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FIXED POINT THEOREMS IN MODULAR SPACES WITH SIMULATION FUNCTIONS AND ALTERING DISTANCE FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we introduce three different contractive conditions and prove some new fixed point theorems in ω-complete modular spaces via altering distance functions and stimulation functions. Moreover, we give an application on solving integral equation. Our results generalize and extend some results in [3, 19, 32].

1. Introduction

As well as we know, functional analysis is made up of two main methods which are variational methods, degree methods and fixed point methods. It is well known that fixed point method is a very useful tool on solving the differential equations, integral equations and so on. Fixed point theory has been studied by many scholars in different spaces for example, partial metric spaces [14, 31, 33], G-metric spaces [6, 7, 17], metric-like spaces [2, 8, 30, 36, 37], modular spaces [1, 3, 12, 19] and so on. Especially, In 1950, Nakano [23] initiated the theory of modular spaces and further it was generalized and redefined by Musielak and Orlicz [21, 22] in 1959. In 2010, Chistyakov [10] introduced the notions of modular spaces which is considered to be an interesting generalization of metric spaces. At the same time, the author proved some fixed point theorems in ω-complete modular spaces. The purpose of this paper is to prove some new fixed point theorems for compatible contractive mappings and cycle $\mathbb{Z}_\alpha$-contractive mappings in ω-complete modular spaces via altering distance function. Our results generalize and extend some results in [3, 19, 32].

Throughout this paper, Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and

$$\Theta_1 = \Big\{\phi | \phi : \mathbb{R}^+ \to \mathbb{R}^+ \text{is Lebesgue integral, summable on each compact subset of } \mathbb{R}^+ \text{and} \int_0^\epsilon \phi(t)dt > 0 \text{ for each } \epsilon > 0\Big\}$$

Let Φ be the set of all non-decreasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ that satisfy the following conditions:

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(1) φ is lower semi-continuous on $[0, +\infty)$;
(2) $\varphi(0) = 0$;
(3) $\varphi(s) > 0$ for each $s > 0$.

Let X be a nonempty set. Throughout this paper for a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y), \tag{1.1}$$

for all $\lambda > 0$ and $x, y \in X$. Now, we recall some basic concepts, notations and known results of modular spaces which will be used in the sequel.

Definition 1.1 ([10])**.** A mapping $\omega_\lambda : X \times X \to [0, \infty]$ where X is a nonempty set, is said to be a modular metric on X if for any $x, y, z \in X$ the following three conditions hold true:

(ω1) $\omega_\lambda(x, y) = 0$ if and only if $x = y$, for all $\lambda > 0$;
(ω2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$ and $x, y \in X$;
(ω3) $\omega_{\lambda+\mu}(x, z) \leq \omega_\lambda(x, y) + \omega_\mu(y, z)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If $\omega_\lambda(x, y) = \omega(x, y)$ dose not depend on $\lambda > 0$ and assumes only finite values, then axiom (ω1)-(ω3) mean that ω is a metric on X.

If instead of (ω1) we have only the condition ($\omega 1'$)

$$\omega_\lambda(x, x) = 0 \text{ for all } \lambda > 0,\ x \in X,$$

then ω_λ is said to be a pseudomodular (metric) on X. A pseudomodular metric ω_λ on X is said to be regular if the following weaker version of (ω1) is satisfied:

$$x = y \text{ if and only if } w_\lambda(x, y) = 0 \text{ for some } \lambda > 0.$$

Definition 1.2 ([10])**.** Let ω be a pseudomodular on X. Fix $x_0 \in X$, the set

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

is said to be modular space (around x_0).

Definition 1.3 ([10])**.** Let X_ω be a modular metric space.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$, as $n \to \infty$ for some $\lambda > 0$;
(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to \infty$ for some $\lambda > 0$;
(3) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence in C is a convergent sequence and its limit is in C.

