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STRONG CONVEGENCE THEOREMS BY HYBRID METHODS FOR THE SPLIT FEASIBILITY PROBLEM IN BANACH SPACES

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Abstract. In this paper, we consider the split feasibility problem in Banach spaces. Then using the hybrid method and the shrinking projection method in mathematical programming, we prove strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. It seems that such theorems are first in Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [4] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [3] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [3] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0 \right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem in Hilbert spaces; see, for instance, [1, 3, 5, 7, 17].

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On the other hand, in 2003, Nakajo and Takahashi [8] proved the following strong convergence theorem by using the hybrid method in mathematical programming. Let C be a nonempty, closed and convex subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0, 1]$ is chosen so that $0 \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Takahashi, Takeuchi and Kubota [16] also obtained the following result by using the shrinking projection method:

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x \in H$. For $C_1 = C$ and $x_1 \in C$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, , \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \limsup_{n\to\infty} \alpha_n < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

In this paper, motivated by these problems and results, we consider the split feasibility problem in Banach spaces. Then using the hybrid method and the shrinking projection method in mathematical programming, we prove two strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. It seems that such theorems are first in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [13] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [11].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [11] and [12]. We know the following result.

Lemma 2.1 ([11]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.2 ([11]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C$.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [6] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [6]. The following lemma was proved by Tsukada [18].

Lemma 2.3 ([18]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main results

In this section, using the hybrid method in mathematical programming, we first prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

Theorem 3.1. *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_D Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$ and $0 < r\|A\|^2 < 2$. Then $\{x_n\}$ converges strongly to a point $z_1 \in C \cap A^{-1}D$, where $z_1 = P_{C\cap A^{-1}D} x_1$.

Proof. Since

$$\|y_n - z\|^2 \leq \|x_n - z\|^2$$

$$\Longleftrightarrow \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0,$$

it follows that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that Q_n is closed and convex. Then $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show that $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in C \cap A^{-1}D$. Then $z = P_C z$ and $Az = P_D Az$. Since P_C is nonexpansive, we have that for $z \in C \cap A^{-1}D$,

$$\begin{aligned}
\|z_n - z\|^2 &= \left\|P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big) - P_Cz\right\|^2 \\
&\leq \|x_n - rA^*J_F(Ax_n - P_DAx_n) - z\|^2 \\
&= \|x_n - z - rA^*J_F(Ax_n - P_DAx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, rA^*J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + \|rA^*J_F(Ax_n - P_DAx_n)\|^2 \\
&\leq \|x_n - z\|^2 - 2r\langle Ax_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + r^2\|A\|^2\|J_F(Ax_n - P_DAx_n)\|^2 \qquad (3.1) \\
&= \|x_n - z\|^2 - 2r\langle Ax_n - P_DAx_n + P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&\quad + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\
&= \|x_n - z\|^2 - 2r\|Ax_n - P_DAx_n\|^2 \\
&\quad - 2r\langle P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2r\|Ax_n - P_DAx_n\|^2 + r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 \\
&= \|x_n - z\|^2 + r(r\|A\|^2 - 2)\|Ax_n - P_DAx_n\|^2 \\
&\leq \|x_n - z\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|y_n - z\| &= \|\alpha_n x_n + (1-\alpha_n)z_n - z\| \\
&\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|z_n - z\| \\
&\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\
&\leq \|x_n - z\|.
\end{aligned}$$

Therefore, $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. Let us show that $C \cap A^{-1}D \subset Q_n$ for all $n \in \mathbb{N}$. It is obvious that $C \cap A^{-1}D \subset Q_1$. Suppose that $C \cap A^{-1}D \subset Q_k$ for some $k \in \mathbb{N}$. Then $C \cap A^{-1}D \subset C_k \cap Q_k$. From $x_{k+1} = P_{C_k \cap Q_k}x_1$, we have that

$$\langle x_{k+1} - z, x_1 - x_{k+1}\rangle \geq 0, \quad \forall z \in C_k \cap Q_k$$

and hence

$$\langle x_{k+1} - z, x_1 - x_{k+1}\rangle \geq 0, \quad \forall z \in C \cap A^{-1}D.$$

Then $C \cap A^{-1}D \subset Q_{k+1}$. We have by mathematical induction that $C \cap A^{-1}D \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that $C \cap A^{-1}D \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $C \cap A^{-1}D$ is nonempty, closed and convex, there exists $z_1 \in C \cap A^{-1}D$ such that $z_1 = P_{C \cap A^{-1}D}x_1$. From $x_{n+1} = P_{C_n \cap Q_n}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_n \cap Q_n$. Since $z_1 \in C \cap A^{-1}D \subset C_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_1\|. \tag{3.2}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of Q_n, we have that $x_n = P_{Q_n}x_1$. From $x_{n+1} = P_{C_n\cap Q_n}x_1$ we have $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in Q_n$ we have that

$$\langle x_n - x_{n+1}, x_1 - x_n\rangle \geq 0.$$

This implies from (2.3) that

$$0 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

From $x_{n+1} \in C_n$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (3.3) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

