



# SEMISMOOTH NEWTON METHODS FOR THE CONE SPECTRUM OF LINEAR TRANSFORMATIONS RELATIVE TO LORENTZ CONES

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ABSTRACT. We propose two semismooth Newton methods for seeking the eigenvalues of a linear transformation relative to Lorentz cones, by the natural equation reformulation and the normal equation reformulation, respectively, and establish their local quadratic convergence results under suitable conditions. The convergence analysis shows that the method based on the natural equation formulation is not influenced by the asymmetry of linear transformations, but the one based on the normal equation formulation suffers from this easily. Numerical experiments indicate that the method based on the normal equation formulation is very effective for the Lorentz eigenvalue problem of  $\mathbf{Z}$ -transformations, and the method based on the natural equation formulation is very promising for those Lorentz eigenvalue problems of general linear asymmetric transformations with dimension less than 200, if one aims at finding at least one Lorentz eigenvalue.

## 1. INTRODUCTION

Let  $\mathbb{K}$  be a closed convex cone in a real Hilbert space  $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ , and  $\mathbb{K}^*$  be the positive dual cone of  $\mathbb{K}$ . For a given continuous linear transformation  $\mathcal{A} : \mathbb{X} \rightarrow \mathbb{X}$ , a real number  $\lambda$  is called  $\mathbb{K}$ -eigenvalue of  $\mathcal{A}$  if there is a nonzero vector  $x \in \mathbb{X}$  such that

$$(1.1) \quad x \in \mathbb{K}, \quad \mathcal{A}(x) - \lambda x \in \mathbb{K}^*, \quad \text{and} \quad \langle x, \mathcal{A}(x) - \lambda x \rangle = 0.$$

This class of problems first appears with a slightly different formulation in the works of Riddell [24] about nonlinear variational inequalities on a cone, Kučera [19, 20] and Quittner [23] on bifurcation analysis of eigenvalues relative to a cone. Recently, due to the applications arising from contact problems in mechanics [5, 12], there are active interests (see, e.g., [1, 17, 18, 26, 28]) in problem (1.1) with  $\mathbb{K}$  being the nonnegative orthant cone  $\mathbb{R}_+^n$  in the Euclidean space  $\mathbb{R}^n$ , which is known as *the Pareto eigenvalue problem*, and some efficient algorithms are designed such as

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the spectral projected gradient algorithm [18], the power iteration method [4], and the nonsmooth algorithm [1]. However, to the best of our knowledge, there are very few numerical methods proposed for seeking the eigenvalues of  $\mathcal{A}$  relative to a nonpolyhedral cone except for [4] where the spectral projected gradient algorithm [18] is applied for the circular eigenvalue problem.

In this paper, we are concerned with numerical algorithms for the problem of seeking the eigenvalues of a linear transformation on  $\mathbb{R}^n$  relative to Lorentz cones. This problem, called *the Lorentz eigenvalue problem*, is to find a  $\lambda \in \mathbb{R}$  and a nonzero  $x \in \mathbb{R}^n$  such that

$$(1.2) \quad x \in K, \quad \mathcal{A}(x) - \lambda x \in K, \quad \langle x, \mathcal{A}(x) - \lambda x \rangle = 0,$$

where  $\mathcal{A}$  is a given linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , and  $K$  is the Cartesian product of Lorentz cones or second-order cones (SOCs). In other words,  $K$  can be expressed as

$$(1.3) \quad K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_r},$$

where  $r, n_1, \dots, n_r \geq 1$ ,  $n_1 + \cdots + n_r = n$ , and  $K^{n_i}$  is the Lorentz cone in  $\mathbb{R}^{n_i}$  defined by

$$K^{n_i} := \{(x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} \mid x_{i1} \geq \|x_{i2}\|\}.$$

It is known that (1.2) has a nonempty solution set, see [27, Corollary 2.1] or [26, Theorem 2.5]. Moreover, Seeger and Torki [26] presented a characterization for its cone spectrum

$$\sigma(\mathcal{A}, K) := \{\lambda \in \mathbb{R} \mid (\lambda, x) \text{ solves (1.2) for some } x \neq 0\}.$$

Recently, for a class of special linear transformations, i.e., **Z**-transformation, Zhou and Gowda [33] established the finiteness of the general symmetric cone spectrum.

Although seeking the solution to problem (1.2) is equivalent to finding a zero of a nonlinear second-order cone complementarity system (see Sec. 3), the existing methods such as the merit function method [7] and the smoothing Newton methods [9, 8, 10] developed for general nonlinear second-order cone complementarity problems, are not suitable for finding the solution of (1.2), since their global convergence conditions and stationary point conditions are typically not satisfied by the nonlinear mapping  $F(x, \lambda) \equiv \mathcal{A}(x) - \lambda x$ . Motivated by the efficiency of nonsmooth algorithms for the Pareto eigenvalue problem [1], in this paper we propose two semismooth Newton methods via the natural equation reformulation and the normal equation reformulation of (1.2), respectively, and establish their local quadratic convergence under suitable conditions. The convergence analysis results show that the method based on the natural equation formulation may avoid the influence of the asymmetry of  $\mathcal{A}$  well, whereas the method based on the normal equation formulation suffers from it easily. This is also verified by numerical experiments.

Our numerical results show that the method based on the normal equation formulation is very effective for the Lorentz eigenvalue problem of **Z**-transformations such as the Lyapunov transformation and the Stein transformation, and the method

based on the natural equation formulation is very promising for those Lorentz eigenvalue problems of general linear asymmetric transformations whose dimensions are within 200, whenever one aims at finding at least one Lorentz eigenvalue.

This paper is organized as follows. Section 2 recalls some background materials about the Jordan product associated with  $K^n$  and the results that will be used in the subsequent sections. Section 3 gives two nonsmooth system reformulations of (1.2) and the semismooth Newton methods based on them. Section 4 focuses on the local convergence analysis of the two semismooth Newton methods. Section 5 reports the computational experience and numerical results with the two methods solving Lorentz eigenvalue problems of several classes of linear transformations. Finally, we conclude this paper with some remarks.

Throughout this paper,  $I$  denotes an identity matrix of appropriate dimension,  $\mathbb{R}^n$  denotes the space of  $n$ -dimensional real column vectors endowed with the inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\| \cdot \|$ , and  $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$  is identified with  $\mathbb{R}^{n_1 + \cdots + n_r}$ . For any given linear transformation  $\mathcal{A}$  on  $\mathbb{R}^n$ , we denote  $\sigma(\mathcal{A})$  the spectrum of  $\mathcal{A}$ , and  $\hat{\sigma}(\mathcal{A}, \mathbb{K})$  the spectrum of  $\mathcal{A}$  consisting of the eigenvalues whose eigenvectors belong to  $\mathbb{K}$ . An  $n \times n$  real (not necessarily symmetric) matrix  $B$  is said to be *positive semidefinite* if  $\langle x, Bx \rangle \geq 0$  for all  $x \in \mathbb{R}^n$ . A linear transformation  $\mathcal{A}$  on  $\mathbb{X}$  is said to have the **Z-property** [33] or the *cross-negativity* property [29] with respect to  $\mathbb{K}$  if the following implication holds:

$$(1.4) \quad x \in \mathbb{K}, \quad y \in \mathbb{K}^* \quad \text{and} \quad \langle x, y \rangle = 0 \implies \langle \mathcal{A}(x), y \rangle \leq 0,$$

and we call  $\mathcal{A}$  a **Z-transformation** w.r.t. (an acronym for “with respect to”)  $\mathbb{K}$ .

## 2. PRELIMINARIES

We first recall from [11] the definition of Jordan product. The Jordan product of any two vectors  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  is defined as

$$x \circ y := (\langle x, y \rangle, x_1 y_2 + y_1 x_2),$$

and write  $x^2 := x \circ x$ . The Jordan product, unlike the matrix multiplication, is not associative in general. The identity element under this product is  $e_n := (1, 0, \dots, 0) \in \mathbb{R}^n$ , i.e.,  $e_n \circ x = x$  for any  $x \in \mathbb{R}^n$ . For any  $a \in \mathbb{R}^n$ , we denote  $\mathcal{L}_a$  and  $\mathcal{S}_a$  by the *Lyapunov transformation* and the *Stein transformation* on  $\mathbb{R}^n$  associated with  $a$ , respectively, given by

$$\mathcal{L}_a(x) := a \circ x \quad \forall x \in \mathbb{R}^n$$

and

$$\mathcal{S}_a(x) := x + a^2 \circ x - 2a \circ (a \circ x) \quad \forall x \in \mathbb{R}^n.$$

The two linear transformations are symmetric with respect to the inner product  $\langle \cdot, \cdot \rangle$ . It is trivial to verify that  $\mathcal{L}_a$  is a **Z-transformation** w.r.t. the cone  $K^n$ . Observing that  $\mathcal{S}_a(x) = x - \mathcal{Q}_a(x)$  where  $\mathcal{Q}_a := 2(\mathcal{L}_a)^2 - \mathcal{L}_{a^2}$  is the *quadratic representation* associated with  $a$ , and  $\mathcal{Q}_a(K^n) \subseteq K^n$ , we also have that  $\mathcal{S}_a$  is a **Z-transformation**

w.r.t.  $K^n$ .

We also recall from [11] that each  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  has a spectral factorization

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)},$$

where  $\lambda_i(x)$  for  $i = 1, 2$  are the spectral values of  $x$ , and  $u_x^{(i)}$  for  $i = 1, 2$  are the associated spectral vectors, respectively, defined by

$$\lambda_i(x) := x_1 + (-1)^i \|x_2\|, \quad u_x^{(i)} := \frac{1}{2} (1, (-1)^i \bar{x}_2),$$

with  $\bar{x}_2 = \frac{x_2}{\|x_2\|}$  if  $x_2 \neq 0$  and otherwise  $\bar{x}_2$  being an arbitrary vector in  $\mathbb{R}^{n-1}$  with  $\|\bar{x}_2\| = 1$ . If  $x_2 \neq 0$ , the factorization is unique. The trace of  $x$  is defined as  $\text{tr}(x) := \lambda_1(x) + \lambda_2(x)$ . The spectral values of  $x$  and the eigenvalues of  $\mathcal{L}_x$  have the following relationship.

**Lemma 2.1** ([9]). *For any given  $x \in \mathbb{R}^n$ , let  $\lambda_1(x), \lambda_2(x)$  be the spectral values of  $x$ , and  $u_x^{(1)}, u_x^{(2)}$  be the associated spectral vectors. Then,  $\mathcal{L}_x$  has the eigenvalue decomposition*

$$\mathcal{L}_x = U(x) \text{diag}(\lambda_2(x), x_1, \dots, x_1, \lambda_1(x)) U(x)^T$$

where

$$U(x) = \left[ \sqrt{2}u_x^{(2)}; u_x^{(3)}; \dots, u_x^{(n)}; \sqrt{2}u_x^{(1)} \right] \in \mathbb{R}^{n \times n}$$

is an orthogonal matrix, and  $u_x^{(i)}$  for  $i = 3, \dots, n$  have the form of  $(0, \bar{u}_x^i)$  with  $\bar{u}_x^3, \dots, \bar{u}_x^n$  being any unit vectors in  $\mathbb{R}^{n-1}$  that span the linear subspace orthogonal to  $x_2$ .

