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A NOTE ON THE IMPLICIT MIDPOINT RULE FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. The implicit midpoint rule is used to approximate fixed points of nonexpansive mappings via a semi-implicit way. We prove a weak convergence result of this method by relaxing the conditions imposed upon the sequence of parameters that define the method.

1. INTRODUCTION

It has long been an interesting topic of finding fixed points of nonexpansive mappings in both Hilbert and Banach spaces. Several iterative methods have therefore been invented. Two particular and most well known such methods are the Krasnoselskii-Mann (KM) method [8, 14] and the Halpern method [7]. These two methods generate a sequence $\{x_n\}$ via the recursive procedures:

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, \quad n \geq 0 \tag{1.1}$$

and, respectively,

$$x_{n+1} = (1 - t_n)u + t_n T x_n, \quad n \geq 0, \tag{1.2}$$

where $\{t_n\}$ is a sequence of real positive numbers (usually assumed in the interval $[0, 1]$) and T is a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in the domain of T).

Both algorithms (1.1) and (1.2) have extensively been investigated; see [9, 13, 10, 11, 16, 18, 19, 12, 20]) and the references therein. The feature is that the KM method can have only weak convergence (in general) and Halpern's method can always have strong convergence in the Hilbert space setting. We list the following two fundamental convergence theorems (the Banach space versions can be found in Reich [15, 16]).

Theorem 1.1. *Let H be a Hilbert space and C a nonempty closed convex subset of H. Suppose $T : C \to C$ is a nonexpansive mapping with a fixed point. Suppose $\{t_n\}$ is a sequence in $[0, 1]$ such that*

$$\sum_{n=0}^{\infty} t_n(1 - t_n) = \infty. \tag{1.3}$$

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Then the sequence $\{x_n\}$ *generated by the KM method (1.1) converges weakly to a fixed point of* T*.*

Theorem 1.2. *Let* H *be a Hilbert space and* C *a nonempty closed convex subset of* H*. Suppose* $T : C \to C$ *is a nonexpansive mapping with a fixed point. Suppose* $\{t_n\}$ *is a sequence in* $[0,1]$ *such that*

(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=0}^{\infty} t_n = \infty$,
(iii) *either* $\sum_{n=0}^{\infty} |t_n - t_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Then the sequence $\{x_n\}$ *generated by the Halpern method (1.2) converges strongly to a fixed point of* T *which is closest to* u *from the set of fixed points of* T *.*

The development of inventing new iterative methods for finding fixed points of nonexpansive mappings has always been going on. For instance, Halpern's method (1.2) has been adapted [20] to a finite family of nonexpansive mappings. More recently, the KM algorithm (1.1) has been adapted to the implicit midpoint rule [1] motivated by the implicit midpoint rule for differential equations [2, 3, 4, 5, 17].

The implicit midpoint rule (IMR) for nonexpansive mappings [1] generates a sequence $\{x_n\}$ by the following semi-implicit iteration recursion:

(1.4)

Implicit Midpoint Rule (IMR) :
Initialize $x_0 \in C$ arbitrarily and iterate

$$x_{n+1} := (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0,$$

where $t_n \in (0,1)$ for all n.

Notice that IMR (1.4) is well defined, due to the fact that the map, for each fixed $u \in C$ and $t \in (0,1)$,

$$x \mapsto (1-t)u + tT\left(\frac{u+x}{2}\right), \quad x \in C$$

is a contraction of C and thus has a unique fixed point in C.

A convergence result on IMR (1.4) is provided in [1] as follows.

Theorem 1.3. *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping with fixed points. Then the sequence* $\{x_n\}$ *generated by IMR (1.4) converge weakly to a fixed point of* T *provided the sequence* $\{t_n\}$ *satisfies the two conditions:*

(C1) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and some* $a > 0$*,*
(C2) $\liminf_{n\to\infty} t_n > 0$.

[Remark: The condition (C1) is redundant as it is implied by the condition (C2).]

The ergodicity of IMR (1.4) has recently been discussed in [21].

In this note we shall provide an alternative by proving a weak convergence result using the condition (1.3). Namely, we will prove the following result.

Theorem 1.4. *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping with fixed points. Then the sequence* $\{x_n\}$ *generated by IMR (1.4) converge weakly to a fixed point of* T *provided* $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$.

2. Proof of main result

The main result of this note is Theorem 1.4. To prove it, we shall first establish certain properties of IMR (1.4). We begin by recalling the notion of nonexpansive mappings.

Let H is a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad x, y \in C.$$

We shall use $Fix(T)$ to denote the set of fixed points of T. Namely, $Fix(T) = \{x \in C : Tx = x\}$. We always assume that $Fix(T) \neq \emptyset$. [Note that if C is in addition bounded, then $Fix(T) \neq \emptyset$.]

2.1. **Properties of IMR (1.4).** First we quote the following lemma.

Lemma 2.1. [1] *Let $\{x_n\}$ be the sequence generated by IMR (1.4). Then*

(i) $\|x_{n+1} - p\| \le \|x_n - p\|$ *for all* $n \ge 0$ *and* $p \in Fix(T)$.

(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$.

(iii) $\sum_{n=1}^{\infty} t_n(1 - t_n) \left\| x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|^2 < \infty$.

