z\| \\
&= \|x_n - z\| .
\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n - z\|$ exists. This also means that the condition (i) of Lemma 2.4 holds for E. Thus $\{x_n\}, \{y_n\}$ and $\{Sy_n\}$ are bounded. As in the proof of Theorem 4.1, we also have that $\{A^* U A x_n\}$ is bounded. By the inequality (4.5),

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|S y_n - z\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|y_n - z\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n)\{\|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\kappa) \|UAx_n\|^2\}
\end{aligned}$$

$$\le \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 2\kappa)(1-\beta_n) \|UAx_n\|^2 .$$

Thus we have

$$\begin{aligned}\lambda_n(\lambda_n \|A\|^2 - 2\kappa)(1-\beta_n) \|UAx_n\|^2 \\ \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 .\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - z\|$ exists, we have that

$$\lim_{n\to\infty} \|UAx_n\| = 0. \tag{4.6}$$

On the other hand, since J_{λ_n} is firmly nonexpansive and $I - \lambda_n A^*UA$ is nonexpansive, we have from (2.2) that

$$\begin{aligned}2\|y_n - z\|^2 &= 2 \|J_{\lambda_n}(I - \lambda_n A^*UA)x_n - J_{\lambda_n}(I - \lambda_n A^*UA)z\|^2 \\ &\le 2\langle y_n - z, (I - \lambda_n A^*UA)x_n - (I - \lambda_n A^*UA)z\rangle \\ &= 2\langle y_n - z, (I - \lambda_n A^*UA)x_n - z\rangle \\ &= \|y_n - z\|^2 + \|(I - \lambda_n A^*UA)x_n - z\|^2 \\ &\quad - \|y_n - (I - \lambda_n A^*UA)x_n\|^2 \\ &\le \|y_n - z\|^2 + \|x_n - z\|^2 - \|y_n - x_n + \lambda_n A^*UAx_n\|^2 \\ &= \|y_n - z\|^2 + \|x_n - z\|^2 - \|y_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle y_n - x_n, A^*UAx_n\rangle - {\lambda_n}^2 \|A^*UAx_n\|^2 .\end{aligned}$$

Therefore we have

$$\begin{aligned}\|y_n - z\|^2 \le \|x_n - z\|^2 - \|y_n - x_n\|^2 \\ - 2\lambda_n\langle y_n - x_n, A^*UAx_n\rangle - {\lambda_n}^2 \|A^*UAx_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \|Sy_n - z\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \|y_n - z\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\{\|x_n - z\|^2 - \|y_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle y_n - x_n, A^*UAx_n\rangle - {\lambda_n}^2 \|A^*UAx_n\|^2\} \\ &\le \|x_n - z\|^2 - (1-\beta_n) \|y_n - x_n\|^2 - {\lambda_n}^2(1-\beta_n) \|A^*UAx_n\|^2 \\ &\quad - 2\lambda_n(1-\beta_n)\langle y_n - x_n, A^*UAx_n\rangle.\end{aligned}$$

This means that

$$\begin{aligned}(1-\beta_n) \|y_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ - {\lambda_n}^2(1-\beta_n) \|A^*UAx_n\|^2 - 2\lambda_n(1-\beta_n)\langle y_n - x_n, A^*UAx_n\rangle.\end{aligned}$$

Since $\{y_n\}$ and $\{x_n\}$ are bounded, $\lim_{n\to\infty} \|A^*UAx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - z\|$ exists, we have that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Since A is a bounded linear operator, we have that $Ax_{n_j} \rightharpoonup Ax^*$. Since

$$\kappa\|UAx_{n_j} - UAx^*\|^2 \leq \langle UAx_{n_j} - UAx^*, Ax_{n_j} - Ax^*\rangle,$$

we have from (4.6) that $\kappa\|UAx^*\|^2 \leq 0$. This implies $UAx^* = 0$ and hence $x^* \in A^{-1}(U^{-1}0)$. Let us prove that $x^* \in B^{-1}(0)$. Since $y_n = J_{\lambda_n}(I - \lambda_n A^*UA)x_n$, we have that

$$\begin{aligned} y_n &= J_{\lambda_n}(I - \lambda_n A^*UA)x_n \\ &\Leftrightarrow (I - \lambda_n A^*UA)x_n \in y_n + \lambda_n B y_n \\ &\Leftrightarrow x_n - y_n - \lambda_n A^*UAx_n \in \lambda_n B y_n \\ &\Leftrightarrow \frac{1}{\lambda_n}(x_n - y_n - \lambda_n A^*UAx_n) \in By_n. \end{aligned}$$