With the spectral factorization of  $x$ , for any given real-valued function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , we may define a vector-valued function  $f^{\text{soc}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$f^{\text{soc}}(x) := f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}.$$

For example, letting  $f(t) \equiv \max\{0, t\}$ ,  $f^{\text{soc}}(x)$  is the metric projection of  $x$  onto  $K^n$ , written as  $\Pi_{K^n}(x)$  or  $(x)_+$ . The following lemma recalls the  $B$ -subdifferential of  $\Pi_{K^n}(\cdot)$  at a general point. For the definition of the  $B$ -subdifferential, the reader may refer to [6].

**Lemma 2.2** ([21, 22]). *Given a general point  $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , each element  $V \in \partial_B \Pi_{K^n}(z)$  has the following representation:*

(a): *If  $z_1 \neq \pm \|z_2\|$ , then  $\Pi_{K^n}(z)$  is continuously differentiable at  $z$  with*

$$V = \Pi'_{K^n}(z) = \begin{cases} 0 & \text{if } z_1 < -\|z_2\|, \\ I & \text{if } z_1 > \|z_2\|, \\ \frac{1}{2} \begin{bmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & \bar{Z} \end{bmatrix} & \text{if } -\|z_2\| < z_1 < \|z_2\|, \end{cases}$$

where

$$\bar{z}_2 = \frac{z_2}{\|z_2\|}, \quad \bar{Z} := \left( \frac{z_1}{\|z_2\|} + 1 \right) I - \frac{z_1}{\|z_2\|} \bar{z}_2 \bar{z}_2^T.$$

(b): If  $z_2 \neq 0$  and  $z_1 = \|z_2\|$ , then

$$V \in \left\{ I, \frac{1}{2} \begin{bmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & \bar{Z} \end{bmatrix} \right\}, \quad \text{where } \bar{z}_2 := \frac{z_2}{\|z_2\|} \text{ and } \bar{Z} := 2I - \bar{z}_2 \bar{z}_2^T.$$

(c): If  $z_2 \neq 0$  and  $z_1 = -\|z_2\|$ , then

$$V \in \left\{ 0, \frac{1}{2} \begin{bmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & \bar{Z} \end{bmatrix} \right\}, \quad \text{where } \bar{z}_2 := \frac{z_2}{\|z_2\|} \text{ and } \bar{Z} := \bar{z}_2 \bar{z}_2^T.$$

(d): If  $z = 0$ , then either  $V = 0$ ,  $V = I$ , or  $V$  belongs to the set

$$\left\{ \frac{1}{2} \begin{bmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & \bar{Z} \end{bmatrix} \mid \bar{Z} = (z_0 + 1)I - z_0 \bar{z}_2 \bar{z}_2^T \text{ for some } |z_0| \leq 1 \text{ and } \|\bar{z}_2\| = 1 \right\}.$$

To close this section, we present some properties regarding the cone spectrum  $\sigma(\mathcal{A}, K^n)$ .

**Lemma 2.3.** *Let  $\mathcal{A}$  be a linear transformation on  $\mathbb{R}^n$ . Then, the following results hold.*

- (a): *If  $\mathcal{A}$  has the  $\mathbf{Z}$ -property w.r.t.  $K^n$ , then  $\sigma(\mathcal{A}, K^n) = \hat{\sigma}(\mathcal{A}, K^n) \subseteq \sigma(\mathcal{A})$ .*
- (b): *If  $\mathcal{A}$  is symmetric, then  $\sigma(\mathcal{A}, K^n)$  contains at most  $3n - 2$  elements.*
- (c): *If  $\mathcal{A}$  is asymmetric and  $\sigma(\mathcal{A}, K^n)$  is finite, then  $\sigma(\mathcal{A}, K^n)$  has at most  $6n - 5$  elements.*

*Proof.* (a) By the definitions of  $\hat{\sigma}(\mathcal{A}, K^n)$  and  $\sigma(\mathcal{A}, K^n)$ , clearly,  $\hat{\sigma}(\mathcal{A}, K^n) \subseteq \sigma(\mathcal{A}, K^n)$ . Conversely, the proof of [33, Theorem 9] implies that  $\sigma(\mathcal{A}, K^n) \subseteq \hat{\sigma}(\mathcal{A}, K^n)$ . Thus, we have  $\sigma(\mathcal{A}, K^n) = \hat{\sigma}(\mathcal{A}, K^n)$ . By the definition, it is clear that  $\hat{\sigma}(\mathcal{A}, K^n) \subseteq \sigma(\mathcal{A})$ .

(b) For a general linear transformation  $\mathcal{A}$ ,  $\sigma(\mathcal{A}, K^n) = \sigma_{\text{int}}(\mathcal{A}, K^n) \cup \sigma_{\text{bd}}(\mathcal{A}, K^n)$ , where  $\sigma_{\text{int}}(\mathcal{A}, K^n)$  consists of those eigenvalues whose eigenvectors belong to the interior of  $K^n$ , and so  $\sigma_{\text{int}}(\mathcal{A}, K^n) \subseteq \sigma(\mathcal{A})$ , and  $\sigma_{\text{bd}}(\mathcal{A}, K^n)$  consists of those eigenvalues whose eigenvectors are on the boundary of  $K^n$ . Note that  $\sigma_{\text{int}}(\mathcal{A}, K^n)$  contains at most one element when  $\mathcal{A}$  is symmetric since the eigenvectors associated with different eigenvalues are orthogonal. By [26, Corollary 4.5], we know that  $\sigma_{\text{bd}}(\mathcal{A}, K^n)$  has a finite number of elements. Since  $\mathcal{A}$  is symmetric, the cardinality of  $\sigma_{\text{bd}}(\mathcal{A}, K^n)$  equals the total number of solutions of systems S<sub>I</sub>-S<sub>II</sub> in [26, Theorem 4.2], which is at most  $3n - 3$  by the analysis of [26, Corollary 4.4]. Thus,  $\sigma(\mathcal{A}, K^n)$  contains at most  $3n - 2$  elements.

(c) In this case,  $\sigma_{\text{int}}(\mathcal{A}, K^n)$  contains at most  $n$  elements, and the cardinality of  $\sigma_{\text{bd}}(\mathcal{A}, K^n)$  equals the total number of solutions to systems S<sub>I</sub>-S<sub>III</sub> in [26, Theorem 4.2], which is at most  $5n - 5$  by the analysis of [26, Corollary 4.4]. Then, the desired result follows by the same arguments as in part (b).  $\square$

### 3. SEMISMOOTH NEWTON METHODS

For convenience, in the rest of this paper, a linear transformation  $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  will be identified with an  $n \times n$  real matrix  $A$ . We assume  $K = K^n$  throughout this section, and all analysis can be carried over to the case where  $K$  has a structure as in (1.3).

Note that problem (1.2) is equivalent to finding a zero  $(\lambda^*, x^*)$  of the following system

$$(3.1) \quad \begin{cases} x \in K^n, Ax - \lambda x \in K^n, \langle x, Ax - \lambda x \rangle = 0, \\ \varphi(x) = 1, \end{cases}$$

where  $\varphi: K^n \rightarrow \mathbb{R}$  is a normalizing function for the Lorentz cone  $K^n$ . A normalizing function for a closed convex cone  $\mathbb{K}$  is a continuous function  $\varphi: \mathbb{K} \rightarrow \mathbb{R}$  satisfying

- $\varphi(x) > 0$  for all nonzero vector  $x \in \mathbb{K}$ ,
- $\varphi(tx) = t\varphi(x)$  for all  $t > 0$  and  $x \in \mathbb{K}$ ,
- $\mathbb{K}_\varphi := \{x \in \mathbb{K} : \varphi(x) = 1\}$  is compact.

For the cone  $K^n$ , both  $\varphi_1(x) = \text{tr}(x)$  and  $\varphi_2(x) = \|x\|$  are normalizing functions. Recall that a mapping  $\phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be an *SOC-complementarity function* associated with the cone  $K^n$  if it satisfies the following equivalence

$$(3.2) \quad \phi(x, y) = 0 \iff x \in K^n, y \in K^n, \langle x, y \rangle = 0.$$

With such a complementarity function  $\phi$ , we can rewrite (3.1) as a system of equations

$$(3.3) \quad \begin{cases} \phi(x, y) = 0, \\ Ax - \lambda x - y = 0, \\ \varphi(x) - 1 = 0. \end{cases}$$

Two popular choices for  $\phi$  are the natural residual (NR) SOC-complementarity function and the Fischer-Burmeister SOC-complementarity function respectively defined by

$$(3.4) \quad \phi_{\text{NR}}(x, y) := x - (x - y)_+ \quad \forall x, y \in \mathbb{R}^n$$

and

$$(3.5) \quad \phi_{\text{FB}}(x, y) := (x^2 + y^2)^{1/2} - (x + y) \quad \forall x, y \in \mathbb{R}^n,$$

where  $x^{1/2}$  with  $x \in K^n$  is the unique square root of  $x$ , i.e.,  $x^{1/2} \circ x^{1/2} = x$ . The two functions were proved in [25] to have similar favorable properties such as the globally Lipschitz continuity and the strong semismoothness. For the Pareto eigenvalue problem, the computational experience in [1] indicates that the nonsmooth Newton method based on  $\phi_{\text{FB}}$  has better numerical performance than the one based on  $\phi_{\text{NR}}$ . However, for the nonpolyhedral  $K^n$ , we can not expect this result since the  $B$ -subdifferential of  $\phi_{\text{FB}}$  does not enjoy all good properties of that of  $\phi_{\text{NR}}$ ; for example, every element in  $\partial_B \phi_{\text{NR}}(x, y)$  has the form of  $[I - V \ V]$  with  $V$  being an  $n \times n$  symmetric matrix, but every element in  $\partial_B \phi_{\text{FB}}(x, y)$  has the form of  $[U \ V]$  with  $U, V \in \mathbb{R}^{n \times n}$  being asymmetric. In view of this, we in this paper concentrate on the equation formulation involving the NR function  $\phi_{\text{NR}}$ . In addition, due to the linearity of  $\varphi_1$ , we always choose  $\varphi$  as  $\varphi_1$  instead of  $\varphi_2$ . In other words, we focus on the following nonsmooth system

$$(3.6) \quad \Phi(w) = \Phi(x, y, \lambda) := \begin{pmatrix} \phi_{\text{NR}}(x, y) \\ Ax - \lambda x - y \\ \text{tr}(x) - 1 \end{pmatrix} = 0.$$

**Proposition 3.1.** *Let  $\Phi: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  be given by (3.6). Then  $\Phi$  is strongly semismooth and its B-subdifferential at any point  $w = (x, y, \lambda)$  satisfies*

$$\partial_B \Phi(w) \subseteq \left\{ \left[ \begin{array}{ccc|c} U_x & U_y & 0 & \\ A - \lambda I & -I & -x & \\ \hline 2e_n^T & 0 & 0 & \end{array} \right] : [U_x \ U_y] \in \partial_B \phi_{\text{NR}}(x, y) \right\}.$$

Also,  $\Phi$  is Lipschitz continuous on  $\mathbb{S} \times \mathbb{R}^n \times \mathbb{J}$  if  $\mathbb{S} \subseteq \mathbb{R}^n$  and  $\mathbb{J} \subseteq \mathbb{R}$  are bounded.

