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FIXED POINT THEOREMS FOR GENERALIZED HYBRID DEMICONTINUOUS MAPPINGS IN HILBERT SPACES

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Abstract. In this paper we show fixed point theorems for widely more generalized hybrid demicontinuous self and non-self mappings in Hilbert spaces. Using these fixed point theorems, we can directly show a fixed point theorem for demicontinuous pseudocontractive mappings in Hilbert spaces which was proved by Moloney and Weng [20].

1. Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. Kocourek, Takahashi and Yao [17] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping (α, β)-generalized hybrid. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive [23] for $\alpha = 1$ and $\beta = 0$, that is, $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. It is nonspreading [19] for $\alpha = 2$ and $\beta = 1$, that is, $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. It is also hybrid [24] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, that is, $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. They showed fixed point theorems for such mappings; see also Kohsaka and Takahashi [18] and Iemoto and Takahashi [9]. Moreover they showed mean convergence theorems of Baillon's type. Furthermore they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ (1-\alpha+\gamma)\|x - Ty\|^2 \\ &\leq (\beta + (\beta-\alpha)\gamma)\|Tx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ &\quad + (\alpha-\beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2\end{aligned}$$

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for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the authors [14] also defined a class of nonlinear mappings in a Hilbert space which covers contractive mappings and generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0 \end{aligned}$$

for any $x, y \in C$. Furthermore the authors [15] defined a class of nonlinear mappings in a Hilbert space which covers super hybrid mappings and widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any $x, y \in C$. Then we show fixed point theorems for such new mappings in a Hilbert space. Furthermore we show mean convergence theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly show Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [17] for super hybrid mappings.

Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned} &\alpha(1+\gamma)\|Tx - Ty\|^2 + (1 - \alpha(1+\gamma))\|x - Ty\|^2 \\ &\quad \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\qquad - (\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2 \end{aligned}$$

for any $x, y \in C$. Furthermore they showed a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping. Moreover the authors [13, 11, 12] showed fixed point theorems for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore the authors showed mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

On the other hand, Browder and Petryshyn [5] showed that any Lipschitz continuous pseudocontractive mapping has a fixed point in a closed ball of a Hilbert space. A mapping T from C into H is said to be pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Moreover Moloney and Weng [20] showed a fixed point theorem for demicontinuous pseudocontractive mappings in a Hilbert space.

In this paper we show fixed point theorems for widely more generalized hybrid demicontinuous self and non-self mappings in Hilbert spaces. Such theorems are more general and useful than fixed point theorems for pseudcontractive mappings. Using these fixed point theorems, we can directly show Moloney and Weng's fixed point theorem [20] for demicontinuous pseudocontractive mappings in a Hilbert space.

2. Preliminaries

Let H be an infinite dimensional real Hilbert space and let C be a non-empty bounded close convex subset of H. A mapping T from C into H is demicontinuous if, a sequence $\{Tx_n\}$ is convergent to Tx weakly whenever a sequence $\{x_n\}$ in C is convergent to x. A mapping T from C into H is hemicontinuous if the mapping $t \mapsto T((1-t)x + ty)$ is continuous from $[0, 1]$ into H with its weak topology for any $x, y \in C$. Note that, if a mapping is demicontinuous, then it is hemicontinuous. Then in [20] Moloney and Weng showed that any demicontinuous pseudocontractive mapping has a fixed point.

Theorem 2.1 ([20])**.** *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be a demicontinuous pseudocontractive mapping from C into itself. Then T has a fixed point.*

To show the theorem above, they used the following lemmas.

Lemma 2.2 ([20])**.** *Let H be a real Hilbert space, let B be a non-empty closed ball in H and let T be a demicontinuous mapping from B into B. Then there exists a sequence $\{z_n\}$ in B such that*

(i) $\{Tz_n - z_n\}$ *is convergent to* 0 *weakly;*
(ii) $\langle Tz_m - z_m, z_n\rangle = 0$ *if* $m \geq n$*;*
(iii) $\langle Tz_m - z_m, Tz_n - z_n\rangle = 0$ *if* $m \neq n$.