Example 1.4. Let (X, d) is a (pseudo)metric space with (pseudo)metric d, $\lambda > 0$ and $x, y \in X$. then the following are (pseudo)modular metrics

(a) $\omega^a_\lambda(x, y) = d(x, y)/\varphi(\lambda)$ where $\varphi : (0, \infty) \to (0, \infty)$ is a nondecreasing function;
(b) $\omega^b_\lambda(x, y) = \infty$ if $\lambda \leq d(x, y)$, and $\omega^b_\lambda(x, y) = 0$ if $\lambda > d(x, y)$;
(c) $\omega^c_\lambda(x, y) = \infty$ if $\lambda < d(x, y)$, and $\omega^c_\lambda(x, y) = 0$ if $\lambda \geq d(x, y)$.

Definition 1.5 ([13])**.** An altering distance function is a function $\rho : [0, +\infty) \to [0, +\infty)$ which is increasing, continuous and $\rho(t) = 0$ if and only if $t = 0$.

Fixed point problems involving altering distances have also been studied in [13, 20,25,27]. In [26], the following lemma shows that contractive conditions of integral type can be interpreted as contractive conditions involving an altering distance.

Lemma 1.6 ([26])**.** *Let* $\phi \in \Theta_1$. *Define* $\Phi_0(t) = \int_0^t \phi(s)ds$. *Then* Φ_0 *is an altering distance function.*

Very recently, in [15], Khojasteh et al. introduced the notions of simulation functions. Based on the work of [15], many authors used this notions to prove the existence of fixed point in various spaces, such as partial metric spaces [24], quasi-metric spaces [5], b-metric spaces [11] and other related paper [9].

Definition 1.7 ([15])**.** A simulation function is a mapping $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(ξ_1) $\xi(0,0) = 0$;
(ξ_2) $\xi(t,s) < s - t$ for all $t, s > 0$;
(ξ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then
$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

Let $\mathbb{Y}$ be the family of all simulation functions $\xi : [0,\infty)\times[0,\infty) \to \mathbb{R}$ in Definition 1.7.

However, in [29], the authors slightly modified the definition of simulation function which introduced by Khojasteh et al. [15] and enlarged the family of all simulation functions.

Definition 1.8 ([29])**.** A simulation function is a mapping $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:
(ξ_1) $\xi(0,0) = 0$;
(ξ_2) $\xi(t,s) < s - t$ for all $t, s > 0$;
(ξ_3') if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then
$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

Let $\mathbb{Z}$ be the family of all simulation functions $\xi : [0,\infty)\times[0,\infty) \to \mathbb{R}$ in Definition 1.8.

Thereupon, the authors [29] give an example to illustrate that every simulation function in the original Khojasteh et al.'s sense (Definition 1.7) is also a simulation function in Roldán-López-de-Hierro et al.'s sense (Definition 1.8), but the converse is not true.

Lemma 1.9. *Let* $\{t_n\}$ *and* $\{s_n\}$ *be two sequences in* $[0,\infty)$, $\xi \in \mathbb{Y}$. *If* $\xi(t_n, s_n) \geq 0$ *and* $t_n > 0$, $s_n > 0$ *then*
(1) $t_n < s_n$;
(2) *if* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n$, *then* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = 0$.

Proof. Let's proof it by reduction to absurdity.
(1) If $t_n \geq s_n$, then by (ξ_2) we get
$$\xi(t_n, s_n) < s_n - t_n.$$

With the $s_n - t_n \leq 0$, we have $\xi(t_n, s_n) < 0$, a contradiction with $\xi(t_n, s_n) \geq 0$.

(2) If $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, by (ξ_3) we get

$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

which is a contradiction with $\xi(t_n, s_n) \geq 0$.

□

Remark 1.10. From the Lemma 1.9 we can know that if the hypotheses of Lemma 1.9 are satisfied then the $\xi \in \mathbb{Y}$ actually only need $\xi \in \mathbb{Z}$.