*Proof.* Since the mapping  $F(x, y, \lambda) = Ax - \lambda x - y$  is twice continuously differentiable in  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ , it is strongly semismooth. Together with the strong semismoothness of  $\phi_{\text{NR}}$  and  $\text{tr}(x) - 1$ , it follows from [13, Theorem 19] that  $\Phi$  is strongly semismooth. The representation for the B-subdifferential of  $\Phi$  at  $w$  is direct by using [6, Prop. 2.6.2(e)] and noting that  $\text{tr}(x) = 2\langle e_n, x \rangle$  for  $x \in \mathbb{R}^n$ . Using the global Lipschitz continuity of  $\phi_{\text{NR}}$ , it is not hard to verify the last part.  $\square$

Note that the system (3.6) has a compact solution set which by Prop. 3.1 implies that the operator  $\Phi$  is always Lipschitz continuous on the set of its roots.

With the NR SOC complementarity function  $\phi_{\text{NR}}$ , we may present an alternative nonsmooth equation formulation of (1.2), which is based on the following lemma. Since this lemma is a special case of [14, Prop. 1.5.9], we omit its proof.

**Lemma 3.2.** *The vector  $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$  is a solution of (1.2) if and only if there exists a vector  $z \in \mathbb{R}^n$  such that  $x = z_+ \neq 0$  and  $Az_+ - (1 + \lambda)z_+ + z = 0$ .*

Lemma 3.2 means that the solution of (1.2) can be obtained by solving the nonsmooth system

$$(3.7) \quad \Psi(\omega) = \Psi(z, \lambda) := \begin{pmatrix} Az_+ - (1 + \lambda)z_+ + z \\ \text{tr}(z) - 1 \end{pmatrix} = 0$$

in the sense that if  $(\lambda^*, z^*)$  is a root point of (3.7), then  $(\lambda^*, z_+^*)$  is a solution to (1.2), whereas if  $(\lambda^*, x^*)$  is a solution to (1.2), then  $(\lambda^*, (1 + \lambda^*)x^* - Ax^*)$  is a root of (3.7). The equation  $\text{tr}(z) = 1$  in (3.7) is used to guarantee that  $z_+$  is a nonzero vector. The system (3.7) has an advantage over (3.6) that its dimension is same as that of the original problem.

**Proposition 3.3.** *Let  $\Psi: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}$  be defined by (3.7). Then  $\Psi$  is strongly semismooth and its B-subdifferential at any point  $\omega = (z, \lambda)$  satisfies*

$$\partial_B \Psi(\omega) \subseteq \left\{ \left[ \begin{array}{cc|c} (I - V) + (A - \lambda I)V & -z_+ & \\ \hline 2e_n^T & 0 & \end{array} \right] : V \in \partial_B \Pi_K(z) \right\}.$$

Also,  $\Psi$  is Lipschitz continuous in  $\mathbb{S} \times \mathbb{J}$  if  $\mathbb{S} \subseteq \mathbb{R}^n$  and  $\mathbb{J} \subseteq \mathbb{R}$  are bounded.

*Proof.* Note that  $Az_+ - (1 + \lambda)z_+$  is strongly semismooth by [13, Theorem 19] since it is a composition of the twice continuously differentiable function  $Ax - (1 + \lambda)x$  and the strongly semismooth function  $\Pi_K(z)$ . The first part then follows by [13, Theorem 19]. The second part is direct by the expression of  $\Psi$  and [6, Prop. 2.6.2(e)], and the last part can be easily verified by using the global Lipschitz continuity of  $\Pi_K(z)$ .  $\square$

It is not difficult to verify that the system (3.7) has a compact solution set. Hence, from Prop. 3.3,  $\Psi$  is always Lipschitz continuous on the set of its roots.

In view of Props. 3.1 and 3.3, we may apply the nonsmooth Newton method [30, 31, 32] for the strongly semismooth systems (3.6) and (3.7), which have the following iterations:

$$(3.8) \quad w^{k+1} := w^k - W_k^{-1}\Phi(w^k), \quad W_k \in \partial_B\Phi(w^k), \quad k = 0, 1, 2, \dots,$$

and

$$(3.9) \quad \omega^{k+1} := \omega^k - M_k^{-1}\Psi(\omega^k), \quad M_k \in \partial_B\Psi(\omega^k), \quad k = 0, 1, 2, \dots$$

In the next section, we show that the two methods are quadratically convergent under suitable conditions if the starting points  $w^0$  and  $\omega^0$  are chosen to be sufficiently close to  $(x^*, y^*, \lambda^*)$  and  $((1 + \lambda^*)x^* - Ax^*, \lambda^*)$ , respectively, where  $(\lambda^*, x^*)$  is a solution of (1.2).

#### 4. CONVERGENCE RESULTS

To obtain the local convergence results of the methods (3.8)-(3.9), the key is to establish the nonsingularity of the  $B$ -subdifferential of  $\Phi$  and  $\Psi$  at their respective zeros. In general, there are two ways to achieve this goal. One is to exploit the algebraic technique such as in [21], and the other is to use the perturbation analysis technique such as in [3, 34]. In this section, we make use of the algebraic technique as in [21] to prove the nonsingularity of  $\partial_B\Phi(w^*)$  and  $\partial_B\Psi(\omega^*)$  under suitable conditions, where  $w^*$  and  $\omega^*$  are the solution of (3.6) and (3.7), respectively. Since the nonsmooth systems (3.6) and (3.7) are different from the one studied in [21], the results of this section cannot be obtained directly from [21].

Let  $(\lambda^*, x^*) \in \mathbb{R} \times \mathbb{R}^n$  be an arbitrary solution of (1.2) and  $y^* \equiv Ax^* - \lambda^*x^*$ . Write  $x^* = (x_1^*, \dots, x_r^*)$  and  $y^* = (y_1^*, \dots, y_r^*)$  with  $x_i^*, y_i^* \in \mathbb{R}^{n_i}$ . We first study the nonsingularity of the  $B$ -subdifferential  $\partial_B\Phi(w^*)$  with  $w^* = (x^*, y^*, \lambda^*)$ . For this purpose, we partition the index set  $\{1, 2, \dots, r\}$  into  $J_I \cup J_B \cup J_0 \cup J_{B0} \cup J_{0B} \cup J_{00}$  with

$$(4.1) \quad \begin{aligned} J_I &:= \{i \in \{1, \dots, r\} \mid x_i^* \in \mathbf{int}K^{n_i}, y_i^* = 0\}, \\ J_B &:= \{i \in \{1, \dots, r\} \mid x_i^* \in \mathbf{bd}K^{n_i}, y_i^* \in \mathbf{bd}K^{n_i}\}, \\ J_0 &:= \{i \in \{1, \dots, r\} \mid x_i^* = 0, y_i^* \in \mathbf{int}K^{n_i}\}, \\ J_{B0} &:= \{i \in \{1, \dots, r\} \mid x_i^* \in \mathbf{bd}^+K^{n_i}, y_i^* = 0\}, \\ J_{0B} &:= \{i \in \{1, \dots, r\} \mid x_i^* = 0, y_i^* \in \mathbf{bd}^+K^{n_i}\}, \\ J_{00} &:= \{i \in \{1, \dots, r\} \mid x_i^* = 0, y_i^* = 0\}, \end{aligned}$$

where  $\mathbf{int}K^{n_i}$  and  $\mathbf{bd}K^{n_i}$  denote the interior and the boundary of  $K^{n_i}$ , respectively, and  $\mathbf{bd}^+K^{n_i} = \mathbf{bd}K^{n_i} \setminus \{0\}$ . From Lemma 2.2 and [21], we readily have the following result, where “ $\star$ ” denotes a real number from the interval  $(0, 1)$ ,  $Q_i = (q_i, \hat{Q}_i, q'_i) \in \mathbb{R}^{n_i \times n_i}$  is an orthogonal matrix,  $\bar{Q}_i = (\hat{Q}_i, q'_i) \in \mathbb{R}^{n_i \times (n_i-1)}$  and  $\tilde{Q}_i = (q_i, \hat{Q}_i) \in \mathbb{R}^{n_i \times (n_i-1)}$ .



**Lemma 4.1.** *Let  $U_i \in \partial_B \Pi_{K^{n_i}}(x_i^* - y_i^*)$  for  $i = 1, \dots, r$ . Then the following results hold.*

- (a): *If  $i \in J_I$ , then  $U_i = I$ ; if  $i \in J_0$ , then  $U_i = 0$ ;*
- (b): *If  $i \in J_B$ , then  $U_i = Q_i D_i Q_i^T$  with  $D_i = \text{diag}(0, \star, \dots, \star, 1)$  and  $Q_i = (q_i, \hat{Q}_i, q'_i)$ ;*
- (c): *If  $i \in J_{B0}$ , then  $U_i = I$ , or  $U_i = Q_i D_i Q_i^T$  with  $D_i = \text{diag}(0, 1, \dots, 1)$  and  $Q_i = (q_i, \bar{Q}_i)$ ;*
- (d): *If  $i \in J_{0B}$ , then  $U_i = 0$ , or  $U_i = Q_i D_i Q_i^T$  with  $D_i = \text{diag}(0, \dots, 0, 1)$  and  $Q_i = (\tilde{Q}_i, q'_i)$ ;*
- (e): *If  $i \in J_{00}$ , then  $U_i = I$ , or  $U_i = 0$ , or  $U_i = Q_i D_i Q_i^T$  with  $D_i$  and  $Q_i$  given by  $D_i = \text{diag}(0, \star, \dots, \star, 1)$  and  $Q_i = (q_i, \hat{Q}_i, q'_i)$ , or by  $D_i = \text{diag}(0, 1, \dots, 1)$  and  $Q_i = (q_i, \bar{Q}_i)$ , or by  $D_i = \text{diag}(0, \dots, 0, 1)$  and  $Q_i = (\tilde{Q}_i, q'_i)$ .*