Remark 2.1. In [20] (ii) is different and the following is used:
(ii) $\langle Tz_n - z_n, z_n\rangle = 0$.
However by the proof of Lemma, see [20, Lemma 1], the Lemma above is correct. In addition, the proof of Theorem 2.1 uses the fact above. Moreover we should note how to create the sequence $\{z_n\}$. If T has a fixed point z, then we can put $z_n = z$ for any n. If not, then $\{z_n\}$ is created as follows. Let M_1 be a 1-dimension subspace of H and let Π_1 be the orthogonal projection of H onto M_1. Then we consider that $\Pi_1 T$ is a mapping from $M_1 \cap B$ into $M_1 \cap B$. Since $\Pi_1 T$ is continuous and $M_1 \cap B$ is compact and convex, by Brouwer's fixed point theorem $\Pi_1 T$ has a fixed point z_1. Moreover, since $\Pi_1(Tz_1 - z_1) = \Pi_1 T z_1 - \Pi_1 z_1 = \Pi_1 T z_1 - z_1 = 0$, we obtain $Tz_1 - z_1 \in M_1^\perp$. Recursively, when an n-dimension subspace M_n of H, the orthogonal projection Π_n of H onto M_n and $z_n \in M_n \cap B$ with $Tz_n - z_n \in M_n^\perp$ are given, let $M_{n+1} = M_n \oplus \{a(Tz_n - z_n) \mid a \in \mathbb{R}\}$ be the $(n+1)$-dimension subspace of H and let Π_{n+1} be the orthogonal projection of H onto M_{n+1}. Then we consider that $\Pi_{n+1} T$ is a mapping from $M_{n+1} \cap B$ into $M_{n+1} \cap B$. Since $\Pi_{n+1} T$ is continuous and $M_{n+1} \cap B$ is compact and convex, by Brouwer's fixed point theorem $\Pi_{n+1} T$ has a fixed point z_{n+1}. Moreover, since $\Pi_{n+1}(Tz_{n+1} - z_{n+1}) = \Pi_{n+1} T z_{n+1} - \Pi_{n+1} z_{n+1} = \Pi_{n+1} T z_{n+1} - z_{n+1} = 0$, we obtain $Tz_{n+1} - z_{n+1} \in M_{n+1}^\perp$.

Lemma 2.3 ([20])**.** *Let H be a real Hilbert space, let C be a non-empty convex subset of H and let T be a hemicontinuous pseudocontractive mapping from C into itself. Suppose that there exists a sequence $\{x_n\}$ in C such that*

(i) $\{Tx_n - x_n\}$ *is convergent to* 0 *weakly;*
(ii) $\langle Tx_n - x_n, x_n\rangle$ *is convergent to* 0*;*
(iii) $\{x_n\}$ *is convergent to* $x \in C$ *weakly.*

Then x *is a fixed point of* T*.*

The following is Ramsey's Theorem:

Ramsey's Theorem. *Let* $\mathcal{V}$ *be the set of all ordered pairs* $(m, n) \in \mathbb{N} \times \mathbb{N}$ *with* $m > n$ *and let* ϕ *be a mapping from* $\mathcal{V}$ *into* $\{0, 1\}$*. Then there exists a countable infinite subset* A *of* $\mathbb{N}$ *such that* $\phi(\mathcal{V} \cap (A \times A))$ *is a singleton.*

Using this theorem, Moloney and Weng [20] obtained the following lemma.

Lemma 2.4 ([20]). *Let* H *be a real Hilbert space and let* W *be an infinite countable subset of* H*. Then for any positive number* δ *there exists an infinite countable subset* V *of* W *such that*

$$\|x - y\|^2 \leq (1 + \delta)(\|x\|^2 + \|y\|^2)$$

for any $x, y \in V$*.*

Remark 2.2. In Lemma 2.4, if W is bounded, then for any positive number δ there exists an infinite countable subset V of W such that

$$\|x - y\|^2 \leq \|x\|^2 + \|y\|^2 + \delta$$

for any $x, y \in V$.

3. Fixed point theorems for self mappings

In this section we consider fixed point theorems for widely more generalized hybrid demicontinuous self mappings. We provide the following lemma instead of Lemma 2.3.

Lemma 3.1. *Let* H *be a real Hilbert space, let* C *be a non-empty convex subset of* H *and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid hemicontinuous mapping from* C *into itself which satisfies the following conditions* (1), (2) *and* (3)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$*;*
(2) $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$.
(3) $2\alpha + \beta + \gamma > 0$.

Suppose that there exists a sequence $\{x_n\}$ *in* C *such that*

(i) $\{Tx_n - x_n\}$ *is convergent to* 0 *weakly;*
(ii) $\langle Tx_n - x_n, x_n\rangle$ *is convergent to* 0*;*
(iii) $\{x_n\}$ *is convergent to* $x \in C$ *weakly.*

Then x *is a fixed point of* T*.*

Proof. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we obtain that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Replacing the variables x and y, we obtain that

$$\alpha\|Tx - Ty\|^2 + \gamma\|x - Ty\|^2 + \beta\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\zeta\|x - Tx\|^2 + \varepsilon\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0$$

for any $x, y \in C$. Therefore we obtain

$$2\alpha\|Tx - Ty\|^2 + (\beta + \gamma)\|x - Ty\|^2 + (\beta + \gamma)\|Tx - y\|^2 + 2\delta\|x - y\|^2 \\ +(\varepsilon + \zeta)\|x - Tx\|^2 + (\varepsilon + \zeta)\|y - Ty\|^2 + 2\eta\|(x - Tx) - (y - Ty)\|^2 \leq 0$$