In [29], the authors show a wide range of examples to highlight the usefulness of Definition 1.8 as follows.

Example 1.11. Let φ and ψ be two altering distance functions such that $\psi(t) < t \leq \varphi(t)$ for all $t > 0$. Then the mapping

$$\xi_1(t, s) = \psi(s) - \varphi(t),$$

for all $t, s \in [0, \infty)$ is a simulation function.

If in the previous example, $\varphi(t) = t$ and $\psi(t) = \mu t$ for all $t \geq 0$, where $\mu \in [0, 1)$, then we obtain the following particular case of simulation function:

$$\xi_B(t, s) = \mu s - t,$$

for all $t, s \in [0, \infty)$.

Example 1.12. If $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\varphi^{-1}(0) = \{0\}$ and we define $\xi_R : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\xi_R(t, s) = s - \varphi(s) - t,$$

for all $s, t \in [0, \infty)$, then ξ_R is a simulation function.

If, in the previous example, φ is continuous, we deduce the following case.

Example 1.13. If $\varphi : [0, \infty) \to [0, \infty)$ is continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$, and we define $\xi_R : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\xi_R(t, s) = s - \varphi(s) - t,$$

for all $s, t \in [0, \infty)$, then ξ_R is a simulation function.

Example 1.14. Let $f, g : [0, \infty) \to [0, \infty)$ be two continuous functions such that with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$ and define $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\xi(t, s) = s - \frac{f(t, s)}{g(t, s)} t,$$

for all $s, t \in [0, \infty)$, then ξ is a simulation function.

Example 1.15. If $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $r > 0$ and $\eta(0) = 0$, and we define $\xi_{BW} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\xi_{BW}(t,s) = \eta(s) - t,$$

for all $s,t \in [0,\infty)$, then ξ_{BW} is a simulation function.

Example 1.16. If $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and for all $\int_0^\varepsilon \phi(u)du > \varepsilon$, for each $\varepsilon > 0$, and we define $\xi_{BW} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\xi_K(t,s) = s - \int_0^t \phi(u)du,$$

for all $s,t \in [0,\infty)$, then ξ_K is a simulation function.

2. Common fixed point results for compatible contractive mappings

In this section, we present some new fixed point theorems for compatible contractive mappings in modular spaces. Before stating our main results, we need to give the following definitions.

Definition 2.1. Let (X,ω_λ) be a modular metric space and Let C be a nonempty subset of X. If $A,S : C \to C$ be two mappings, then A and S are said to be:

(1) commuting if $ASx = SAx$, for all $x \in C$;
(2) weakly commuting if $\omega_\lambda(ASx, SAx) \le \omega_\lambda(Ax, Sx)$, for all $x \in C$;
(3) compatible if $\lim_{n\to\infty} \omega_\lambda(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$;
(4) non-compatible if there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ but

$$\lim_{n\to\infty} \omega_\lambda(ASx_n, SAx_n),$$

is either nonzero or nonexistent;
(5) weakly compatible if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$, for some $x \in C$.

Definition 2.2. Let (X,ω) be a modular metric space, C be an arbitrary subset of X and let A,B,S,T be mappings from C into C. Then

(1) the pair (A,S) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$$

for some $z \in C$;
(2) the pairs (A,S) and (B,T) are said to share the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in C$;

(3) the pair (A, S) is said to have the common limit range property with respect to the mapping S (denoted by (CLR_S)) if there exists a sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$$

with $z = Sv$, for some $z, v \in C$;

(4) the pairs (A, S) and (B, T) are said to have the common limit range property (with respect to mappings S and T)(denoted by (CLR_{ST})), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

with $z = Sv = Tu$, for some $z, v, u \in C$.

Definition 2.3 ([4])**.** Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

(1) $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \dots, m\}$;
(2) $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \dots, n\}$;
(3) $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.

Now, we give the following lemma.