Based on Lemma 4.1(c)–(e), we further partition the index sets  $J_{B0}$ ,  $J_{0B}$ ,  $J_{00}$  into  $J_{B0} = J_{B0}^1 \cup J_{B0}^2$ ,  $J_{0B} = J_{0B}^1 \cup J_{0B}^2$ , and  $J_{00} = J_{00}^1 \cup J_{00}^2 \cup J_{00}^3 \cup J_{00}^4 \cup J_{00}^5$ , respectively, with

$$\begin{aligned}
J_{B0}^1 &:= \{i \mid U_i = I\}, \quad J_{B0}^2 = J_{B0} \setminus J_{B0}^1; \quad J_{0B}^1 := \{i \mid U_i = 0\}, \quad J_{0B}^2 = J_{0B} \setminus J_{0B}^1; \\
J_{00}^1 &:= \{i \mid U_i = I\}, \quad J_{00}^2 := \{i \mid U_i = 0\}, \\
J_{00}^3 &:= \left\{ i \mid U_i = Q_i D_i Q_i^T \text{ with } D_i = \text{diag}(0, \star, \dots, \star, 1) \text{ and } Q_i = (q_i, \hat{Q}_i, q'_i) \right\}, \\
(4.2) \quad J_{00}^4 &:= \left\{ i \mid U_i = Q_i D_i Q_i^T \text{ with } D_i = \text{diag}(0, 1, \dots, 1) \text{ and } Q_i = (q_i, \bar{Q}_i) \right\}, \\
J_{00}^5 &:= \left\{ i \mid U_i = Q_i D_i Q_i^T \text{ with } D_i = \text{diag}(0, \dots, 0, 1) \text{ and } Q_i = (\tilde{Q}_i, q'_i) \right\}.
\end{aligned}$$

**Proposition 4.2.** *Let  $B \in \mathbb{R}^{n \times n}$  and  $u, v \in \mathbb{R}^n$ . Suppose that  $U^a, U^b \in \mathbb{R}^{n \times n}$  are two symmetric positive semidefinite matrices such that their sum  $U^a + U^b$  is positive definite and  $U^a, U^b$  have a common basis of eigenvectors, so that there exist an orthogonal matrix  $Q \in \mathbb{R}^{n \times n}$  and diagonal matrices  $D^a = \text{diag}(a_1, \dots, a_n), D^b = \text{diag}(b_1, \dots, b_n)$  satisfying  $U^a = Q D^a Q^T, U^b = Q D^b Q^T$  and  $a_j \geq 0, b_j \geq 0, a_j + b_j > 0$  for all  $j = 1, 2, \dots, n$ . Let the index set  $\{1, 2, \dots, n\}$  be partitioned as  $\{1, 2, \dots, n\} = \alpha \cup \beta \cup \gamma$  with*

$$\alpha := \{j \mid a_j > 0, b_j = 0\}, \quad \beta := \{j \mid a_j > 0, b_j > 0\}, \quad \gamma := \{j \mid a_j = 0, b_j > 0\},$$

and let  $Q_\alpha, Q_\beta$  and  $Q_\gamma$  denote the submatrices of  $Q$  consisting of the columns from  $Q$  corresponding to the index sets  $\alpha, \beta$  and  $\gamma$ , respectively. Denote

$$\tilde{B}_{\beta\beta} = Q_\beta^T B Q_\beta, \quad \tilde{B}_{\beta\gamma} = Q_\beta^T B Q_\gamma, \quad \tilde{B}_{\gamma\gamma} = Q_\gamma^T B Q_\gamma, \quad \tilde{B}_{\gamma\beta} = Q_\gamma^T B Q_\beta,$$

and

$$\tilde{u}_\beta = Q_\beta^T u, \quad \tilde{u}_\gamma = Q_\gamma^T u, \quad \tilde{v}_\beta = Q_\beta^T v, \quad \tilde{v}_\gamma = Q_\gamma^T v.$$

Assume that the following two conditions hold:

- (a): *The matrix  $\tilde{B}_{\gamma\gamma}$  is nonsingular and  $\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta}$  is positive semidefinite;*
- (b):  *$\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma \neq 0$  and the matrix  $\frac{[\tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma - \tilde{v}_\beta][\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} - \tilde{u}_\beta^T]}{\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma}$  is positive semidefinite.*

Then the matrix  $W = \begin{bmatrix} U^a & U^b & 0 \\ B & -I & -v \\ 2u^T & 0 & 0 \end{bmatrix}$  is nonsingular.

*Proof.* By the expression of  $W$ , it is easy to verify that  $W$  is nonsingular if and only if

$$\widetilde{W} = \begin{bmatrix} D^a & D^b & 0 \\ Q^T B Q & -I & -Q^T v \\ 2u^T Q & 0 & 0 \end{bmatrix}$$

is nonsingular. Let  $\Delta w = (\Delta x, \Delta y, \Delta \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  satisfy  $\widetilde{W} \Delta w = 0$ . Then,

$$(4.3) \quad D^a \Delta x + D^b \Delta y = 0,$$

$$(4.4) \quad (Q^T B Q) \Delta x - \Delta y - (Q^T v) \Delta \lambda = 0,$$

$$(4.5) \quad (u^T Q) \Delta x = 0.$$

From equation (4.3) and the definitions of  $D^a$  and  $D^b$ , it is not hard to obtain

$$(4.6) \quad \Delta x_\alpha = 0, \quad \Delta y_\gamma = 0, \quad D_\beta^a \Delta x_\beta + D_\beta^b \Delta y_\beta = 0,$$

where  $\Delta x = (\Delta x_\alpha, \Delta x_\beta, \Delta x_\gamma)$  and  $\Delta y = (\Delta y_\alpha, \Delta y_\beta, \Delta y_\gamma)$ . We next argue that the given assumptions (a) and (b) imply  $\Delta x_\beta = 0$ , and  $\Delta y_\beta = 0$  then follows from the third equality of (4.6). Assume that  $\Delta x_\beta \neq 0$ . Then, on the one hand, since  $D_\beta^a$  and  $D_\beta^b$  are diagonal and positive definite, the third equality of (4.6) implies

$$(4.7) \quad \langle \Delta x_\beta, \Delta y_\beta \rangle = -\Delta x_\beta^T (D_\beta^b)^{-1} D_\beta^a \Delta x_\beta < 0.$$

On the other hand, by  $\Delta x_\alpha = 0$  and  $\Delta y_\gamma = 0$ , equation (4.4) can be written as

$$(4.8) \quad \begin{aligned} Q_\alpha^T B Q_\beta \Delta x_\beta + Q_\alpha^T B Q_\gamma \Delta x_\gamma - \Delta y_\alpha - (Q_\alpha^T v) \Delta \lambda &= 0, \\ Q_\beta^T B Q_\beta \Delta x_\beta + Q_\beta^T B Q_\gamma \Delta x_\gamma - \Delta y_\beta - (Q_\beta^T v) \Delta \lambda &= 0, \\ Q_\gamma^T B Q_\beta \Delta x_\beta + Q_\gamma^T B Q_\gamma \Delta x_\gamma - (Q_\gamma^T v) \Delta \lambda &= 0. \end{aligned}$$

From the second equality in (4.8) and the notations in the proposition, it follows that

$$\langle \Delta x_\beta, \Delta y_\beta \rangle = \Delta x_\beta^T \widetilde{B}_{\beta\beta} \Delta x_\beta + \Delta x_\beta^T \widetilde{B}_{\beta\gamma} \Delta x_\gamma - \Delta x_\beta^T \widetilde{v}_\beta \Delta \lambda.$$

Since  $\widetilde{B}_{\gamma\gamma} = Q_\gamma^T B Q_\gamma$  is nonsingular, from the third equation of (4.8), we have

$$(4.9) \quad \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{B}_{\gamma\beta} \Delta x_\beta + \Delta x_\gamma - \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{v}_\gamma \Delta \lambda = 0.$$

Combining the last two equations gives

$$(4.10) \quad \begin{aligned} \langle \Delta x_\beta, \Delta y_\beta \rangle &= \Delta x_\beta^T \left[ \widetilde{B}_{\beta\beta} - \widetilde{B}_{\beta\gamma} \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{B}_{\gamma\beta} \right] \Delta x_\beta \\ &\quad + \Delta x_\beta^T \left[ \widetilde{B}_{\beta\gamma} \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{v}_\gamma - \widetilde{v}_\beta \right] \Delta \lambda. \end{aligned}$$

In addition, from equation (4.9) and  $(u^T Q) \Delta x = \widetilde{u}_\beta^T \Delta x_\beta + \widetilde{u}_\gamma^T \Delta x_\gamma = 0$ , we obtain

$$\widetilde{u}_\gamma^T \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{v}_\gamma \Delta \lambda = \widetilde{u}_\gamma^T \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{B}_{\gamma\beta} \Delta x_\beta - \widetilde{u}_\beta^T \Delta x_\beta,$$

which together with  $\widetilde{u}_\gamma^T \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{v}_\gamma \neq 0$  yields

$$(4.11) \quad \Delta \lambda = \left( \widetilde{u}_\gamma^T \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{v}_\gamma \right)^{-1} \left[ \widetilde{u}_\gamma^T \widetilde{B}_{\gamma\gamma}^{-1} \widetilde{B}_{\gamma\beta} - \widetilde{u}_\beta^T \right] \Delta x_\beta.$$

Substituting this into equation (4.10) and using assumptions (a) and (b) lead to

$$\begin{aligned} \langle \Delta x_\beta, \Delta y_\beta \rangle &= \Delta x_\beta^T \left[ \tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} \right] \Delta x_\beta \\ &\quad + \left( \tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma \right)^{-1} \Delta x_\beta^T \left[ \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma - \tilde{v}_\beta \right] \left[ \tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} - \tilde{u}_\beta^T \right] \Delta x_\beta \\ &\geq 0. \end{aligned}$$

This clearly contradicts inequality (4.7). Thus, we prove  $\Delta x_\beta = 0$  and  $\Delta y_\beta = 0$ , and  $\Delta \lambda = 0$  then follows from (4.11). Together with (4.8), we readily get  $\Delta x_\gamma = 0$  and  $\Delta y_\alpha = 0$ . Consequently,  $\Delta w = (\Delta x, \Delta y, \Delta \lambda) = 0$ . The proof is complete.  $\square$

**Remark 4.3.** When the vectors  $u$  and  $v$  satisfy  $(\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} - \tilde{u}_\beta^T)(\tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma - \tilde{v}_\beta) = 0$ , the assumptions (a) and (b) in Prop. 4.2 can be replaced by (a) and  $\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma \neq 0$ . When  $B$  is symmetric, by [16, Corollary 6.3.4], the assumptions (a) and (b) can be replaced by

(c): The matrix  $\tilde{B}_{\gamma\gamma}$  is nonsingular,  $\tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma \neq 0$  and

$$\lambda_{\min} \left( \tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} \right) \geq \frac{\| \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma - \tilde{v}_\beta \| \| \tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} - \tilde{u}_\beta^T \|}{| \tilde{u}_\gamma^T \tilde{B}_{\gamma\gamma}^{-1} \tilde{v}_\gamma |}.$$

In fact, if  $B$  is symmetric positive definite, the assumption (a) automatically holds.