for any $x, y \in C$ and hence T is a $(2\alpha, \beta + \gamma, \beta + \gamma, 2\delta, \varepsilon + \zeta, \varepsilon + \zeta, 2\eta)$-widely more generalized hybrid mapping. Let $z(t) = (1 - t)x + tTx$, where $t \in [0, 1]$. Since C is convex, $z(t) \in C$. Since T is hemicontinuous, there exists $t_0 \in (0, 1)$ such that

$$\langle Tz(t_0) - z(t_0), Tx - x\rangle \geq \frac{1}{2}\|Tx - x\|^2.$$

By (i), (ii) and (iii), for any positive number ρ there exists natural number n such that

$$\begin{aligned} |\langle Tx_n - x_n, Tz(t_0) - z(t_0)\rangle| &< \rho, \\ |\langle Tx_n - x_n, x_n - z(t_0)\rangle| &< \rho, \\ |\langle x_n - x, Tz(t_0) - z(t_0)\rangle| &< \rho. \end{aligned}$$

Since $2\eta \geq -(2\alpha + \beta + \gamma + \varepsilon + \zeta)$, $\alpha + \beta + \gamma + \delta \geq 0$ and $2\alpha + \beta + \gamma > 0$, we obtain

$$\begin{aligned} 0 \;\geq\;& 2\alpha\|Tz(t_0) - Tx_n\|^2 + (\beta + \gamma)\|z(t_0) - Tx_n\|^2 \\ & +(\beta + \gamma)\|Tz(t_0) - x_n\|^2 + 2\delta\|z(t_0) - x_n\|^2 \\ & +(\varepsilon + \zeta)\|z(t_0) - Tz(t_0)\|^2 + (\varepsilon + \zeta)\|x_n - Tx_n\|^2 \\ & +2\eta\|(z(t_0) - Tz(t_0)) - (x_n - Tx_n)\|^2 \\ \geq\;& 2\alpha\|Tz(t_0) - Tx_n\|^2 + (\beta + \gamma)\|z(t_0) - Tx_n\|^2 \\ & +(\beta + \gamma)\|Tz(t_0) - x_n\|^2 + 2\delta\|z(t_0) - x_n\|^2 \\ & +(\varepsilon + \zeta)\|z(t_0) - Tz(t_0)\|^2 + (\varepsilon + \zeta)\|x_n - Tx_n\|^2 \\ & -(2\alpha + \beta + \gamma + \varepsilon + \zeta)\|(z(t_0) - Tz(t_0)) - (x_n - Tx_n)\|^2 \\ =\;& 2\alpha(\|Tz(t_0) - z(t_0)\|^2 + \|z(t_0) - x_n\|^2 + \|x_n - Tx_n\|^2 \\ & \qquad +2\langle Tz(t_0) - z(t_0), z(t_0) - x_n\rangle \\ & \qquad +2\langle z(t_0) - x_n, x_n - Tx_n\rangle \\ & \qquad +2\langle x_n - Tx_n, Tz(t_0) - z(t_0)\rangle) \\ & +(\beta + \gamma)(\|z(t_0) - x_n\|^2 + \|x_n - Tx_n\|^2 \\ & \qquad +2\langle z(t_0) - x_n, x_n - Tx_n\rangle) \\ & +(\beta + \gamma)(\|Tz(t_0) - z(t_0)\|^2 + \|z(t_0) - x_n\|^2 \\ & \qquad +2\langle Tz(t_0) - z(t_0), z(t_0) - x_n\rangle) \\ & +2\delta\|z(t_0) - x_n\|^2 \\ & -(2\alpha + \beta + \gamma)\|z(t_0) - Tz(t_0)\|^2 \\ & -(2\alpha + \beta + \gamma)\|x_n - Tx_n\|^2 \\ & +2(2\alpha + \beta + \gamma + \varepsilon + \zeta)\langle z(t_0) - Tz(t_0), x_n - Tx_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= 2(\alpha+\beta+\gamma+\delta)\|z(t_0)-x_n\|^2 \\
&\qquad +2(2\alpha+\beta+\gamma)\langle Tz(t_0)-z(t_0), z(t_0)-x_n\rangle \\
&\qquad +2(2\alpha+\beta+\gamma)\langle z(t_0)-x_n, x_n-Tx_n\rangle \\
&\qquad -2(\beta+\gamma+\varepsilon+\zeta)\langle x_n-Tx_n, Tz(t_0)-z(t_0)\rangle \\
&\geq 2(2\alpha+\beta+\gamma)\langle Tz(t_0)-z(t_0), z(t_0)-x\rangle \\
&\qquad -2(2\alpha+\beta+\gamma)|\langle x_n-x, Tz(t_0)-z(t_0)\rangle| \\
&\qquad -2(2\alpha+\beta+\gamma)|\langle Tx_n-x_n, x_n-z(t_0)\rangle| \\
&\qquad -2|\beta+\gamma+\varepsilon+\zeta||\langle Tx_n-x_n, Tz(t_0)-z(t_0)\rangle| \\
&\geq (2\alpha+\beta+\gamma)t_0\|Tx-x\|^2 - 2(2(2\alpha+\beta+\gamma)+|\beta+\gamma+\varepsilon+\zeta|)\rho.
\end{aligned}
$$