Lemma 2.4. *Let (X, ω_λ) be a modular metric space and C be a nonempty subsets of X. A, B, S and T be self mappings of C. Suppose that the following hypotheses hold:*

(i) *the pair (A, S) satisfies the (CLR_S) property (or the pair (B, T) satisfies the (CLR_T) property);*
(ii) *$A(C) \subset T(C)$ (or $B(C) \subset S(C)$);*
(iii) *$T(C)$ or $S(C)$ is a closed subset of C;*
(iv) *$\{By_n\}$ converges for every sequence $\{y_n\}$ in C whenever $\{Ty_n\}$ converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in C whenever $\{Sx_n\}$ converges);*
(v) *there exists $\xi \in \mathbb{Z}$ such that*

$$\xi\big(\rho(\omega_\lambda(Ax, By)), \rho(M_\omega(x, y))\big) \geq 0, \tag{2.1}$$

for all $x, y \in C$, where ρ is an altering distance function and

$$M_\omega(x, y) = \max\Big\{\omega_\lambda(Ax, Sx), \omega_\lambda(By, Ty), \omega_\lambda(Sx, Ty), \frac{\omega_\lambda(Ax, Ty) + \omega_\lambda(By, Sx)}{2}\Big\}.$$

Then the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. Since the pair (A, S) satisfies the (CLR_S) property, there exists a sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(C)$. By (ii), $A(C) \subset T(C)$(wherein $T(C)$ is a closed subset of C), and for each $\{x_n\} \subset C$, there corresponds a sequence $\{y_n\} \subset C$ such that $Ax_n = Ty_n$. Hence,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z,$$

where $z \in S(C) \cap T(C)$. Thus, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. By (iv), the sequence $\{By_n\}$ converges and in all we need to show that $By_n \to z$ as

$n \to \infty$ in the rest of paper. Now, putting $x = x_n$ and $y = y_n$ in condition (2.1), we have

$$0 \le \xi\left(\rho(\omega_\lambda(Ax_n, By_n)), \rho(M_\omega(x_n, y_n))\right), \tag{2.2}$$

where

$$M_\omega(x_n, y_n) = \max\Big\{\omega_\lambda(Ax_n, Sx_n), \omega_\lambda(By_n, Ty_n), \omega_\lambda(Sx_n, Ty_n), \frac{\omega_\lambda(Ax_n, Ty_n) + \omega_\lambda(By_n, Sx_n)}{2}\Big\}. \tag{2.3}$$

If we assume that $By_n \to m$ as $n \to \infty$. Thus taking limit as $n \to \infty$ we have

$$\lim_{n\to\infty} \omega_\lambda(Ax_n, By_n) = \omega_\lambda(m, z), \tag{2.4}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} M_\omega(x_n, y_n)\\
&\quad = \lim_{n\to\infty} \max\{\omega_\lambda(Ax_n, Sx_n), \omega_\lambda(By_n, Ty_n), \omega_\lambda(Sx_n, Ty_n),\\
&\quad \frac{\omega_\lambda(Ax_n, Ty_n) + \omega_\lambda(By_n, Sx_n)}{2}\}\\
&\quad = \max\left\{\omega_\lambda(z, z), \omega_\lambda(m, z), \omega_\lambda(z, z), \frac{\omega_\lambda(z, z) + \omega_\lambda(m, z)}{2}\right\}\\
&\quad = \omega_\lambda(m, z).
\end{aligned} \tag{2.5}$$

From the Lemma 1.9 we get $\omega_\lambda(m, z) = 0$ that is $z = m$. Hence both the pairs (A, S) and (B, T) share the (CLR_{ST}) property. The proof is completed. ☐

Next, we are ready to state and prove the following theorems.