Now applying Lemma 4.1 and Prop. 4.2, we establish the nonsingularity of  $\partial_B \Phi(w^*)$ .

**Theorem 4.4.** Let  $(\lambda^*, x^*)$  be a solution of (1.2),  $B = A - \lambda^* I$  and  $e = (e_1, \dots, e_r)$ . Let the (block) index sets  $J_I, J_B, J_0, J_{B0}, J_{0B}, J_{00}$  be given by (4.1). Suppose that for any partitioning  $J_{B0} = J_{B0}^1 \cup J_{B0}^2$ ,  $J_{0B} = J_{0B}^1 \cup J_{0B}^2$ , and  $J_{00} = J_{00}^1 \cup J_{00}^2 \cup J_{00}^3 \cup J_{00}^4 \cup J_{00}^5$  given as in (4.2), the two conditions in Proposition 4.2 hold with

$$\begin{aligned} \tilde{B}_{\beta\beta} &= Q_\beta^T B Q_\beta, \quad \tilde{B}_{\beta\gamma} = Q_\beta^T B Q_\gamma, \quad \tilde{B}_{\gamma\gamma} = Q_\gamma^T B Q_\gamma, \quad \tilde{B}_{\gamma\beta} = Q_\gamma^T B Q_\beta, \\ \tilde{u}_\beta &= Q_\beta^T e, \quad \tilde{u}_\gamma = Q_\gamma^T e, \quad \tilde{v}_\beta = Q_\beta^T x^*, \quad \tilde{v}_\gamma = Q_\gamma^T x^*, \end{aligned}$$

where  $Q_\gamma = (I(i \in J_I \cup J_{B0}^1 \cup J_{00}^1) \quad q'_i(i \in J_B \cup J_{0B}^2 \cup J_{00}^3 \cup J_{00}^5) \quad \bar{Q}_i(i \in J_{B0}^2 \cup J_{00}^4))$  and  $Q_\beta = (\hat{Q}_i(i \in J_B \cup J_{00}^3))$ . Then, all matrices  $W \in \partial_B \Phi(w^*)$  are nonsingular.

*Proof.* Choose  $W \in \partial_B \Phi(w^*)$  arbitrarily. Then, by Prop. 3.1 and the expression of  $\phi_{\text{NR}}$ ,

$$W = \begin{bmatrix} I - U & U & 0 \\ B & -I & -x^* \\ 2e^T & 0 & 0 \end{bmatrix}$$

for a suitable block diagonal matrix  $U = \text{diag}(U_1, \dots, U_r)$  with  $U_i \in \partial_B \Pi_{K^{n_i}}(x_i^* - y_i^*)$ . By Lemma 4.1, clearly,  $I - U$  and  $U$  are symmetric positive semidefinite and their sum equals  $I$ . Thus, to apply Prop. 4.2, it suffices to identify the index sets  $\beta$  and  $\gamma$ . From Lemma 4.1 and the partitions of  $J_{B0}$ ,  $J_{0B}$  and  $J_{00}$ , we see that the following indices  $j$  belong to the index set  $\beta$  in Prop. 4.2:

- All middle indices belonging to a block index  $i \in J_B \cup J_{00}^3$ , with  $\hat{Q}_i$  consisting of the middle  $n_i - 2$  columns of the corresponding orthogonal matrix  $Q_i$ ;

and the following indices  $j$  belong to the index set  $\gamma$  in Prop. 4.2:

- All indices  $j$  belonging to one of the block indices  $i \in J_I \cup J_{B0}^1 \cup J_{00}^1$ . The corresponding orthogonal matrix is  $Q_i = I$ .
- The last index of each block index  $i \in J_B \cup J_{0B}^2 \cup J_{00}^3 \cup J_{00}^5$ , with  $q'_i$  being the last column of the corresponding orthogonal matrix  $Q_i$ ;
- The last  $n_i - 1$  indices  $j$  belonging to a block index  $i \in J_{B0}^2 \cup J_{00}^4$ , with  $\bar{Q}_i$  consisting of the last  $n_i - 1$  columns of the corresponding orthogonal matrix  $Q_i$ .

Using these identifications and Prop. 4.2, the desired result readily follows.  $\square$

Next, we study the nonsingularity of B-subdifferential  $\partial_B \Psi(\omega^*)$  with  $\omega^* = (z^*, \lambda^*)$ , where  $z^* = (1 + \lambda^*)x^* - Ax^*$  and  $(\lambda^*, x^*)$  be an arbitrary solution of (1.2). Let  $y^* \equiv Ax^* - \lambda^*x^*$ . Then, by noting that  $z^* = (z_1^*, \dots, z_r^*)$  with  $z_i^* \in \mathbb{R}^{n_i}$ ,

$$z^* = x^* - (Ax^* - \lambda^*x^*) = x^* - y^* \text{ and } (z^*)_+ = x^* = [x^* - (Ax^* - \lambda^*x^*)]_+,$$

it is not difficult to deduce from the definition of index sets in (4.1) that

$$(4.12) \quad \begin{aligned} J_I &= \{i \mid z_i^* \in \mathbf{int}K^{n_i}\}, \\ J_0 &= \{i \mid z_i^* \in \mathbf{int}(-K^{n_i})\}, \\ J_B &= \{i \mid z_i^* \notin K^{n_i} \cup (-K^{n_i})\}, \\ J_{B0} &= \{i \mid z_i^* \in \mathbf{bd}^+K^{n_i}\}, \\ J_{0B} &= \{i \mid z_i^* \in \mathbf{bd}^+(-K^{n_i})\}, \\ J_{00} &= \{i \mid z_i^* = 0\}. \end{aligned}$$

The following proposition plays a key role in establishing the nonsingularity of  $\partial_B \Psi(\omega^*)$ .

**Proposition 4.5.** *Let  $B \in \mathbb{R}^{n \times n}$  and  $u, v \in \mathbb{R}^n$ . Suppose that  $V^a, V^b \in \mathbb{R}^{n \times n}$  are two symmetric positive semidefinite matrices such that their sum  $V^a + V^b$  is positive definite and  $V^a, V^b$  have a common basis of eigenvectors, so that there exist an orthogonal matrix  $Q \in \mathbb{R}^{n \times n}$  and diagonal matrices  $D^a = \text{diag}(a_1, \dots, a_n), D^b = \text{diag}(b_1, \dots, b_n)$  satisfying  $V^a = QD^aQ^T, V^b = QD^bQ^T$  and  $a_j \geq 0, b_j \geq 0, a_j + b_j > 0$  for all  $j = 1, 2, \dots, n$ . Let the index set  $\{1, 2, \dots, n\}$  be partitioned as  $\{1, 2, \dots, n\} = \alpha \cup \beta \cup \gamma$  with*

$$\alpha := \{j \mid a_j > 0, b_j = 0\}, \quad \beta := \{j \mid a_j > 0, b_j > 0\}, \quad \gamma := \{j \mid a_j = 0, b_j > 0\},$$

and let  $Q_\alpha, Q_\beta$  and  $Q_\gamma$  denote the submatrices of  $Q$  consisting of the columns from  $Q$  corresponding to the index sets  $\alpha, \beta$  and  $\gamma$ , respectively. Then, the matrix

$$M = \begin{bmatrix} V^a + BV^b & -v \\ 2u^T & 0 \end{bmatrix}$$

is nonsingular under the following conditions (a) and (b), or the conditions (a') and (b').

- (a): The matrices  $(D_\beta^a + \tilde{B}_{\beta\beta}D_\beta^b)$  and  $[\tilde{B}_{\gamma\gamma} - \tilde{B}_{\gamma\beta}D_\beta^b(D_\beta^a + \tilde{B}_{\beta\beta}D_\beta^b)^{-1}\tilde{B}_{\beta\gamma}]$  are nonsingular.
- (a'): The matrices  $\tilde{B}_{\gamma\gamma}$  and  $[D_\beta^a + (\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma}\tilde{B}_{\gamma\gamma}^{-1}\tilde{B}_{\gamma\beta})D_\beta^b]$  are nonsingular.

**(b):**  $\tilde{u}^T(D^a + Q^T B Q D^b)^{-1} \tilde{v} \neq 0$ .

Here,  $\tilde{B}_{\beta\gamma}, \tilde{B}_{\gamma\beta}, \tilde{B}_{\beta\beta}, \tilde{B}_{\gamma\gamma}$  are defined as in Proposition 4.2, and  $\tilde{u} = Q^T u, \tilde{v} = Q^T v$ . When  $B$  is symmetric,  $M$  is nonsingular under (b) and one of the following conditions:

**(c):**  $\tilde{B}_{\beta\beta}$  is positive semidefinite and  $[\tilde{B}_{\gamma\gamma} - \tilde{B}_{\gamma\beta} D_\beta^b (D_\beta^a + \tilde{B}_{\beta\beta} D_\beta^b)^{-1} \tilde{B}_{\beta\gamma}]$  is nonsingular.

**(c')**: The matrix  $\tilde{B}_{\gamma\gamma}$  is nonsingular and  $[\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta}]$  is positive semidefinite.

*Proof.* By the expression of  $M$ , it is easy to verify that  $M$  is nonsingular if and only if

$$\tilde{M} = \begin{bmatrix} D^a + Q^T B Q D^b & -Q^T v \\ 2u^T Q & 0 \end{bmatrix}$$

is nonsingular. Let  $\Delta\omega = (\Delta z, \Delta\lambda) \in \mathbb{R}^n \times \mathbb{R}$  be such that  $\tilde{M}\Delta\omega = 0$ . Then,

$$(4.13) \quad (D^a + Q^T B Q D^b) \Delta z - (Q^T v) \Delta\lambda = 0,$$

$$(4.14) \quad (u^T Q) \Delta z = 0.$$

We first argue that under assumption (a) or (a'), the matrix  $(D^a + Q^T B Q D^b)$  is nonsingular. Indeed, let  $(D^a + Q^T B Q D^b)\xi = 0$  and write  $\xi = (\xi_\alpha, \xi_\beta, \xi_\gamma)$ . Then, by the definitions of  $D^a$  and  $D^b$ , the equation  $(D^a + Q^T B Q D^b)\xi = 0$  can be rewritten as

$$(4.15) \quad \begin{aligned} D_\alpha^a \Delta\xi_\alpha + Q_\alpha^T B Q_\beta D_\beta^b \Delta\xi_\beta + Q_\alpha^T B Q_\gamma D_\gamma^b \Delta\xi_\gamma &= 0, \\ D_\beta^a \Delta\xi_\beta + Q_\beta^T B Q_\beta D_\beta^b \Delta\xi_\beta + Q_\beta^T B Q_\gamma D_\gamma^b \Delta\xi_\gamma &= 0, \\ Q_\gamma^T B Q_\beta D_\beta^b \Delta\xi_\beta + Q_\gamma^T B Q_\gamma D_\gamma^b \Delta\xi_\gamma &= 0. \end{aligned}$$