Since B is monotone, we have that for $(u, v) \in B$,

$$\langle y_n - u, \frac{1}{\lambda_n}(x_n - y_n - \lambda_n A^*UAx_n) - v\rangle \geq 0$$

and hence

$$\langle y_n - u, \frac{x_n - y_n}{\lambda_n} - A^*UAx_n - v\rangle \geq 0.$$

Replacing n by n_j, we have that

$$\langle y_{n_j} - u, \frac{x_{n_j} - y_{n_j}}{\lambda_{n_j}} - A^*UAx_{n_j} - v\rangle \geq 0.$$

Since $x_{n_j} - y_{n_j} \to 0$, $0 < a \leq \lambda_{n_j} \leq b$, $y_{n_j} \rightharpoonup x^*$ and $A^*UAx_{n_j} \to 0$, we have that $\langle x^* - u, -v\rangle \geq 0$. Since B is maximal monotone, we have that $0 \in Bx^*$. Let us show $x^* \in F(S)$. Putting $c = \lim_{n\to\infty} \|x_n - z\|$, we have

$$\begin{aligned} \limsup_{n\to\infty} \|Sy_n - z\| &\leq \limsup_{n\to\infty} \|y_n - z\| \\ &\leq \limsup_{n\to\infty} \|x_n - z\| \leq c. \end{aligned}$$

On the other hand, we have that

$$\lim_{n\to\infty} \|\beta_n(x_n - z) + (1 - \beta_n)(Sy_n - z)\| = \lim_{n\to\infty} \|x_{n+1} - z\| = c.$$

From Lemma 2.2, we have that

$$\lim_{n\to\infty} \|(x_n - z) - (Sy_n - z)\| = \lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{4.7}$$

From

$$\|y_n - Sy_n\| \leq \|y_n - x_n\| + \|x_n - Sy_n\|,$$

we have

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{4.8}$$

Since $y_{n_j} \rightharpoonup x^*$, we have from Lemma 2.7 and (4.8) that $x^* \in F(S)$. Therefore, we obtain that

$$x^* \in E = F(S) \cap B^{-1}(0) \cap A^{-1}(U^{-1}0).$$

This implies that the condition (ii) of Lemma 2.4 holds for E. We have from Lemma 2.4 that there exists $z^* \in E$ such that $x_n \rightharpoonup z^*$ as $n \to \infty$. Moreover, since

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \in \mathbb{N},\ z \in E,$$

by Lemma 2.3 there exists some $z_0 \in E$ such that $P_E x_n \to z_0$. The property of metric projection implies that

$$\langle z^* - P_E x_n, x_n - P_E x_n \rangle \leq 0.$$

Therefore, we have

$$\langle z^* - z_0, z^* - z_0 \rangle = \|z^* - z_0\|^2 \leq 0.$$

This means that $z^* = z_0$, i.e., $x_n \rightharpoonup z^* = \lim_{n\to\infty} P_E(x_n)$. □

5. Applications

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x \rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. By Rockafellar [22], it is shown that ∂f is maximal monotone. Let C be a non-empty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x, \quad \forall x \in H,\ \lambda > 0.$$

We know that $J_\lambda x = P_C x$ for all $x \in H$ and $\lambda > 0$; see [28]. Using Theorem 4.1, we first obtain the following weak convergence theorem which was proved by Takahashi, Xu and Yao [32].