We have from (3.1) that for any $z \in C \cap A^{-1}D$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)z_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\quad + (1-\alpha_n)r(r\|A\|^2 - 2)\|Ax_n - P_DAx_n\|^2 \\
&\leq \|x_n - z\|^2 + (1-\alpha_n)r(r\|A\|^2 - 2)\|Ax_n - P_DAx_n\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
(1-\alpha_n)r(2-r\|A\|^2)\|Ax_n - P_DAx_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$ and $0 \leq \alpha_n \leq a < 1$, we have that

$$\lim_{n\to\infty} \|Ax_n - P_DAx_n\|^2 = 0. \tag{3.5}$$

We also have that $\|y_n - x_n\| = \|\alpha_n x_n + (1-\alpha_n)z_n - x_n\| = (1-\alpha_n)\|z_n - x_n\|$. From $\|y_n - x_n\| \to 0$ and $0 \leq \alpha_n \leq a < 1$, we have that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From (3.4) $\{y_{n_i}\}$ converges weakly to w. Furthermore, from (3.6)

$\{z_{n_i}\}$ converges weakly to w. We have from $z_{n_i} \in C$ and $x_{n_i} \rightharpoonup w$ that $w \in C$. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using this and $\lim_{n\to\infty} \|Ax_n - P_D Ax_n\| = 0$, we have that $P_D Ax_{n_i} \rightharpoonup Aw$. Since P_D is the metric projection of F onto D, we have from [2] and [13] that $\langle P_D Ax_n - P_D Aw, J_F(Ax_n - P_D Ax_n)\rangle \geq 0$ and

$$\langle P_D Aw - P_D Ax_n, J_F(Aw - P_D Aw)\rangle \geq 0$$

and hence

$$\langle P_D Ax_n - P_D Aw, J_F(Ax_n - P_D Ax_n) - J_F(Aw - P_D Aw)\rangle \geq 0.$$

Since $P_D Ax_{n_i} \rightharpoonup Aw$ and $\|J_F(Ax_n - P_D Ax_n)\| \to 0$, we have that

$$-\|Aw - P_D Aw\|^2 = \langle Aw - P_D Aw, -J_F(Aw - P_D Aw)\rangle \geq 0$$

and hence $Aw = P_D Aw$. This implies that $w \in C \cap A^{-1}D$.

From $z_1 = P_{C\cap A^{-1}D} x_1$, $w \in C \cap A^{-1}D$ and (3), we have that

$$\begin{aligned}\|x_1 - z_1\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_1\|.\end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_1\|.$$

Since H satisfies the Kadec-Klee property, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_1.$$

Therefore, we have $x_n \to w = z_1$. This completes the proof. □

Next, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [16], we prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

Theorem 3.2. *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$ and $C_1 = H$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_D Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$ and $0 < r\|A\|^2 < 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D} u$.

Proof. It is obvious that $C \cap A^{-1}D \subset C_1 = H$ and C_1 is closed and convex. Suppose that $C \cap A^{-1}D \subset C_k$ and C_k is closed and convex for some $k \in \mathbb{N}$. Since $z_k = P_C\Big(x_k - rA^*J_F(Ax_k - P_DAx_k)\Big)$ with $0 < r\|A\|^2 < 2$, as in the proof of Theorem 3.1, we have that for $z \in C \cap A^{-1}D$,

$$\begin{aligned}
\|z_k - z\|^2 &= \|P_C\Big(x_k - rA^*J_F(Ax_k - P_DAx_k)\Big) - P_Cz\|^2 \\
&\le \|x_k - rA^*J_F(Ax_k - P_DAx_k) - z\|^2 \\
&\le \|x_k - z\|^2 - 2r\|Ax_k - P_DAx_k\|^2 \\
&\qquad - 2r\langle P_DAx_k - Az, J_F(Ax_k - P_DAx_k)\rangle + r^2\|A\|^2\|Ax_k - P_DAx_k\|^2 \\
&\le \|x_k - z\|^2 - 2r\|Ax_k - P_DAx_k\|^2 + r^2\|A\|^2\|Ax_k - P_DAx_k\|^2 \\
&= \|x_k - z\|^2 + r(r\|A\|^2 - 2)\|Ax_k - P_DAx_k\|^2 \\
&\le \|x_k - z\|^2
\end{aligned} \tag{3.7}$$

and hence

$$\begin{aligned}
\|y_k - z\|^2 &= \|\alpha_k x_k + (1 - \alpha_k)z_k - z\|^2 \\
&\le \|x_k - z\|^2.
\end{aligned}$$