Lemma 2.2. *Let $\{x_n\}$ be the sequence generated by IMR (1.4). Then $\{\|x_n - Tx_n\|\}$ is decreasing; hence,* $\lim_{n\to\infty} \|x_n - Tx_n\|$ *exists.*

Proof. As a matter of fact, we have

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \left\|(1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right) - Tx_{n+1}\right\| \\
&\le (1 - t_n)\|x_n - Tx_{n+1}\| + t_n \left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - Tx_{n+1}\right\| \\
&\le (1 - t_n)\|x_n - Tx_{n+1}\| + \frac{t_n}{2}\|x_n - x_{n+1}\| \\
&\le (1 - t_n)(\|x_n - Tx_n\| + \|Tx_n - Tx_{n+1}\|) + \frac{t_n}{2}\|x_n - x_{n+1}\| \\
&\le (1 - t_n)(\|x_n - Tx_n\| + \|x_n - x_{n+1}\|) + \frac{t_n}{2}\|x_n - x_{n+1}\| \\
&= (1 - t_n)\|x_n - Tx_n\| + (1 - \frac{1}{2}t_n)\|x_n - x_{n+1}\|. \qquad (2.1)
\end{aligned}$$

However, by definition of IMR (1.4), we have

$$\begin{aligned}
\|x_n - x_{n+1}\| &= t_n \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\
&\le t_n \left(\|x_n - Tx_n\| + \left\|Tx_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right) \\
&\le t_n \left(\|x_n - Tx_n\| + \left\|x_n - \left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right)
\end{aligned}$$

$$= t_n\left(\|x_n - Tx_n\| + \frac{1}{2}\|x_n - x_{n+1}\|\right).$$

It turns out that

$$\|x_n - x_{n+1}\| \le \frac{t_n}{1-\frac{1}{2}t_n}\|x_n - Tx_n\|. \tag{2.2}$$

Substituting (2.2) into (2.1) yields

$$\|x_{n+1} - Tx_{n+1}\| \le \|x_n - Tx_n\|. \tag{2.3}$$

□

Lemma 2.3. *There holds the relation*

$$\|x_n - Tx_n\| \le \frac{3}{2}\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|. \tag{2.4}$$

Proof. We have

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\le t_n\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + (1-t_n)\|x_n - Tx_n\| + t_n\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - Tx_n\right\| \\
&\le t_n\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + (1-t_n)\|x_n - Tx_n\| + \frac{1}{2}t_n\|x_n - x_{n+1}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \frac{1}{2}\|x_n - x_{n+1}\| \\
&= (1 + \frac{1}{2}t_n)\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\
&\le \frac{3}{2}\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|.
\end{aligned}$$

□

Combining Lemma 2.1(iii) and Lemma 2.3 immediately yields

Lemma 2.4. *Let $\{x_n\}$ be generated by IMR (1.4). Then*

$$\sum_{n=0}^{\infty} t_n(1-t_n)\|x_n - Tx_n\|^2 < \infty. \tag{2.5}$$

2.2. **Proof of Theorem 1.4.** We need the demiclosedness principle for nonexpansive mappings in order to prove Theorem 1.4.

Lemma 2.5. [6] *Let C be a nonempty closed convex subset of a Hilbert space H and let $V : C \to H$ be a nonexpansive mapping with a fixed point. Assume $\{z_n\}$ is a sequence in C such that $z_n \to z$ weakly and $(I - V)z_n \to 0$ strongly. Then $(I - T)z = 0$; i.e., $z \in Fix(T)$.*

We use the notation $\omega_w(z_n)$ to denote the set of all weak cluster points of the sequence $\{z_n\}$.

The following result is easily proved (see [10]).

Lemma 2.6. *Let K be a nonempty closed convex subset of a Hilbert space H and let $\{z_n\}$ be a bounded sequence in H. Assume*

(i) $\lim_{n\to\infty} \|z_n - p\|$ *exists for all* $p \in K$,

(ii) $\omega_w(z_n) \subset K$.

Then $\{z_n\}$ weakly converges to a point in K.

We are now in a position to prove Theorem 1.4, the main result of this paper.

Proof of Theorem 1.4. Since $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$, we derive from Lemma 2.4 that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ which together with Lemma 2.2 implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.6}$$

Now Lemma 2.5 ensures that $\omega_w(x_n) \subset Fix(T)$. By virtue of Lemma 2.1(i), we can apply Lemma 2.6 with $K := Fix(T)$ to conclude that $\{x_n\}$ must converge weakly to a point in $Fix(T)$. □

3. Concluding remarks

IMR (1.4) is a semi-implicit iterative method for nonexpansive mappings. We raise the following questions for further investigations:

(1) Can IMR (1.4) be strongly convergent? As a semi-implicit method, strong convergence would be hoped. However, this is unclear yet.

(2) How IMR (1.4) is comparable with the KM method (1.1)? Again as a semi-implicit method, faster convergence of IMR than KM would be expected. Numerical experiments would help to convince it.

(3) What is the Banach space version of Theorem 1.4? Such a version for Theorem 1.3 is partially obtained in [22].

(4) In IMR (1.4), the midpoint between x_n and x_{n+1} is used (hence the name of implicit midpoint rule). One may wonder whether or not the midpoint can be replaced with any convex combination of x_n and x_{n+1}, that is, if we define

$$x_{n+1} := (1-t_n)x_n + t_n T\left(\lambda x_n + (1-\lambda)x_{n+1}\right), \quad n \geq 0,$$

with $\lambda \in (0,1)$. What is then the convergence of the sequence $\{x_n\}$?

(5) Develop IMR (1.4) to accommodate the case of finding a common fixed point of a finite family of nonexpansive mappings.

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