If assumption (a) holds, then from the nonsingularity of  $D_\beta^a + \tilde{B}_{\beta\beta} D_\beta^b$  and the second equation of (4.14), it follows that

$$(4.16) \quad \Delta\xi_\beta = - [D_\beta^a + \tilde{B}_{\beta\beta} D_\beta^b]^{-1} \tilde{B}_{\beta\gamma} D_\gamma^b \Delta\xi_\gamma.$$

Substituting it into the third equation of (4.14) yields that

$$[\tilde{B}_{\gamma\gamma} - \tilde{B}_{\gamma\beta} D_\beta^b (D_\beta^a + \tilde{B}_{\beta\beta} D_\beta^b)^{-1} \tilde{B}_{\beta\gamma}] D_\gamma^b \Delta\xi_\gamma = 0.$$

By assumption (a) and the nonsingularity of  $D_\gamma^b$ , we then get  $\Delta\xi_\gamma = 0$ , and  $\Delta\xi_\beta = 0$  then follows from (4.15). Since  $D_\alpha^a$  is symmetric positive definite, we have  $\Delta\xi_\alpha = 0$  from the first equation of (4.14). Thus, we prove that  $(D^a + Q^T B Q D^b)$  is nonsingular under assumption (a). If assumption (a') holds, then from the third equation of (4.14),

$$(4.17) \quad \Delta\xi_\gamma = -(D_\gamma^b)^{-1} B_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta} D_\beta^b \Delta\xi_\beta.$$

Substituting into the second equation of (4.14) and using assumption (a') yields that  $\Delta\xi_\beta = 0$ , and then  $\Delta\xi_\gamma = 0$  follows from (4.16), and  $\Delta\xi_\alpha = 0$  follows from the first equality of (4.14). Thus,  $(D^a + Q^T B Q D^b)$  is also nonsingular under assumption

(a'). Now, from equation (4.12) it follows that  $\Delta z = (D^a + Q^T B Q D^b)^{-1} \tilde{v} \Delta \lambda$ . Together with (4.13) and assumption (b), we get  $\Delta \lambda = 0$ , and so  $\Delta z = 0$  follows. Consequently,  $\tilde{M}$  is nonsingular.

When  $B$  is symmetric, clearly,  $\tilde{B}_{\beta\beta}$  and  $(\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta})$  are symmetric. If  $\tilde{B}_{\beta\beta}$  is positive semidefinite, then by using [16, Theorem 7.6.3] and noting that  $D_\beta^a$  and  $D_\beta^b$  are positive definite diagonals, we have that  $I + \tilde{B}_{\beta\beta} D_\beta^b (D_\beta^a)^{-1}$  is nonsingular, and so is  $(D_\beta^a + \tilde{B}_{\beta\beta} D_\beta^b)$ . This shows that under this case, condition (c) implies condition (a). Similarly, if  $(\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta})$  is positive semidefinite, then  $(D_\beta^a + (\tilde{B}_{\beta\beta} - \tilde{B}_{\beta\gamma} \tilde{B}_{\gamma\gamma}^{-1} \tilde{B}_{\gamma\beta}) D_\beta^b)$  is nonsingular, that is, condition (c') implies condition (a'). The proof is completed.  $\square$

**Theorem 4.6.** *Let  $(\lambda^*, x^*)$  be a solution of (1.2),  $B = A - \lambda^* I$ ,  $z^* = (I - B)x^*$  and  $e = (e_1, \dots, e_r)$ . Let the (block) index sets  $J_I, J_0, J_B, J_{B_0}, J_{0B}, J_{00}$  be given by (4.12). Suppose that for any partitioning  $J_{B_0} = J_{B_0}^1 \cup J_{B_0}^2$ ,  $J_{0B} = J_{0B}^1 \cup J_{0B}^2$ , and  $J_{00} = J_{00}^1 \cup J_{00}^2 \cup J_{00}^3 \cup J_{00}^4 \cup J_{00}^5$  given as in (4.2), the conditions (a)-(b) or (a')-(b) in Prop. 4.5 hold with*

$$\tilde{B}_{\beta\beta} = Q_\beta^T B Q_\beta, \quad \tilde{B}_{\beta\gamma} = Q_\beta^T B Q_\gamma, \quad \tilde{B}_{\gamma\gamma} = Q_\gamma^T B Q_\gamma, \quad \tilde{B}_{\gamma\beta} = Q_\gamma^T B Q_\beta, \quad u = e, \quad v = x^*,$$

where  $Q_\gamma = (I(i \in J_I \cup J_{B_0}^1 \cup J_{00}^1) \quad q'_i(i \in J_B \cup J_{0B}^2 \cup J_{00}^3 \cup J_{00}^5) \quad \bar{Q}_i(i \in J_{B_0}^2 \cup J_{00}^4))$  and  $Q_\beta = (\hat{Q}_i(i \in J_B \cup J_{00}^3))$ . Then, all matrices  $M \in \partial_B \Psi(\omega^*)$  are nonsingular.

If  $A$  is symmetric, the conditions (a) and (a') can be replaced by the conditions (c) and (c'), respectively, of Prop. 4.5 with the above matrices and vectors.

*Proof.* Choose  $M \in \partial_B \Psi(\omega^*)$  arbitrarily. From Prop. 3.3, it follows that

$$M = \begin{bmatrix} (I - V) + BV & -(z^*)_+ \\ 2e^T & 0 \end{bmatrix}$$

for a suitable block diagonal matrix  $V = \text{diag}(V_1, \dots, V_r)$  with  $V_i \in \partial_B \Pi_{K^{n_i}}(z_i^*) = \partial_B \Pi_{K^{n_i}}(x_i^* - y_i^*)$ . By Lemma 4.1, clearly,  $I - V$  and  $V$  are symmetric positive semidefinite and their sum equals  $I$ . To apply Prop. 4.5 with  $u = e$  and  $v = (z^*)_+ = x^*$ , it suffices to identify the index sets  $\beta$  and  $\gamma$ . This is same as the proof of Theorem 4.4.  $\square$

**Remark 4.7.** Comparing Prop. 4.5 with Prop. 4.2, we see that when  $A$  is symmetric, the condition (c') of Prop. 4.5 is same as the condition (a) of Prop. 4.2. This means that the nonsingularity of  $\partial_B \Psi(\omega^*)$  requires a stronger condition than that of  $\partial_B \Phi(w^*)$ . Furthermore, the asymmetry of  $A$  has a remarkable influence on the nonsingularity of  $\partial_B \Psi(\omega^*)$ , and so the successful convergence of the method (3.9), but it does not give an influence on the nonsingularity of  $\partial_B \Phi(w^*)$ .

Now applying Theorems 4.4-4.6 and [31], we immediately obtain the following result.

**Theorem 4.8.** *Let  $(\lambda^*, x^*)$  be an arbitrary solution of (1.2). Let  $z^* = (1 + \lambda^*)x^* - Ax^*$  and  $y^* = Ax^* - \lambda^*x^*$ . If the conditions (a)-(b) in Theorem 4.4 hold for the partitions in (4.2), then the method (3.8) applied to  $\Phi(w) = 0$  is locally quadratically convergent; if the conditions (a)-(b) or (a')-(b) in Theorem 4.6 hold for the*

partitions in (4.2), then the method (3.9) applied to  $\Psi(\omega) = 0$  is locally quadratically convergent.

## 5. NUMERICAL EXPERIMENTS

In this section, we apply the semismooth Newton methods (3.8) and (3.9), abbreviated as the SNM1 and the SNM2, respectively, for solving Lorentz eigenvalue problems. During the testing, we do not adopt any globalization strategy for the two semismooth Newton methods, taking into account that Lorentz eigenvalue problems generally have many solutions and the aim is to seek as many solutions as possible. All numerical experiments were carried out on a PC with a Processor 2.80 GHz Intel Pentium(R) 4 and 512(Mb) memory, and the codes were all written in Matlab 6.5. The two methods were stopped if

$$(5.1) \quad \|\Phi(w^k)\| (\|\Psi(\omega^k)\|) \leq 10^{-8} \quad \text{or} \quad k > k_{max},$$

where  $k_{max}$  denotes the maximum number of iterations allowed for the methods. We tested the Lorentz eigenvalue problems for three classes of linear transformations.

**5.1. Testing on Z-transformations.** The experiment tests the numerical performance of the SNM1 and the SNM2 for solving the Lorentz eigenvalue problems of **Z**-transformations, more specifically, the Lyapunov transformation  $\mathcal{L}_a$  and the Stein transformation  $\mathcal{S}_a$ . During the tests, corresponding to the Cartesian structure of  $K$ , we generated  $\mathcal{L}_a$  and  $\mathcal{S}_a$  in the following way:

$$\mathcal{L}_a = \text{diag}(\mathcal{L}_{a_1}, \mathcal{L}_{a_2}, \dots, \mathcal{L}_{a_r}), \quad \mathcal{S}_a = \text{diag}(\mathcal{S}_{a_1}, \mathcal{S}_{a_2}, \dots, \mathcal{S}_{a_r}),$$

where each  $a_i \in \mathbb{R}^{n_i}$  was generated randomly such that their elements are uniformly distributed in the interval  $[-1, 1]$ . From Lemma 2.3(a), we have that

$$\sigma(\mathcal{L}_{a_i}, K^{n_i}) = \hat{\sigma}(\mathcal{L}_{a_i}, K^{n_i}) \quad \text{and} \quad \sigma(\mathcal{S}_{a_i}, K^{n_i}) = \hat{\sigma}(\mathcal{S}_{a_i}, K^{n_i}), \quad i = 1, 2, \dots, r.$$

Suppose that  $a_i \in \mathbb{R}^{n_i}$  has the spectral decomposition  $a_i = a_{i1}c_{i1} + a_{i2}c_{i2}$ , where  $\{c_{i1}, c_{i2}\}$  is the corresponding Jordan frame. From Lemma 2.1, it is easy to see that

$$\hat{\sigma}(\mathcal{L}_{a_i}, K^{n_i}) = \{a_{i1}, a_{i2}\} \quad \text{and} \quad \hat{\sigma}(\mathcal{S}_{a_i}, K^{n_i}) = \{1 - a_{i1}^2, 1 - a_{i2}^2\}.$$

This means that the spectrums  $\sigma(\mathcal{L}_a, K)$  and  $\sigma(\mathcal{S}_a, K)$  have at most  $2r$  eigenvalues.