Since ρ is arbitrary and $(2\alpha+\beta+\gamma)t_0 > 0$, we obtain $\|Tx-x\| = 0$ and hence x is a fixed point of T. □

Theorem 3.2. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid demicontinuous mapping from C into itself which satisfies the following conditions* (1), (2) *and* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$;
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta \geq 0$;
(3) $\beta+\gamma+\varepsilon+\zeta \geq 0$ *and* $\beta+\gamma+2(\varepsilon+\zeta+\eta) < 0$, *or*
$\beta+\gamma+\varepsilon+\zeta < 0$ *and* $-\beta-\gamma+2\eta \leq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$.

Proof. Since T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping,

$$
\begin{aligned}
&\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \\
&\qquad +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0
\end{aligned}
$$

for any $x, y \in C$. Replacing the variables x and y, we obtain

$$
\begin{aligned}
&\alpha\|Tx-Ty\|^2+\gamma\|x-Ty\|^2+\beta\|Tx-y\|^2+\delta\|x-y\|^2 \\
&\qquad +\zeta\|x-Tx\|^2+\varepsilon\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0
\end{aligned}
$$

for any $x, y \in C$. Therefore we obtain

$$
\begin{aligned}
&2\alpha\|Tx-Ty\|^2+(\beta+\gamma)\|x-Ty\|^2+(\beta+\gamma)\|Tx-y\|^2+2\delta\|x-y\|^2 \\
&\qquad +(\varepsilon+\zeta)\|x-Tx\|^2+(\varepsilon+\zeta)\|y-Ty\|^2+2\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0
\end{aligned}
$$

for any $x, y \in C$ and hence T is a $(2\alpha, \beta+\gamma, \beta+\gamma, 2\delta, \varepsilon+\zeta, \varepsilon+\zeta, 2\eta)$-widely more generalized hybrid mapping. By (3), if $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta) < 0$, then

$$-\beta-\gamma+2\eta = -(\beta+\gamma+\varepsilon+\zeta)+\varepsilon+\zeta+2\eta \leq \beta+\gamma+2(\varepsilon+\zeta+\eta) < 0;$$

if $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta \leq 0$, then

$$\beta+\gamma+2(\varepsilon+\zeta+\eta) < -(\beta+\gamma+\varepsilon+\zeta)+\varepsilon+\zeta+2\eta = -\beta-\gamma+2\eta \leq 0.$$

Therefore we obtain $-\beta-\gamma+2\eta \le 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta) < 0$. Moreover, if $\beta+\gamma+\varepsilon+\zeta \ge 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta) < 0$, then

$$\varepsilon+\zeta+2\eta < -(\beta+\gamma+\varepsilon+\zeta) \le 0;$$

if $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta \le 0$, then

$$\varepsilon+\zeta+2\eta \le \beta+\gamma+\varepsilon+\zeta < 0.$$

Therefore we obtain $\varepsilon+\zeta+2\eta < 0$. Moreover, by (2) and $\varepsilon+\zeta+2\eta < 0$ we obtain $2\alpha+\beta+\gamma > 0$. Since C is weakly sequentially compact, to show that T has a fixed point, by Lemma 3.1 we show that there exists a sequence $\{x_n\}$ in C such that

(i) $\{Tx_n - x_n\}$ is convergent to 0 weakly;

(ii) $\langle Tx_n - x_n, x_n\rangle$ is convergent to 0.

Let B be a closed ball containing C and let P be a metric projection of H onto C. Since TP is a demicontinuous mapping from B into B, there exists a sequence $\{z_n\}$ in B satisfying the conditions (i), (ii) and (iii) in Lemma 2.2. Put $x_n = Pz_n$. Note that $\{x_n\} \subset C$ and

$$\begin{aligned} Tx_n - x_n &= TPz_n - z_n + z_n - Pz_n, \\ \langle Tx_n - x_n, x_n\rangle &= \langle TPz_n - z_n, z_n\rangle + \langle TPz_n - z_n, Pz_n - z_n\rangle \\ &\qquad + \langle z_n - Pz_n, z_n\rangle - \|Pz_n - z_n\|^2. \end{aligned}$$