Theorem 2.5. *Let (X, ω_λ) be a modular metric space and C be a nonempty subsets of X. A, B, S and T be self mappings of C satisfying the hypothesis (v) of Lemma 2.4. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S, and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(C) \cap T(C)$. Since $z \in S(C)$, there exist a point $u \in C$ such that $Su = z$. Choose $x = u$ and $y = y_n$ in condition (2.1), one has

$$\xi\big(\rho(\omega_\lambda(Au, By_n)), \rho(M_\omega(u, y_n))\big) \ge 0 \tag{2.6}$$

where

$$M_\omega(u, y_n) = \max\Big\{\omega_\lambda(Au, Su), \omega_\lambda(By_n, Ty_n), \omega_\lambda(Su, Ty_n), \frac{\omega_\lambda(Au, Ty_n) + \omega_\lambda(By_n, Su)}{2}\Big\}.$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} \omega_\lambda(Au, By_n)) = \omega_\lambda(Au, z), \tag{2.7}$$

and

$$\begin{aligned}\lim_{n\to\infty} & M_\omega(u, y_n) \\ &= \lim_{n\to\infty} \max\{\omega_\lambda(Au, Sx_n), \omega_\lambda(By_n, Ty_n), \omega_\lambda(Su, Ty_n), \\ &\quad \frac{\omega_\lambda(Au, Ty_n) + \omega_\lambda(By_n, Su)}{2}\} \\ &= \max\left\{\omega_\lambda(Au, z), \omega_\lambda(z, z), \omega_\lambda(z, z), \frac{\omega_\lambda(Au, z) + \omega_\lambda(z, z)}{2}\right\} \\ &= \omega_\lambda(Au, z).\end{aligned}$$

From the Lemma 1.9 (2). it follows easily that $Au = z$. Therefore $Au = Su = z$ which implies that u is a coincidence point of the pair (A, S).

As $z \in T(C)$, there exists a point $v \in C$ such that $Tv = z$. Putting $x = x_n$ and $y = v$ in condition (2.1), one has

$$\xi\big(\rho(\omega_\lambda(Ax_n, Bv)), \rho(M_\omega(x_n, v))\big) \geq 0 \tag{2.8}$$

where

$$\begin{aligned}M_\omega(x_n, v) = \max\Big\{&\omega_\lambda(Ax_n, Sx_n), \omega_\lambda(Bv, Tv), \omega_\lambda(Sx_n, Tv), \\ &\frac{\omega_\lambda(Ax_n, Tv) + \omega_\lambda(Bv, Sx_n)}{2}\Big\}.\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \omega_\lambda(Ax_n, Bv)) = \omega_\lambda(z, Bv), \tag{2.9}$$

and

$$\begin{aligned}\lim_{n\to\infty} M_\omega(x_n, v) &= \limsup_{n\to\infty} \max\Big\{\omega_\lambda(Ax_n, Sx_n), \omega_\lambda(Bv, Tv), \omega_\lambda(Sx_n, Tv), \\ &\qquad \frac{\omega_\lambda(Ax_n, Tv) + \omega_\lambda(Bv, Sx_n)}{2}\Big\} \\ &= \max\left\{\omega_\lambda(z, z), \omega_\lambda(Bv, z), \omega_\lambda(z, z), \frac{\omega_\lambda(z, z) + \omega_\lambda(Bv, z)}{2}\right\} \\ &= \omega_\lambda(z, Bv).\end{aligned}$$

It follows easily that $Bv = z$ by Lemma 1.9 (2). Therefore $Bv = Tv = z$ which implies that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible, $Au = Su$ and $Bv = Tv$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$. Putting $x = z$ and $y = v$ in condition (2.1), we have

$$\xi\big(\rho(\omega_\lambda(Az, z)), \rho(M_\omega(z, v))\big) = \xi\big(\rho(\omega_\lambda(Az, Bv)), \rho(M_\omega(z, v))\big) \geq 0, \tag{2.10}$$

where

$$\begin{aligned}M_\omega(z, v) &= \max\left\{\omega_\lambda(Az, Sz), \omega_\lambda(Bv, Tv), \omega_\lambda(Sz, Tv), \frac{\omega_\lambda(Az, Tv) + \omega_\lambda(Bv, Sz)}{2}\right\} \\ &= \omega_\lambda(Az, z).\end{aligned}$$

