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JENSEN'S INEQUALITY AND CHEBYSHEV'S INEQUALITY ARE BROTHERS

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ABSTRACT. In this note, we consider Jensen's inequality and Chebyshev's inequality, and show that these inequalities have the same parent.

1. Introduction

The inequality

$$\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2 \quad (a, b \in \mathbf{R})$$

is exactly the same with the inequality

$$\frac{a \cdot a + b \cdot b}{2} \geq \frac{a+b}{2} \cdot \frac{a+b}{2} \quad (a, b \in \mathbf{R}).$$

The former is a special case of Jensen's inequality which represents the convexity of the function $y = x^2$ and the latter is a special case of Chebyshev's inequality which is implied from the rearrangement inequality. The above example suggests that there is some relationship between these inequalities, and such an example may be found elsewhere.

This short paper is inspired by above observation. We show that Jensen's inequality and Chebyshev's inequality have same parent as mentioned in the title.

2. Preliminary and Key theorem

The finite form of Jensen's inequality asserts that if $t_1, \ldots, t_n$ are positive numbers with $\sum_{i=1}^n t_i = 1$ and φ is a covex (resp. concave) function on a real interval I, then

$$J(n): \quad \sum_{i=1}^n t_i\varphi(x_i) \geq \varphi\left(\sum_{i=1}^n t_i x_i\right) \quad \left(resp. \quad \sum_{i=1}^n t_i\varphi(x_i) \leq \varphi\left(\sum_{i=1}^n t_i x_i\right)\right)$$

holds for all $x_1, \ldots, x_n \in I$. We can without loss of generality assume the additional condition $x_1 \leq \cdots \leq x_n$ in the above inequality. We also note that Jensen's inequality for $n = 2$ represents the unevenness of the function φ. Therefore Jensen's inequality has the following property :

$$J(2) \Rightarrow J(n) \quad (\forall n \geq 3)$$

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The φ-mean inequality stated in [1, Theorem 1] also has such a property. We can see that Chebyshev's inequality for sequences of real numbers has a similar property. In fact, this inequality may be stated as follows :

$$\frac{x_1y_1 + \cdots + x_ny_n}{n} \geq \frac{x_1 + \cdots + x_n}{n} \cdot \frac{y_1 + \cdots + y_n}{n}$$

holds whenever both $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are simultaneously monotone increasing or monotone decreasing. Note that Chebyshev's inequality for $n = 2$ is a rewrite of the special rearrangement inequality :

$$ax + by \geq ay + bx \quad (a \leq b, x \leq y).$$

Since Chebyshev's inequality for $n \geq 3$ can be shown by applying repeatedly the above rearrangement inequality, we see that Chebyshev's inequality has the same property as Jensen's inequality.

Now Chebyshev's inequality for $n = 2$ asserts that if $a \leq b$ and $x \leq y$, then

$$\frac{ax + by}{2} \geq \frac{a + b}{2} \cdot \frac{x + y}{2}$$

holds. If we regard a and b in the above inequality as functions, the following lemma is envisioned.

Lemma 2.1 (see [2, Lemma 1]). *Let φ and ψ be two functions on a real interval I such that $\psi - \varphi$ is monotone increasing on I and ψ is convex on I. Then*

$$((1-t)\varphi + t\psi)((1-t)x + ty) \leq (1-t)\varphi(x) + t\psi(y)$$

holds for all $t \in \boldsymbol{R}$ with $0 < t < 1$ and $x, y \in I$ with $x \leq y$.

The following theorem follows from Lemma 2.1, and it is the key for obtaining our conclusion.

Theorem 2.2 (see [2, Theorem 1]). *Let I and J be two interval of $\boldsymbol{R}$. Let $n \geq 2$ and $t_1, \ldots, t_n > 0$ with $\sum_{i=1}^n t_i = 1$. Suppose that $\varphi_1, \ldots, \varphi_n$ are real-valued functions on I such that $\sum_{i=1}^k t_i(\varphi_{k+1} - \varphi_i)$ is monotone increasing on I and φ_{k+1} is convex on I for each $k = 1, \ldots, n-1$, and that $\psi_1, \ldots, \psi_n$ are functions from J to I such that $\sum_{i=1}^k t_i(\psi_{k+1} - \psi_i) \geq 0$ on J for each $k = 1, \ldots, n-1$. Then*

$$\sum_{i=1}^n t_i(\varphi_i \circ \psi_i) \geq \sum_{i=1}^n t_i\varphi_i \circ \sum_{i=1}^n t_i\psi_i$$

holds on J, where $\circ$ denotes the composition of functions.

3. Conclusion

Let $\{x_1, \ldots, x_n\}$ and $\{y_i, \ldots, y_n\}$ be two monotone increasing or monotone decreasing sequences in $\mathbf{R}$. In Theorem 2.2, put $I = J = \mathbf{R}$, $\varphi_i(x) = x_ix$ $(resp. -x_ix)$ and $\psi_i(x) = y_i$ $(resp. -y_i)$ $(i = 1, \ldots, n, x \in \mathbf{R})$ for the increasing case (resp. the decreasing case). Then we can easily see that all conditions in Theorem 2.2 are satisfied. Hence we obtain from Theorem 2.2 the following weighted Chebyshev's inequality :

$$\sum_{i=1}^{n} t_i x_i y_i \geq \left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{i=1}^{n} t_i y_i\right) (t_1, \ldots, t_n > 0 : \sum_{i=1}^{n} t_i = 1).$$

We note that this is a restate of [2, Proof of Corollary 1].

Next let φ be convex function on a real interval I, $x_1, \ldots, x_n \in I$ with $x_1 \leq \cdots \leq x_n$ and $t_1, \ldots, t_n > 0$ with $\sum_{i=1}^{n} t_i = 1$. put $J = I, \varphi_i = \varphi$ $(i = 1, \ldots, n)$ and $\psi(x) = x_i (x \in I, i = 1, \ldots, n)$ in Theorem 2.2. Then we can easily see that conditions in Theorem 2.2 are satisfied, and hence the inequality in Theorem 2.2 holds. However since

$$\sum_{i=1}^{n} t_i(\varphi_i \circ \psi_i)(x) = \sum_{i=1}^{n} t_i \varphi(\psi_i(x)) = \sum_{i=1}^{n} t_i \varphi(x_i)$$

and

$$\left(\sum_{i=1}^{n} t_i \varphi_i \circ \sum_{i=1}^{n} t_i \psi_i\right)(x) = \varphi\left(\sum_{i=1}^{n} t_i \psi_i(x)\right) = \varphi\left(\sum_{i=1}^{n} t_i x_i\right)$$

hold for all $x \in I$, it follows that the inequality in Theorem 2.2 is just Jensen's inequality $J(n)$.

Thus, we see that Theorem 2.2 is the common parent of Jensen's inequality and Chebyshev's inequality.

References

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