Since $\{TPz_n - z_n\}$ is convergent to 0 weakly, $\langle TPz_n - z_n, z_n\rangle = 0$ and, $\|TPz_n - z_n\|$ and $\|z_n\|$ are bounded, if $\liminf_{n\to\infty}\|Pz_n - z_n\| = 0$, then there exists a subsequence of $\{x_n\}$ satisfying (i) and (ii). Assume that there exists a positive number ω such that $\|Pz_n - z_n\| \ge \omega$ for any n. Since $\|Pz_n - TPz_n\|$ is bounded, by Remark 2.2 there exists an infinite subset $\{Pz_{n(i)} - TPz_{n(i)}\}$ of $\{Pz_n - TPz_n\}$ such that

$$\begin{aligned} &\|(Pz_{n(i)} - TPz_{n(i)}) - (Pz_{n(j)} - TPz_{n(j)})\|^2 \\ &\qquad \le \|Pz_{n(i)} - TPz_{n(i)}\|^2 + \|Pz_{n(j)} - TPz_{n(j)}\|^2 + \omega^2 \end{aligned}$$

for any i and for any j. Note that

$$\|Px - Py\| \le \|x - y\|$$

for any $x, y \in H$ and

$$\|x - Py\|^2 + \|Py - y\|^2 \le \|x - y\|^2$$

for any $x \in C$ and for any $y \in H$. Furthermore, note that, for any $x, y, w, z \in H$,

$$\begin{aligned} \|x - y - (w - z)\|^2 &= \|x - y\|^2 + \|w - z\|^2 - 2\langle x - y, w - z\rangle \\ &= \|x - y\|^2 + \|w - z\|^2 \\ &\qquad - \|x - z\|^2 - \|y - w\|^2 + \|x - w\|^2 + \|y - z\|^2. \end{aligned}$$

Since $-\beta-\gamma+2\eta \le 0$, $\varepsilon+\zeta+2\eta < 0$, $\beta+\gamma+2(\varepsilon+\zeta+\eta) < 0$, $2\alpha+\beta+\gamma > 0$, $\alpha+\beta+\gamma+\delta \ge 0$, $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta \ge 0$, and by Lemma 2.2 $\langle TPz_m - z_m, z_n\rangle = 0$ if $m \ge n$ and $\langle TPz_m - z_m, TPz_n - z_n\rangle = 0$ if $m \ne n$, we obtain

$$\begin{aligned} 0 \ \ge\ & 2\alpha\|TPz_{n(i)} - TPz_{n(j)}\|^2 + (\beta+\gamma)\|Pz_{n(i)} - TPz_{n(j)}\|^2 \\ & + (\beta+\gamma)\|TPz_{n(i)} - Pz_{n(j)}\|^2 + 2\delta\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\ & + (\varepsilon+\zeta)\|Pz_{n(i)} - TPz_{n(i)}\|^2 + (\varepsilon+\zeta)\|Pz_{n(j)} - TPz_{n(j)}\|^2 \end{aligned}$$

$$
\begin{aligned}
& +2\eta\|(Pz_{n(i)} - TPz_{n(i)}) - (Pz_{n(j)} - TPz_{n(j)})\|^2 \\
= \;& (2\alpha+\beta+\gamma)\|TPz_{n(i)} - TPz_{n(j)}\|^2 \\
& +(\beta+\gamma+2\delta)\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\
& +(\beta+\gamma+\varepsilon+\zeta)\|Pz_{n(i)} - TPz_{n(i)}\|^2 \\
& +(\beta+\gamma+\varepsilon+\zeta)\|Pz_{n(j)} - TPz_{n(j)}\|^2 \\
& +(-\beta-\gamma+2\eta)\|(Pz_{n(i)} - TPz_{n(i)}) - (Pz_{n(j)} - TPz_{n(j)})\|^2 \\
\geq \;& (2\alpha+\beta+\gamma)\|TPz_{n(i)} - TPz_{n(j)}\|^2 \\
& +(\beta+\gamma+2\delta)\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\
& +(\beta+\gamma+\varepsilon+\zeta)\|Pz_{n(i)} - TPz_{n(i)}\|^2 \\
& +(\beta+\gamma+\varepsilon+\zeta)\|Pz_{n(j)} - TPz_{n(j)}\|^2 \\
& +(-\beta-\gamma+2\eta)(\|Pz_{n(i)} - TPz_{n(i)}\|^2 + \|Pz_{n(j)} - TPz_{n(j)}\|^2 + \omega^2) \\
= \;& (2\alpha+\beta+\gamma)\|TPz_{n(i)} - TPz_{n(j)}\|^2 \\
& +(\beta+\gamma+2\delta)\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\
& +(\varepsilon+\zeta+2\eta)\|Pz_{n(i)} - TPz_{n(i)}\|^2 \\
& +(\varepsilon+\zeta+2\eta)\|Pz_{n(j)} - TPz_{n(j)}\|^2 \\
& +(-\beta-\gamma+2\eta)\omega^2 \\
\geq \;& (2\alpha+\beta+\gamma)(\|TPz_{n(i)} - z_{n(i)}\|^2 + \|Pz_{n(i)} - Pz_{n(j)}\|^2 + \|TPz_{n(j)} - z_{n(j)}\|^2 \\
& \qquad -2|\langle TPz_{n(i)} - z_{n(i)}, z_{n(i)} - z_{n(j)}\rangle| \\
& \qquad -2|\langle TPz_{n(j)} - z_{n(j)}, z_{n(i)} - z_{n(j)}\rangle| \\
& \qquad -2|\langle TPz_{n(i)} - z_{n(i)}, TPz_{n(j)} - z_{n(j)}\rangle|) \\
& +(\beta+\gamma+2\delta)\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\
& +(\varepsilon+\zeta+2\eta)(\|TPz_{n(i)} - z_{n(i)}\|^2 - \|Pz_{n(i)} - z_{n(i)}\|^2) \\
& +(\varepsilon+\zeta+2\eta)(\|TPz_{n(j)} - z_{n(j)}\|^2 - \|Pz_{n(j)} - z_{n(j)}\|^2) \\
& +(-\beta-\gamma+2\eta)\omega^2 \\
\geq \;& (2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta)\|TPz_{n(i)} - z_{n(i)}\|^2 \\
& +2(\alpha+\beta+\gamma+\delta)\|Pz_{n(i)} - Pz_{n(j)}\|^2 \\
& +(2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta)\|TPz_{n(j)} - z_{n(j)}\|^2 \\
& -2(2\alpha+\beta+\gamma)|\langle TPz_{n(i)} - z_{n(i)}, z_{n(i)} - z_{n(j)}\rangle| \\
& -(\beta+\gamma+2(\varepsilon+\zeta+\eta))\omega^2 \\
\geq \;& -2(2\alpha+\beta+\gamma)|\langle TPz_{n(i)} - z_{n(i)}, z_{n(i)} - z_{n(j)}\rangle| - (\beta+\gamma+2(\varepsilon+\zeta+\eta))\omega^2
\end{aligned}
$$