If $z \neq Az$, then $\omega_\lambda(Az, z) > 0$. By (2.10) and ($\xi 2$), we have

$$0 \leq \xi\big(\rho(\omega_\lambda(Az, z)), \rho(M_\omega(z, v))\big) < \rho(\omega_\lambda(Az, z)) - \rho(\omega_\lambda(Az, z)) = 0, \tag{2.11}$$

which is a contradiction. It follows easily that $Sz = Az = z$ and therefore z is a common fixed point of the pair (A, S). Next, we put $x = u$ and $y = z$ in condition (2.1). So we have

$$\xi\big(\rho(\omega_\lambda(z, Bz)), \rho(M_\omega(u, z))\big) = \xi\big(\rho(\omega_\lambda(Au, Bz)), \rho(M_\omega(u, z))\big) \geq 0, \tag{2.12}$$

where

$$\begin{aligned} M_\omega(u, z) &= \max\Big\{\omega_\lambda(Au, Su), \omega_\lambda(Bz, Tz), \omega_\lambda(Su, Tz), \frac{\omega_\lambda(Au, Tz) + \omega_\lambda(Bz, Su)}{2}\Big\} \\ &= \omega_\lambda(z, Bz). \end{aligned}$$

If $z \neq Bz$, then $\omega_\lambda(z, Bz) > 0$. By (2.12) and (ξ_2), we have

$$0 \leq \xi\big(\rho(\omega_\lambda(z, Bz)), \rho(M_\omega(u, z))\big) < \rho(\omega_\lambda(z, Bz)) - \rho(\omega_\lambda(z, Bz)) = 0, \tag{2.13}$$

which is a contradiction. It follows easily that $Bz = Tz = z$. Therefore z is a common fixed point of the pair (B, T) and hence z is a common fixed point of A, B, S and T. The uniqueness of the common fixed point is an easy consequence of condition (2.1) and so, to avoid repetition, we omit the details. The proof is completed. □

Theorem 2.6. *Let (X, ω_λ) be a modular metric space and C be a nonempty subsets of X. A, B, S and T be self mappings of C satisfying all hypotheses of Lemma 2.4. Then A, B, S, and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. At first, we assured that the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in C such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(C) \cap T(C)$. The rest of the proof runs on the lines of the proof of Theorem 2.5, therefore the details are avoided. The proof is completed. □

Next, by choosing A, B, S and T suitably, we can deduce some corollaries for a pair as well as for a triode of self mappings. First of all, based on Example 1.12, and Lemma 1.6 we can get the following corollary.

Corollary 2.7. *Let (X, ω_λ) be a modular metric space and C be a nonempty subsets of X. A, B, S and T be self mappings of C. Assume that C is ω-complete. Suppose that the following hypotheses hold:*

(1) *the pair (A, S) satisfies the (CLR_S) property (or the pair (B, T) satisfies the (CLR_T) property);*
(2) *$A(C) \subset T(C)$ (or $B(C) \subset S(C)$);*
(3) *$T(C)$ or $S(C)$ is a closed subset of C;*
(4) *$\{By_n\}$ converges for every sequence $\{y_n\}$ in C whenever $\{Ty_n\}$ converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in C whenever $\{Sx_n\}$ converges);*
(5) *there exist $\varphi \in \Phi$, $\phi \in \Theta_1$ such that*

$$\int_0^{\omega_\lambda(Ax,By)} \phi(t)dt \leq \int_0^{M_\omega(x,y)} \phi(t)dt - \varphi\Big(\int_0^{M_\omega(x,y)} \phi(t)dt\Big),$$

for all $x, y \in X$, where ρ is an altering distance function and

$$M_\omega(x,y) = \max\Big\{\omega_\lambda(Ax,Sx), \omega_\lambda(By,Ty), \omega_\lambda(Sx,Ty), \frac{\omega_\lambda(Ax,Ty)+\omega_\lambda(By,Sx)}{2}\Big\}.$$

Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, if (A,S) and (B,T) are weakly compatible, then it has a unique common fixed point in C.