Given the transformation  $\mathcal{L}_a$  or  $\mathcal{S}_a$ , we applied the SNM1 for solving system (3.6) with a starting point  $w^0 = (x^0, y^0, \lambda^0)$ , where  $x^0 = (x_1^0, \dots, x_r^0)$  was generated randomly such that the entries of each subvector  $x_i^0$  are uniformly distributed in the interval  $[-1, 1]$ , and

$$(5.2) \quad \lambda^0 = \frac{\langle Ax^0, x^0 \rangle}{\langle x^0, x^0 \rangle}, \quad y^0 = Ax^0 - \lambda^0 x^0;$$

while we used the SNM2 to solve system (3.7) with a starting point  $(z^0, \lambda^0)$ , where  $z^0 = (z_1^0, \dots, z_r^0)$  was generated randomly so that the entries of each subvector  $z_i^0$  are uniformly distributed in the interval  $[-1, 1]$ , and  $\lambda^0$  was given by the following formula

$$(5.3) \quad \lambda^0 = \frac{\langle A(z^0)_+, (z^0)_+ \rangle}{\langle (z^0)_+, (z^0)_+ \rangle}.$$

The maximum number of iterations  $k_{max}$  in (5.1) was chosen as 100 for this experiment.

TABLE 1. The percentage of convergence for Lorentz eigenvalue problems of  $\mathcal{L}_a$

Dimension		SNM1		SNM2		Dimension		SNM1		SNM2	
$r$	$n_r$	%	Aiter	%	Aiter	$r$	$n_r$	%	Aiter	%	Aiter
1	100	97.4	7.4	100	10.4	5	40	100	9.1	100	10.6
1	200	97.5	7.5	100	11.8	10	30	100	9.6	100	10.7
1	300	97.1	7.5	99.9	12.7	10	10	100	9.1	100	8.2
1	500	97.4	7.6	100	14.2	50	6	100	8.7	100	8.7
5	80	100	9.5	100	11.3	100	3	100	6.7	100	6.5

Table 1 reports the percentage of convergence of the SNM1 and the SNM2 estimated by using a sample of  $10^3$  random pairs  $(\mathcal{L}_a, x^0)$  and  $(\mathcal{L}_a, z^0)$ , respectively; and Table 2 reports the percentage of convergence of the two methods estimated by using a sample of  $10^3$  random pairs  $(\mathcal{S}_a, x^0)$  and  $(\mathcal{S}_a, z^0)$ , respectively. In these tables,  $r$  and  $n_r$  denotes the number of Lorentz cones and the dimension of each Lorentz cone, respectively, % column gives the percentage of convergence for  $10^3$  randomly generated pairs, and **Aiter** column lists the average iteration required by those problems with successful convergence.

TABLE 2. The percentage of convergence for Lorentz eigenvalue problems of  $\mathcal{S}_a$

Dimension		SNM1		SNM2		Dimension		SNM1		SNM2	
$r$	$n_r$	%	Aiter	%	Aiter	$r$	$n_r$	%	Aiter	%	Aiter
1	100	71.9	8.5	100	7.8	5	40	99.9	10.7	100	9.2
1	200	70.5	8.5	100	8.0	10	30	100	11.3	100	9.8
1	300	68.1	8.5	100	8.1	10	10	100	9.7	100	8.2
1	500	74.7	8.7	100	8.3	50	6	99.7	8.9	100	8.2
5	80	100	11.2	100	9.2	100	3	100	5.8	100	6.6



From Table 1, we see that for the Lorentz eigenvalue problems involving the Lyapunov transformation, the SNM2 can solve almost all test problems, and the SNM1 has at least 97% successful convergence. From Table 2, for the Lorentz eigenvalue problems involving the Stein transformation, the SNM2 solves all test problems successfully, whereas the SNM1 can solve successfully those problems with  $n_r \leq 80$ , and for the problems with  $n_r \geq 100$ , it has about 70% successful convergence. This indicates that the SNM2 has much more desirable numerical performance than the SNM1 for the Lorentz problems involving a  $\mathbf{Z}$ -transformation, in terms of the percentage of successful convergence and the average number of iterations. Particularly, from these two tables, it seems that the successful convergence of the two methods for this class of Lorentz eigenvalue problems is not influenced by the dimension of problems, and the average number of iterations does not have a remarkable increase with the dimension of Lorentz cones.

**5.2. Testing on general symmetric transformations.** This subsection includes two experiments where the first one tests whether the SNM1 and the SNM2 can seek effectively the eigenvalues of a general linear symmetric transformation relative to  $K = K^n$  with  $n \in \{3, 4, 5, 6, 7, 8, 9\}$ , and the second one tests the percentage of successful convergence with the two methods solving the Lorentz eigenvalue problems involving general linear symmetric transformations.

In the first experiment, corresponding to each  $K^n$ , we used a sample of  $10^3$  matrices  $A = (\tilde{A} + \tilde{A}^T)/2$ , where each  $\tilde{A}$  was generated randomly such that its elements are uniformly distributed in  $[-1, 1]$ . For each random  $A$ , we considered 100 random starting points  $(x^0, y^0, \lambda^0)$  for the SNM1 and 100 random starting points  $(z^0, \lambda^0)$  for the SNM2, respectively, where the entries of  $x^0$  and  $z^0$  are uniformly distributed in  $[-100, 100]$ , and  $\lambda^0$  and  $y^0$  for the SNM1 are given by (5.2), and  $\lambda^0$  for the SNM2 is given by

$$(5.4) \quad \lambda^0 = \langle Az^0, z^0 \rangle / \langle z^0, z^0 \rangle.$$

Since  $(z^0)_+$  may equal zero under this case, we do not employ (5.3) to determine  $\lambda^0$  for the SNM2. The two methods were terminated once one of the conditions in (5.1) was satisfied with  $k_{max} = 100$ . The numerical results are summarized in Table 3, where **Nsol** column denotes the number of eigenvalues, the numbers in each column give the number of problems which were tested to have the corresponding number of eigenvalues from 100 initial points, the numbers in the bracket of each column represent the results for SNM2, % row gives the percentage of successful convergence for  $10^3$  randomly generated pairs, **AN<sub>e</sub>** row gives the average number of eigenvalues found relative to  $K^n$ , calculated by (taking  $n = 3$  for example)  $0.399 + 2 \times 0.002 + 3 \times 0.453 + 4 \times 0.001 + 5 \times 0.145$ , and **AN<sub>s</sub>** row lists the average number of eigenvalues for which the eigenvectors found belong to the boundary of  $K^n$ . Specifically, a eigenvector  $x^* = (x_1^*, x_2^*) \in \mathbb{R} \times \mathbb{R}^{n-1}$  corresponding to  $\lambda^*$  is said to be on the boundary of  $K^n$  if  $|\lambda_1(x^*)| = |x_1^* - \|x_2^*\|| \leq 10^{-8}$ .

From Table 3, it is easy to see that the SNM1 finds successfully at least one eigenvalue for all test problems corresponding to each  $K^n$  from 100 initial points, whereas the SNM2 fails to find a eigenvalue for some test problems when  $n \geq 5$  from 100 initial points. The numbers in **AN<sub>e</sub>** row show that the average number

TABLE 3. The number of eigenvalues involving general linear symmetric transformations

Nsol	Dimension ( $n$ )						
	3	4	5	6	7	8	9
1	399(399)	400(403)	417(421)	457(465)	506(510)	600(576)	659(573)
2	2(8)	6(48)	5(54)	3(61)	1(63)	4(56)	– (42)
3	453(454)	432(397)	400(360)	369(318)	354(291)	283(224)	241(188)
4	1(6)	2(26)	4(27)	6(23)	3(22)	4(22)	3(12)
5	145(133)	140(112)	130(113)	129(100)	103(71)	78(60)	69(47)
6	– (–)	2(7)	5(7)	5(5)	2(7)	– (3)	– (3)
7	– (–)	18(7)	32(15)	28(17)	25(12)	27(13)	19(10)
8	– (–)	– (–)	4(1)	– (–)	– (–)	– (1)	–(1)
9	– (–)	– (–)	3(1)	3(2)	5(1)	4 (–)	6 (–)
10	– (–)	– (–)	– (–)	– (–)	1 (–)	– (–)	– (–)
11	– (–)	– (–)	– (–)	– (–)	– (–)	– (–)	– (1)
12	– (–)	– (–)	– (–)	– (–)	– (–)	– (–)	1 (–)
%	100(100)	100(100)	100(99.9)	100(99.1)	100(97.6)	100(95.5)	100(87.7)
$\mathbf{AN}_e$	2.49(2.47)	2.55(2.45)	2.61(2.47)	2.49(2.30)	2.34(2.09)	2.09(1.86)	1.94(1.61)
$\mathbf{AN}_s$	64.3(64.4)	72.0(73.5)	77.9(80.0)	82.3(83.8)	85.8(86.9)	87.9(88.0)	91.4(91.2)

Note: the notation “–” means that solving  $10^3$  problems from 100 initial points can not yield such amount of eigenvalues.

of eigenvalues found by the two methods decreases when  $n$  becomes large, which means that the gap between the number of Lorentz eigenvalues found in practice and the estimated upper bound  $3n - 2$  becomes larger and larger with  $n$  increasing. The numbers in  $\mathbf{AN}_s$  row indicate that the average number of eigenvectors on the boundary of  $K^n$  increases with  $n$  becoming large, and the SNM2 can find more eigenvectors on the boundary of  $K^n$  than the SNM1.

In the second experiment, we generated the symmetric matrix  $A$  in the following way

$$A = \text{diag}(A_1, A_2, \dots, A_r) \quad \text{with} \quad A_i = \frac{1}{2}(H_i + H_i^T),$$

where each  $H_i \in \mathbb{R}^{n_i \times n_i}$  was generated randomly such that its elements are uniformly distributed in  $[-1, 1]$ . By Lemma 2.3(b),  $\sigma(A, K)$  has at most  $\sum_{i=1}^r (3n_i - 2)$  elements. Given a transformation  $\mathcal{A}(x) = Ax$  with  $A$  generated as above, we applied the SNM1 for solving (3.6) with a starting point  $w^0 = (x^0, y^0, \lambda^0)$ , where  $x^0 = (x_1^0, \dots, x_r^0)$  was generated randomly in the same way as in Subsection 5.1,

TABLE 4. The percentage of convergence for general linear symmetric transformations

Dimension		SNM1		SNM2		Dimension		SNM1		SNM2	
$r$	$n_r$	%	Aiter	%	Aiter	$r$	$n_r$	%	Aiter	%	Aiter
1	50	93	81.2	–	–	5	40	91.4	76.6	1.2	25
1	100	75	116.7	–	–	10	30	91.8	70.3	5.0	23.7
1	150	57.8	124.6	–	–	10	10	84.2	33.6	74.4	20.1
1	200	42.2	134.8	–	–	50	6	88.8	24.9	83.6	18.1
5	80	80.6	109.4	–	–	100	4	91.6	15.3	92.8	13.1

Note: the notation “–” means that the corresponding method fails for all test problems.

and  $\lambda^0$  and  $y^0$  were given by (5.2); and we used the SNM2 to solve (3.7) with a starting point  $(z^0, \lambda^0)$ , where  $z^0 = (z_1^0, \dots, z_r^0)$  was generated in the same way as in Subsection 5.1, and  $\lambda^0$  was given by (5.3). During the tests, we chose  $k_{max}$  as 300. Table 4 reports the percentage of convergence estimated with a sample of 500 random  $(A, x^0)$  and  $(A, z^0)$  for the SNM1 and the SNM2, respectively.