for any i and for any j with $i < j$. Therefore we obtain

$$2(2\alpha+\beta+\gamma)|\langle TPz_{n(i)} - z_{n(i)}, z_{n(i)} - z_{n(j)}\rangle| \geq -(\beta+\gamma+2(\varepsilon+\zeta+\eta))\omega^2$$

for any i and for any j with $i < j$. Let $\{z_{m(i)}\}$ be a subsequence of $\{z_{n(i)}\}$ and let z be the weak limit of $\{z_{m(i)}\}$. Then we obtain

$$\begin{aligned}&\langle TPz_{m(i)} - z_{m(i)}, z_{m(i)} - z_{m(j)}\rangle \\ &\quad = \langle TPz_{m(i)} - z_{m(i)}, z_{m(i)} - z\rangle + \langle TPz_{m(i)} - z_{m(i)}, z - z_{m(j)}\rangle.\end{aligned}$$

Since by Lemma 2.2 $\langle TPz_{m(i)} - z_{m(i)}, z_{m(i)}\rangle = 0$ and $\{TPz_n - z_n\}$ is convergent to 0 weakly, we obtain $|\langle TPz_{m(i)} - z_{m(i)}, z_{m(i)} - z\rangle| = |\langle TPz_{m(i)} - z_{m(i)}, z\rangle|$ is as small as necessary for sufficiently large $m(i)$. Since z is the weak limit of $\{z_{(m(i)}\}$, there exists $m(j)$ such that $m(i) < m(j)$ and $|\langle TPz_{m(i)} - z_{m(i)}, z - z_{m(j)}\rangle|$ is as small as necessary for any $m(i)$. Therefore we obtain

$$0 \geq -(\beta + \gamma + 2\varepsilon + 2\zeta + 2\eta)\omega^2$$

and it is a contradiction, that is, $\liminf_{n\to\infty} \|Pz_n - z_n\| = 0$ holds.

Next suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of T. Then

$$\begin{aligned} 0 &\geq \alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\ &\qquad + \varepsilon\|p_1 - Tp_1\|^2 + \zeta\|p_2 - Tp_2\|^2 + \eta\|(p_1 - Tp_1) - (p_2 - Tp_2)\|^2 \\ &= (\alpha + \beta + \gamma + \delta)\|p_1 - p_2\|^2 \end{aligned}$$

and hence $p_1 = p_2$. Therefore a fixed point of T is unique. □

Remark 3.1. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions (1), (2) and (3):

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta > 0$;
(3) $\varepsilon + \zeta + 2\eta \geq 0$.

Then by [12, Theorem 3.1] T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$. Note that T does not have to be demicontinuous.

Since any pseudocontractive mapping is a $(1, 0, 0, -1, 0, 0, -1)$-widely more generalized hybrid mapping, it satisfies (1), (2) and (3) of Theorem 3.2. Therefore by Theorem 3.2 we can directly show Theorem 2.1.