Remark 2.8. The Corollary 2.7. generalizes the results of Vetro et al. [32] which generalized the results of Zhang and Song [35].

If we chose $B = A$ and $T = S$, then we can get the following corollary.

Corollary 2.9. *Let (X, ω_λ) be a modular metric space and C be a nonempty subsets of X. A and S be self mappings of C. Suppose that the following hypotheses hold:*

(i) *the pair (A,S) satisfies the (CLR_S) property,*

(ii) *there exist $\xi \in \mathbb{Z}$ such that*

$$\xi\big(\rho(\omega_\lambda(Ax,Ay)), \rho(M_\omega(x,y))\big) \geq 0, \tag{2.14}$$

for all $x, y \in C$, where ρ is an altering distance function and

$$M_\omega(x,y) = \max\Big\{\omega_\lambda(Ax,Sx), \omega_\lambda(Ay,Sy), \omega_\lambda(Sx,Sy), \frac{\omega_\lambda(Ax,Sy)+\omega_\lambda(Ay,Sx)}{2}\Big\}.$$

Then the pairs (A,S) have a coincidence point. Moreover, if (A,S) is weakly compatible, then it has a unique common fixed point in C.

Now, we utilize Definition 2.3 to prove a common fixed point theorem for six mappings in a modular metric space.

Theorem 2.10. *Let (X, ω) be a modular metric space and C be a nonempty subsets of X. A, B, H, R, S and T be self mappings of C. Suppose that the following hypotheses hold:*

(1) *the pairs (A,SR) and (B,TH) satisfies the $(CLR_{(SR)(TH)})$ property;*

(2) *there exists $\xi \in \mathbb{Z}$ such that*

$$\xi\big(\rho(\omega_\lambda(Ax,By)), \rho(M_\omega(x,y))\big) \geq 0, \tag{2.15}$$

for all $x, y \in C$, where ρ is an altering distance function and

$$M_\omega(x,y) = \max\Big\{\omega_\lambda(Ax,SRx), \omega_\lambda(By,THy), \omega_\lambda(SRx,THy), \frac{\omega_\lambda(Ax,THy)+\omega_\lambda(By,SRx)}{2}\Big\}.$$

Then the pairs (A,SR) and (B,TH) have a coincidence point each. Moreover, A, B, H, R, S, and T have a unique common fixed point provided both the pairs (A,SR) and (B,TH) commute pairwise, that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$.

Proof. Since the pairs (A,SR) and (B,TH) are commuting pairwise, obviously both the pairs are weakly compatible. By Theorem 2.6, A, B, SR, and TH have a unique common fixed point z in C. Now, we show that z is the unique common fixed point of the self mappings A, B, H, R, S of C, and T . Putting $x = Rz$ and $y = z$ in condition (2.15), we get

$$\xi\big(\rho(\omega_\lambda(Rz,z)),\rho(M_\omega(Rz,z))\big) = \xi\big(\rho(\omega_\lambda(A(Rz),Bz)),\rho(M_\omega(Rz,z))\big) \geq 0, \tag{2.16}$$

where

$$\begin{aligned} M_\omega(Rz,z) = \max\Big\{&\omega_\lambda(A(Rz),SR(Rz)),\omega_\lambda(Bz,THz),\omega_\lambda(SR(Rz),THz),\\ &\frac{\omega_\lambda(A(Rz),THz)+\omega_\lambda(Bz,SR(Rz))}{2}\Big\}\\ =\ &\omega_\lambda(Rz,z). \end{aligned}$$