Theorem 5.1 ([32]). *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone operator and let $J_\lambda = (I + \lambda B)^{-1}$ be its resolvent of index $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following:

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty, \quad 0 < a \leq \lambda_n \leq \frac{1}{\|A\|^2} \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then $x_n \rightharpoonup z_0 \in B^{-1}0 \cap A^{-1}F(T)$, where $z_0 = \lim_{n\to\infty} P_{B^{-1}0 \cap A^{-1}F(T)} x_n$.

Proof. Suppose that T is nonexpansive. Then $U = I - T$ is $\frac{1}{2}$-inverse strongly monotone. Thus we obtain the desired result by Theorem 4.1. □

We also have the following theorem from Theorem 4.2.

Theorem 5.2 ([32]). *Let H_1 and H_2 be real Hilbert spaces and let C be a non-empty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that $D(B) \subset C$ and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $V : C \to C$ be a generalized hybrid mapping and let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $S := F(V) \cap B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in C$, generate a sequence $\{x_n\}$ by the algorithm*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)V(J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n), \quad \forall n \in \mathbb{N},$$

where the sequences of parameters $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following:

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}.$$

Then $\{x_n\}$ converges weakly to a point $z_0 \in S$, where $z_0 = \lim_{n\to\infty} P_S x_n$.

Proof. A generalized hybrid mapping $V : C \to C$ is widely more generalized hybrid. Since T is nonexpansive, then $U = I - T$ is $\frac{1}{2}$-inverse strongly monotone. Thus we have the desired result from Theorem 4.2. □

Next, we deal with the equilibrium problem with an inverse strongly monotone mapping in Hilbert spaces. Let C be a non-empty, closed and convex subset of a real Hilbert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [2] and [6].

Lemma 5.3 ([2]). *Let C be a non-empty, closed and convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 5.4 ([6]). *For $r > 0$ and $x \in H$, define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}.$$

Then, the following hold:

(i) T_r *is single-valued;*
(ii) T_r *is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$*;*
(iv) $EP(f)$ *is closed and convex.*

Takahashi, Takahashi and Toyoda [25] showed the following.

Lemma 5.5 ([25]). *Let C be a no-nempty, closed and convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthermore,

$$T_r x = (I + rA_f)^{-1}x, \quad \forall r > 0.$$

We obtain the following theorem from Theorem 4.1.

Theorem 5.6. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5. Let $U : H_2 \to H_2$ be a κ-inverse strongly monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}(U^{-1}0) \neq \emptyset$. For $x_1 = x \in H_1$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{\lambda_n}(I - \lambda_n A^* U A)x_n), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following:

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty, \quad 0 < a \leq \lambda_n \leq \frac{2\kappa}{\|A\|^2} \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

Then, $x_n \rightharpoonup z_0 \in EP(f) \cap A^{-1}(U^{-1}0)$, where $z_0 = \lim_{n\to\infty} P_{EP(f)\cap A^{-1}(U^{-1}0)}x_n$.

Proof. Define A_f for the bifunction f and set $B = A_f$ in Theorem 4.1. Thus we have the desired result from Theorem 4.1. □

As in the proof of Theorems 5.2 and 5.6, we obtain the following result from Theorem 4.2.

Theorem 5.7. *Let H_1 and H_2 be Hilbert spaces. Let C be a non-empty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 5.5.*

Let $S : C \to C$ be a generalized hybrid mapping and let $U : H_2 \to H_2$ be a κ-inverse strongly monotone mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap EP(f) \cap A^{-1}(U^{-1}0) \neq \emptyset$. For $x_1 = x \in C$, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S T_{\lambda_n}(I - \lambda_n A^* U A) x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following:

$$0 < c \leq \beta_n \leq d < 1 \quad \text{and} \quad 0 < a \leq \lambda_n \leq b < \frac{2\kappa}{\|A\|^2}.$$

Then, $x_n \rightharpoonup z_0 \in F(S) \cap EP(f) \cap A^{-1}(U^{-1}0)$, where

$$z_0 = \lim_{n\to\infty} P_{F(S)\cap EP(f)\cap A^{-1}(U^{-1}0)} x_n.$$

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