Theorem 3.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid demicontinuous mapping from C into itself which satisfies the following conditions* (1), (2) *and* (3):

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$;
(3) *there exists* $\lambda \in [0, 1)$ *such that*
$\beta + \gamma + \varepsilon + \zeta \geq 0$ *and* $\beta + \gamma + 2(\varepsilon + \zeta + \eta) + (2\alpha + \beta + \gamma)\lambda < 0$, *or*
$\beta + \gamma + \varepsilon + \zeta < 0$ *and* $-\beta - \gamma + 2\eta + (2\alpha + \beta + \gamma)\lambda \leq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$.

Proof. Since T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping, it is a $(2\alpha,\beta+\gamma,\beta+\gamma,2\delta,\varepsilon+\zeta,\varepsilon+\zeta,2\eta)$-widely more generalized hybrid mapping. Let $S=(1-\lambda)T+\lambda I$. Then S is a $\left(\frac{2\alpha}{1-\lambda},\frac{\beta+\gamma}{1-\lambda},\frac{\beta+\gamma}{1-\lambda},-\frac{2\lambda}{1-\lambda}(\alpha+\beta+\gamma)+2\delta,\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2},\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2},\frac{2\eta+2\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping and $F(S)=F(T)$. Moreover we obtain

$$\begin{aligned}
&\frac{2\alpha}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+\left(-\frac{2\lambda}{1-\lambda}(\alpha+\beta+\gamma)+2\delta\right)\\
&\quad=2(\alpha+\beta+\gamma+\delta)\geq 0,\\
&\frac{4\alpha}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}+\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}+\frac{4\eta+4\alpha\lambda}{(1-\lambda)^2}\\
&\quad=\frac{2(2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta)}{(1-\lambda)^2}\geq 0,\\
&\frac{\beta+\gamma}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}+\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}\\
&\quad=\frac{2(\beta+\gamma+\varepsilon+\zeta)}{(1-\lambda)^2},\\
&\frac{\beta+\gamma}{1-\lambda}+\frac{\beta+\gamma}{1-\lambda}+2\left(\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}+\frac{\varepsilon+\zeta+(\beta+\gamma)\lambda}{(1-\lambda)^2}+\frac{2\eta+2\alpha\lambda}{(1-\lambda)^2}\right)\\
&\quad=\frac{2(\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda)}{(1-\lambda)^2},\\
&-\frac{\beta+\gamma}{1-\lambda}-\frac{\beta+\gamma}{1-\lambda}+\frac{4\eta+4\alpha\lambda}{(1-\lambda)^2}\\
&\quad=\frac{2(-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda)}{(1-\lambda)^2}.
\end{aligned}$$

Therefore by Theorem 3.2 we obtain the desired result. □

Remark 3.2. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions (1), (2) and (3):

(1) $\alpha+\beta+\gamma+\delta\geq 0$;
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta>0$;
(3) there exists $\lambda\in[0,1)$ such that $\varepsilon+\zeta+2\eta+(2\alpha+\beta+\gamma)\lambda\geq 0$.

Then by [12, Theorem 3.3] T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta>0$. Note that T does not have to be demicontinuous.

4. Fixed point theorems for non-self mappings

In this section we consider fixed point theorems for widely more generalized hybrid demicontinuous non-self mappings.

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid* *demicontinuous mapping from C into H which satisfies the following conditions* (1), (2) *and* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$*;*
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta \geq 0$*;*
(3) *there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and,*
$\beta+\gamma+\varepsilon+\zeta \geq 0$ *and* $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda < 0$*, or*
$\beta+\gamma+\varepsilon+\zeta < 0$ *and* $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda \leq 0$*.*

Suppose that for any $x \in C$ there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x+m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$.

Proof. Since T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping, it is a $(2\alpha,\beta+\gamma,\beta+\gamma,2\delta,\varepsilon+\zeta,\varepsilon+\zeta,2\eta)$-widely more generalized hybrid mapping. Let $S = (1-\lambda)T+\lambda I$. Since

$$\begin{aligned} Sx &= (1-\lambda)Tx+\lambda x \\ &= (1-\lambda)(x+m(y-x))+\lambda x \\ &= (1-(1-\lambda)m)x+(1-\lambda)my \in C \end{aligned}$$

for any $x \in C$, S is a mapping from C into itself. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Therefore by Theorem 3.3 we obtain the desired result. □

Remark 4.1. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following conditions (1), (2) and (3):

(1) $\alpha+\beta+\gamma+\delta \geq 0$;
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta > 0$;
(3) there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $\varepsilon+\zeta+2\eta+(2\alpha+\beta+\gamma)\lambda \geq 0$.

Suppose that for any $x \in C$ there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x+m(y-x)$. Then by [12, Theorem 4.1] T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$. Note that T does not have to be demicontinuous.