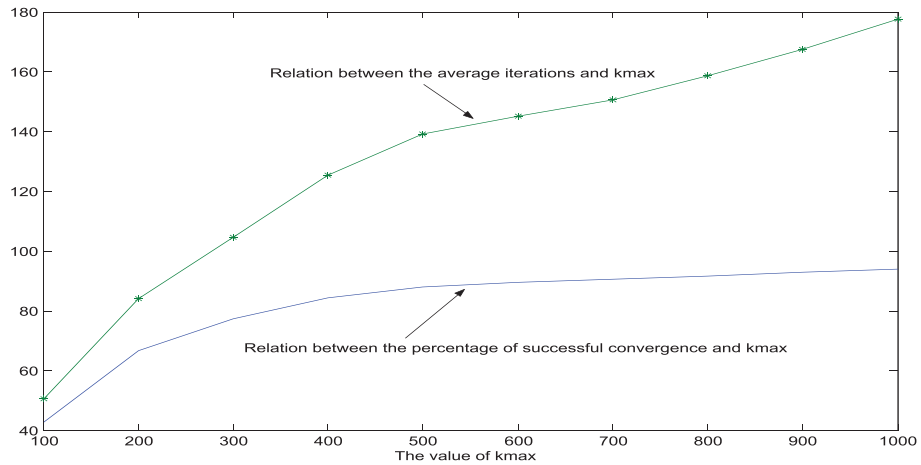


FIGURE 1. The percentage of convergence and the average iteration v.s.  $k_{max}$  for SNM1.

From Table 4, we see that the SNM1 has 75% successful convergence for those test problems with  $n_r \leq 100$ , whereas the SNM2 has an acceptable successful convergence only for those problems with  $n_r \leq 10$ . In addition, the average iterations required by the SNM1 and the SNM2 increase with the value of  $n_r$ . This means that their percentage of successful convergence may be improved by setting  $k_{max}$  to be a larger value. Figure 1 depicts the relation between the percentage of successful convergence of the SNM1 and  $k_{max}$ , as well as the relation between the average

number of iterations and  $k_{max}$ , for the Lorentz eigenvalue problems relative to  $K^{100}$  with  $A$  generated as above. From Figure 1, we see that the percentage of successful convergence is indeed improved by increasing the value of  $k_{max}$ , but the improving ratio is lower than the increasing speed of the average number of iterations. This implies that the performance of the SNM1 for the Lorentz eigenvalue problems of general linear symmetric transformations, in terms of the percentage of successful convergence and the number of iterations, becomes worse when the dimension of problems increases. This phenomenon also appears in the spectral gradient method and the power iteration method for the Pareto eigenvalue problems [4].

**5.3. Testing on asymmetric transformations.** Similar to Subsection 5.2, this subsection includes two experiments where the first one tests whether the SNM1 and the SNM2 can seek effectively the eigenvalues of a general linear asymmetric transformation relative to  $K = K^n$  with  $n \in \{3, 4, 5, 6, 7, 8, 9\}$ , and the second one tests the percentage of successful convergence with the two methods solving the Lorentz eigenvalue problems involving a general linear asymmetric transformation.

TABLE 5. The number of eigenvalues involving linear asymmetric transformations

Nsol	Dimension						
	3	4	5	6	7	8	9
1	681(605)	756(599)	765(544)	791(490)	865(487)	910(441)	916(377)
2	– (69)	– (67)	1(74)	1(75)	–(63)	–(38)	– (39)
3	281(210)	211(141)	210(134)	185(95)	115(38)	79(30)	5(29)
4	–(15)	1(18)	– (9)	–(12)	1(11)	1 (7)	– (5)
5	38(23)	29(13)	23(11)	20(4)	18(7)	9(3)	79 (–)
6	– (–)	1(1)	1(1)	– (–)	1 (–)	– (–)	–(–)
7	– (–)	2(–)	–(–)	3 (–)	– (–)	1 (–)	–(–)
%	100(92.2)	100 (83.9)	100(77.3)	100(67.6)	100 (60.6)	100(51.9)	100 (45.0)
$AN_e$	1.71(1.55)	1.56(1.30)	1.53(1.19)	1.47(0.99)	1.31(0.81)	1.20(0.65)	1.33(0.56)
$AN_s$	72.0(69.1)	80.1(77.1)	85.2(82.0)	88.6 (85.5)	92.0(88.6)	95.8(92.3)	85.3(92.2)

Note: the notation “–” has the same meaning as in Table 3.

In the first experiment, corresponding to each  $K^n$ , we used a sample of  $10^3$  matrices  $A$  with each  $A$  being generated randomly so that its elements are uniformly distributed in  $[-1, 1]$ . For each random  $A$ , we considered 100 random initial points  $(x^0, y^0, \lambda^0)$  for the SNM1, with the entries of  $x^0$  being uniformly distributed in  $[-100, 100]$  and the corresponding  $(\lambda^0, y^0)$  being given by (5.2); while we considered 100 random initial points  $(z^0, \lambda^0)$  for the SNM2, with the entries of  $z^0$  being uniformly distributed in  $[-100, 100]$  and the corresponding  $\lambda^0$  being given by (5.4). In this experiment, the maximum number of iterations  $k_{max}$  was chosen as 100. The

numerical results are reported in Table 5.

From the % row of Table 5, we see that the SNM1 finds successfully at least one eigenvalue for all test problems from 100 initial points, but the SNM2 always fails to finding an eigenvalue for some problems from 100 initial points. The  $\mathbf{AN}_e$  row and  $\mathbf{AN}_s$  row show that the average number of eigenvalues found by the two methods decreases when  $n_r$  becomes large, whereas the number of corresponding eigenvectors on the boundary of  $K^n$  increases. Comparing Table 5 with Table 3, we conclude that the two methods find fewer solutions for the Lorentz eigenvalue problems of a general linear asymmetric transformation, although the estimated upper bound for them is larger than that of the former.

In the second experiment, we consider the test problems with  $A = \text{diag}(A_1, \dots, A_r)$ , where each  $A_i \in \mathbb{R}^{n_i \times n_i}$  was randomly generated such that its elements are uniformly distributed in the interval  $[-1, 1]$ . By Lemma 2.3(c), if the cone spectrum  $\sigma(A_i, K^{n_i})$  is finite, then the maximal cardinality of  $\sigma(A, K)$  is  $\sum_{i=1}^r (6n_i - 5)$ . Given a transformation  $\mathcal{A}(x) = Ax$  with  $A$  generated as above, we applied the SNM1 for solving (3.6) with a starting point  $w^0 = (x^0, y^0, \lambda^0)$ , where  $x^0 = (x_1^0, \dots, x_r^0)$  was generated randomly in the same way as in Subsection 5.1, and  $\lambda^0$  and  $y^0$  were determined by (5.2); and we applied the SNM2 for solving (3.7) with a starting point  $(z^0, \lambda^0)$ , where  $z^0 = (z_1^0, \dots, z_r^0)$  was generated in the same way as in Subsection 5.1, and  $\lambda^0$  was given by (5.3). In this experiment, we chose  $k_{max}$  as 300. Table 6 reports the percentage of convergence estimated by using a sample of 500 random pairs  $(A, x^0)$  and  $(A, z^0)$  for the SNM1 and the SNM2, respectively.

TABLE 6. The percentage of convergence for general linear asymmetric transformations

Dimension		SNM1		SNM2		Dimension		SNM1		SNM2	
$r$	$n_r$	%	Aiter	%	Aiter	$r$	$n_r$	%	Aiter	%	Aiter
1	50	98.2	37.0	–	–	5	40	98.2	34.8	–	–
1	100	97	51.0	–	–	10	50	98.0	38.9	–	–
1	150	94.2	68.3	–	–	10	10	93.2	20.8	–	–
1	200	89.4	76.7	–	–	50	8	95.2	19.9	–	–
5	80	97.8	45.5	–	–	100	4	96.4	12.5	–	–

The notation “–” means that the corresponding method fails for all test problems.

From Table 6, the SNM1 has about 90% successful convergence for all test problems with  $n_r \leq 200$ , whereas the SNM2 fails for all these problems. Also, the performance of the SNM2 can not be improved even if we increase  $k_{max}$  to 500.

Comparing with Table 4, we conclude that the SNM1 has better numerical performance for the Lorentz eigenvalue problems of a general linear asymmetric transformation than those involving a general linear symmetric transformation, though the percentage of successful convergence also decreases and the average iteration increases when  $n_r$  becomes large.

To sum up the numerical experience of last three subsections, we have the conclusions:

- For the Lorentz eigenvalue problems of Lyapunov transformations and Stein transformations, the SNM2 has better numerical performance than the SNM1 by the percentage of successful convergence and the average number of iterations.
- For the Lorentz eigenvalue problems of general linear symmetric transformations, the SNM1 has much better performance than the SNM2 in terms of the percentage of successful convergence, though their successful convergence depends on the maximum number of iterations allowed. Since the improving ratio of the percentage of successful convergence is lower than the increasing speed of the average number of iterations, the two methods are only suitable for this class of small-scaled problems
- For the Lorentz eigenvalue problems of general linear asymmetric transformations, the SNM1 has much better performance than the SNM2 in terms of the percentage of successful convergence and the average number of iterations. Also, it seems a little strange that the SNM1 has better percentage of successful convergence for the Lorentz eigenvalue problems of general linear asymmetric transformations than those of general linear symmetric transformations.

## 6. CONCLUSIONS

We proposed two semismooth Newton methods for seeking the cone spectrum of a linear transformation relative to Lorentz cones, via the natural equation reformulation and the normal equation reformulation of (1.2), respectively. The local quadratic convergence results were established under suitable conditions. Although the local convergence conditions are not easy to verify, the computational experience shows that the two local semismooth methods are very effective for the Lorentz problems of  $\mathbf{Z}$ -transformations, if we do not aim at identifying all Lorentz eigenvalues, and particularly the method based on the normal equation reformulation successfully solves almost all test problems. For the Lorentz eigenvalue problems with a general linear asymmetric transformation (or symmetric transformation), the method based on the natural equation reformulation is promising if the dimension of test problems is within 200 (or 100).

From the numerical results in Tables 3 and 5, we see that there is a big gap between the existing theoretical estimation for the number of Lorentz eigenvalues of a linear symmetric or asymmetric transformations and that obtained from numerical computations. Therefore, the improved theoretical estimation for the number of Lorentz eigenvalues and more effective numerical algorithms are still worthwhile to

explore in the future.

Finally, we want to point out that the two semismooth Newton methods can be easily extended to seek the eigenvalues of a linear transformation relative to a symmetric cone associated with a general Euclidean Jordan algebra, and it is also interesting to study the corresponding semismooth methods for seeking the eigenvalues of a linear transformation relative to positive semidefinite cones.

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