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A FIEXD POINT THEOREM FOR MUTIVALUED NONEXPANSIVE MAPPINGS OF A METRIC SPACE WITH CERTAIN KIND OF CONVEXITY

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Abstract. We prove a fixed point theorem for mutivalued nonexpansive mappings of a metric space with certain kind of convexity that extends Kijima's result [3].

1. Introduction

In 1970, Takahashi [6] introduced a notion of convexity in metric spaces and studied some fixed point theorems in such spaces. Let X be a metric space and $I = [0, 1]$. A function $W : X \times X \times I \to X$ is said to be a convex structure on X, if for each $(x, y, \lambda) \in X \times X \times I$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \quad \text{for all} \quad u \in X.$$

X together with a convex structure is called a convex metric space.

On the other hand, in 1987, Kijima [2] extended, in some sense, the notion of convex metric spaces, i.e., a metric space such that, for each pair $x, y \in X$, there exists $z \in X$ that satisfies

$$(*) \qquad d\,(z, u) \leq \frac{d\,(x, u) \;+\; d\,(y, u)}{2} \qquad \text{for all} \quad u \in X.$$

And he proved a common fixed point theorem for left reversible semigroups of nonexpansive mappings. Also, in 1992, Kijima [3] proved the following fixed point theorem by the idea of (ϵ, n)-sequence.

Theorem 1.1. *Let X be a bounded metric space with metric d, satisfying (*) and T be a nonexpansive self-map of X. T has a fixed point if and only if there exists a nonexpansive self-map S of X such that $ST = TS$ and the range of S is contained in a certain nonempty compact subset of X. Especially, if the range of T^k for a certain positive integer k is contained in a certain nonempty compact subset of X, then T has a fixed point in X.*

In this paper, we prove a fixed point theorem that extends Kijima's result [3].

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2. Preliminaries

Let X be a metric space with metric d such that, for each pair $x, y \in X$, there exists $z \in X$ that satisfies

$$(*) \qquad d(z,u) \leq \frac{d(x,u) + d(y,u)}{2} \qquad \text{for all} \quad u \in X.$$

Let $\mathcal{B}C(X)$ be the family of all nonempty bounded closed subsets of X and $\mathcal{K}(X)$ be the family of all nonempty compact subsets of X. The Hausdorff metric $\mathcal{H}$ for $\mathcal{B}C(X)$ is defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in B} d(x,A),\ \sup_{x\in A} d(x,B)\right\}$$

for $A, B \in \mathcal{B}C(X)$, where

$$d(x,A) = \inf\{\ d(x,y)\ :\ y \in A\}$$

for $x \in X$ and $A \in \mathcal{B}C(X)$. A mapping $T : X \multimap X$ is said to be nonexpansive if T assigns each $x \in X$ to an element Tx of $\mathcal{B}C(X)$ and satisfies that

$$\mathcal{H}(Tx, Ty) \leq d(x,y)$$

for every $x, y \in X$. We put $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$.

3. Main result

We need the following definition [1].

Definition 3.1. Let T be a multivalued nonexpansive mapping of X into $BC(X)$, and f be a nonexpansive mapping of X into itself. f and T is said to be commutative if $f(Tx) \subset T(f(x))$ for all $x \in X$.

Also, we have the following theorem that extends Theorem [3].

Theorem 3.2. *Let X be a metric space with metric d satisfying (*), T be a multivalued nonexpansive mapping of X into $BC(X)$ and f be a nonexpansive mapping of X into itself. T has a fixed point in X if and only if there exists a nonexpansive mapping f of X into itself such that f and T are commutative and the range of f is contained in some nonempty compact subset of X.*

Proof. By Theorem 2 [5], we have a sequence $\{x_n\}_{n=1}^{\infty}$ that stisfies $\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} \inf_{y\in Tx_n} d(x_n, y) = 0.$
Furthermore we have that

$$\begin{aligned} 0 \leq d(f(x_n), f(Tx_n)) &= \inf_{y\in f(Tx_n)} d(f(x_n), y) \\ &= \inf_{z\in Tx_n} d(f(x_n), f(z)) \\ &\leq \inf_{z\in Tx_n} d(x_n, z) \\ &= d(x_n, Tx_n) \to 0\ (n\to\infty) \end{aligned}$$

and hence

$$\begin{aligned} 0 \leq d(f(x_n), T(f(x_n))) &= \inf_{y \in T(f(x_n))} d(f(x_n), y) \\ &\leq \inf_{y \in f(T(x_n))} d(f(x_n), y) \\ &= d(f(x_n), f(Tx_n)) \to 0 \ (n \to \infty). \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} d(f(x_n), T(f(x_n))) = 0.$$

On the other hand, since the range of f is contained in a nonempty compact subset of X, there exists a subsequence $\{f(x_{n_i})\}_{i=1}^{\infty} \subset \{f(x_n)\}_{n=1}^{\infty}$ and $y_* \in X$ that satisfy $\lim_{n\to\infty} f(x_{n_i}) = y_*$.

Since

$$\begin{aligned} |d(x, A) - d(y, B)| &\leq |d(x, A) - d(y, A)| + |d(y, A) - d(y, B)| \\ &\leq d(x, y) + \mathcal{H}(A, B) \end{aligned}$$

for all $x, y \in X$ and $A, B \in BC(X)$, we have that

$$\begin{aligned} |d(y_*, Ty_*) - d(f(x_{n_i}), T(f(x_{n_i})))| &\leq d(y_*, f(x_{n_i})) + \mathcal{H}(Ty_*, T(f(x_{n_i}))) \\ &\leq d(y_*, f(x_{n_i})) + d(y_*, f(x_{n_i})) \\ &= 2d(y_*, f(x_{n_i})) \to 0 \ (i \to \infty) \end{aligned}$$

and hence

$$\begin{aligned} 0 &\leq d(y_*, Ty_*) \\ &\leq d(f(x_{n_i}), T(f(x_{n_i}))) + |d(y_*, Ty_*) - d(f(x_{n_i}), T(f(x_{n_i})))| \\ &\to 0 \quad (i \to \infty). \end{aligned}$$

So, we have $d(y_*, Ty_*) = 0$. Therefore, we have $y_* \in Ty_*$. So, we know $F(T) \neq \phi$. This completes the proof of this theorem. ☐

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