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid demicontinuous mapping from C into H which satisfies the following conditions* (1), (2) *and* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$*;*
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta \geq 0$*;*
(3) *there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 1$ *and,*
$\beta+\gamma+\varepsilon+\zeta \geq 0$ *and* $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda < 0$*, or*
$\beta+\gamma+\varepsilon+\zeta < 0$ *and* $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda \leq 0$*.*

Suppose that there exists $t \in \mathbb{R}$ with $t > 0$ such that for any $x \in C$ there exist $m \in M(t)$ and $y \in C$ such that $Tx = x + m(y - x)$, where

$$M(t) = \begin{cases} \left[\frac{1}{1-\lambda}, 0\right] & \text{if } 2\alpha+\beta+\gamma < 0 \text{ and } \lambda > 1, \\ [0, t] & \text{if } 2\alpha+\beta+\gamma < 0 \text{ and } \lambda < 1, \\ [0, t] \text{ or } [-t, 0] & \text{if } 2\alpha+\beta+\gamma = 0, \\ [-t, 0] & \text{if } 2\alpha+\beta+\gamma > 0 \text{ and } \lambda > 1, \\ \left[0, \frac{1}{1-\lambda}\right] & \text{if } 2\alpha+\beta+\gamma > 0 \text{ and } \lambda < 1. \end{cases}$$

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$.

Proof. Suppose that there exists $t \in \mathbb{R}$ with $t > 0$ such that for any $x \in C$ there exist $m \in M(t)$ and $y \in C$ such that $Tx = x+m(y-x)$. We show that there exists $\lambda_1 \in \mathbb{R}$ such that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in M(t)$. Then by Theorem 4.1 we obtain the desired result. In the case of $2\alpha+\beta+\gamma < 0$ and $\lambda > 1$ we obtain for $\lambda_1 = \lambda$ that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in \left[\frac{1}{1-\lambda}, 0\right]$. In the case of $2\alpha+\beta+\gamma > 0$ and $\lambda < 1$ we obtain for $\lambda_1 = \lambda$ that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in \left[0, \frac{1}{1-\lambda}\right]$. In the case of $2\alpha+\beta+\gamma < 0$ and $\lambda < 1$ let $\lambda_1 \in \mathbb{R}$ satisfy $\lambda \leq \lambda_1 < 1$ and $(1-\lambda_1)t \leq 1$. Then we obtain for λ_1 that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in [0, t]$. In the case of $2\alpha+\beta+\gamma > 0$ and $\lambda > 1$ let $\lambda_1 \in \mathbb{R}$ satisfy $1 < \lambda_1 \leq \lambda$ and $-(1-\lambda_1)t \leq 1$. Then we obtain for λ_1 that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in [-t, 0]$. In the case of $2\alpha+\beta+\gamma = 0$, let $\lambda_1 \in \mathbb{R}$ satisfy $\lambda_1 < 1$ and $(1-\lambda_1)t \leq 1$, or, $1 < \lambda_1$ and $-(1-\lambda_1)t \leq 1$. Then we obtain for λ_1 that $\lambda_1 \neq 1$ and, $\beta+\gamma+\varepsilon+\zeta \geq 0$ and $\beta+\gamma+2(\varepsilon+\zeta+\eta)+(2\alpha+\beta+\gamma)\lambda_1 < 0$, or $\beta+\gamma+\varepsilon+\zeta < 0$ and $-\beta-\gamma+2\eta+(2\alpha+\beta+\gamma)\lambda_1 \leq 0$, and $0 \leq (1-\lambda_1)m \leq 1$ for any $m \in [0, t]$ or for any $m \in [-t, 0]$, respectively. □

Remark 4.2. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid demicontinuous mapping from C into H which satisfies the following conditions (1), (2) and (3):

(1) $\alpha+\beta+\gamma+\delta \geq 0$;
(2) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta > 0$;
(3) there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $\varepsilon+\zeta+2\eta+(2\alpha+\beta+\gamma)\lambda \geq 0$.

Suppose that there exists $t \in \mathbb{R}$ with $t > 0$ such that for any $x \in C$ there exist $m \in M(t)$ and $y \in C$ such that $Tx = x + m(y - x)$, where

$$M(t) = \begin{cases} \left[\frac{1}{1-\lambda}, 0\right] & \text{if } 2\alpha + \beta + \gamma > 0 \text{ and } \lambda > 1, \\ [0, t] & \text{if } 2\alpha + \beta + \gamma > 0 \text{ and } \lambda < 1, \\ [0, t] \text{ or } [-t, 0] & \text{if } 2\alpha + \beta + \gamma = 0, \\ [-t, 0] & \text{if } 2\alpha + \beta + \gamma < 0 \text{ and } \lambda > 1, \\ \left[0, \frac{1}{1-\lambda}\right] & \text{if } 2\alpha + \beta + \gamma < 0 \text{ and } \lambda < 1. \end{cases}$$

Then by [12, Theorem 4.2] T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$. Note that T does not have to be demicontinuous and $M(t)$ is different.

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