Therefore, $C \cap A^{-1}D \subset C_{k+1}$. Moreover, since

$$\begin{aligned}
\{z \in H : \|y_k - z\| \le \|x_k - z\|\} &= \{z \in H : \|y_k - z\|^2 \le \|x_k - z\|^2\} \\
&= \{z \in H : \|y_k\|^2 - \|x_k\|^2 \le 2\langle y_k - x_k, z\rangle\},
\end{aligned}$$

$C_{k+1} = \{z \in H : \|y_k - z\| \le \|x_k - z\|\} \cap C_k$ is closed and convex. So $x_{k+1} = P_{C_{k+1}}u_{k+1}$ is well defined. Applying these facts inductively, we obtain that C_n is nonempty, closed and convex and $C \cap A^{-1}D \subset C_n$ for every $n \in \mathbb{N}$. Hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset C \cap A^{-1}D \neq \emptyset$, C_0 is also nonempty. Let $w_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.3, we have $w_n \to w_0 = P_{C_0}u$. Since a metric projection on H is nonexpansive, it follows that

$$\begin{aligned}
\|x_n - w_0\| &\le \|x_n - w_n\| + \|w_n - w_0\| \\
&= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - w_0\| \\
&\le \|u_n - u\| + \|w_n - w_0\|
\end{aligned}$$

and hence $x_n \to w_0$.

Since $w_0 \in C_0 \subset C_{n+1}$, we have $\|y_n - w_0\| \le \|x_n - w_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $y_n \to w_0$. Then we have that

$$\|x_n - y_n\| \le \|x_n - w_0\| + \|w_0 - y_n\| \to 0. \tag{3.8}$$

From $y_n - x_n = \alpha_n x_n + (1 - \alpha_n)z_n - x_n = (1 - \alpha_n)(z_n - x_n)$, we also have that

$$\|y_n - x_n\| = (1 - \alpha_n)\|z_n - x_n\| \ge (1 - a)\|z_n - x_n\|$$

and hence

$$\|z_n - x_n\| \to 0. \tag{3.9}$$

On the other hand, from (3.7) we know that $z \in C \cap A^{-1}D$,

$$\|z_n - z\|^2 \leq \|x_n - z\|^2 + r(r\|A\|^2 - 2)\|Ax_n - P_D Ax_n\|^2.$$

Then we get that

$$\begin{aligned} r(2 - r\|A\|^2)\|Ax_n - P_D Ax_n\|^2 &\leq \|x_n - z\|^2 - \|z_n - z\|^2 \\ &= (\|x_n - z\| - \|z_n - z\|)(\|x_n - z\| + \|z_n - z\|) \\ &\leq \|x_n - z_n\|(\|x_n - z\| + \|z_n - z\|). \end{aligned}$$

Since $0 < r\|A\|^2 < 2$ and $\|x_n - z_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|Ax_n - P_D Ax_n\| = 0. \tag{3.10}$$

Since $\{x_n\}$ converges atrongly to w_0, from (3.9) $\{z_n\}$ converges strongly to w_0. Since $z_n \in C$, we have that $w_0 \in C$. Since A is bounded and linear, we also have that $\{Ax_n\}$ converges strongly to Aw_0. Using this and $\lim_{n\to\infty} \|Ax_n - P_D Ax_n\| = 0$, we have that $P_D Ax_{n_i} \to Aw_0$. Since P_D is the metric projection of F onto D, as in the proof of Theorem 3.1, we have that

$$\langle P_D Ax_n - P_D Aw_0, J_F(Ax_n - P_D Ax_n) - J_F(Aw_0 - P_D Aw_0)\rangle \geq 0.$$

Since $P_D Ax_n \to Aw_0$ and $\|J_F(Ax_n - P_D Ax_n)\| \to 0$, we have that

$$-\|Aw_0 - P_D Aw_0\|^2 = \langle Aw_0 - P_D Aw_0, -J_F(Aw_0 - P_D Aw_0)\rangle \geq 0$$

and hence $Aw = P_D Aw$. Then $w_0 \in A^{-1}D$. This implies that $w_0 \in C \cap A^{-1}D$.

Since $C \cap A^{-1}D$ is nonempty, closed and convex, there exists $z_0 \in C \cap A^{-1}D$ such that $z_0 = P_{C\cap A^{-1}D}u$. From $x_{n+1} = P_{C_{n+1}}u$, we have that

$$\|u - x_{n+1}\| \leq \|u - y\|$$

for all $y \in C_{n+1}$. Since $z_0 \in C \cap A^{-1}D \subset C_{n+1}$, we have that

$$\|u - x_{n+1}\| \leq \|u - z_0\|.$$

From $x_n \to w_0$ we have that

$$\|u - w_0\| \leq \|u - z_0\|. \tag{3.11}$$

From $z_0 = P_{C\cap A^{-1}D}u$, $w_0 \in C \cap A^{-1}D$ and (30), we have $w_0 = z_0$. Therefore, we have $x_n \to w_0 = z_0$. This completes the proof. □

We do not know whether a Hilbert space H in Theorems 3.1 and 3.2 is replaced by a Banach space E or not.

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