If we assume that $Rz \neq z$, then $\omega_\lambda(Rz,z) > 0$. Therefore, by (2.16)and (ξ_2), we get

$$0 \leq \xi\big(\rho(\omega_\lambda(Rz,z)),\rho(M_\omega(Rz,z))\big) < \rho(\omega_\lambda(Rz,z)) - \rho(\omega_\lambda(Rz,z)) = 0, \tag{2.17}$$

which is a contradiction. Thus $Rz = z$ which implies that $S(Rz) = Sz = z$. Similarly, one can prove that $z = Hz$, that is, $T(Hz) = Tz = z$. Hence $z = Az = Bz = Sz = Rz = Tz = Hz$, and z is the unique common fixed point of A, B, H, R, S and T. □

Corollary 2.11. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_h\}_{h=1}^q$ be four finite families of self mappings on subset C of a modular metric space X with $A = A_1A_2\ldots A_m$, $B = B_1B_2\ldots B_n$, $S = S_1S_2\ldots S_p$, and $T = T_1T_2\ldots T_q$ satisfying hypothesis (v) of Lemma 2.4 such that the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property, then (A,S) and (B,T) have a point of coincidence each. Moreover, $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_h\}_{h=1}^q$ have a unique common fixed point if the pairs of families ($\{A_i\}$, $\{S_k\}$) and ($\{B_r\}$, $\{T_h\}$) commute pairwise wherein $i \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,p\}$, $r \in \{1,2,\ldots,n\}$, and $h \in \{1,2,\ldots,q\}$.*

Corollary 2.12. *Let (X,ω_λ) be a modular metric space and C be a nonempty subsets of X. A, B, S and T be self mappings of C. Suppose that the following hypotheses hold*

(1) *the pairs (A^m,S^p) and (B^n,T^q) satisfies the $(CLR_{(S^p)(T^q)}$ property;*
(2) *there exists $\xi \in \mathbb{Z}$ such that*

$$\xi\big(\rho(\omega_\lambda(A^m x,B^n y)),\rho(M_\omega(x,y))\big) \geq 0, \tag{2.18}$$

for all $x,y \in C$, where ρ is an altering distance function and

$$M_\omega(x,y) = \max\Big\{\omega_\lambda(A^m x,S^p x),\omega_\lambda(B^n y,T^q y),\omega_\lambda(S^p x,T^q y), \frac{\omega_\lambda(A^m x,T^q y)+\omega_\lambda(B^n y,S^p x)}{2}\Big\}.$$

Then A, B, S, and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

3. Fixed point theorems for cyclic $\mathbb{Z}_\alpha$-contractive mappings

In this section, we introduce cyclic $\mathbb{Z}_\alpha$-contractive mappings and prove the existence of fixed point in modular metric spaces. Cyclic contractive condition was introduced by Kirk et al. [16] and generalized by many scholars [18,28,34]. The notion of α-admissibility was defined by Salimi et al. [30] and many authors [3,10,13] applied α-admissibility to some various metric spaces and get some generalized fixed point theorems.

Definition 3.1 ([30]). Let $F : X \to X$ and $\alpha : X \times X \to [0, \infty)$. The mapping F is said to be α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Fx, Fy) \geq 1. \tag{3.1}$$

Definition 3.2. Let (X, ω_λ) be a modular space. Let $\alpha : X \times X \to [0, \infty)$ and $F : X \to X$. We say that F is α-continuous on X, if

$$x_n \to x \text{ as } n \to \infty, \qquad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \Rightarrow Fx_n \to Fx.$$

Definition 3.3. Let (X, ω_λ) be a modular space. A is said to be ω-closed subsets of X, if for any sequence $\{x_n\} \subset A$ such that $x_n \to x$ as $n \to \infty$, we have $x \in A$.

Definition 3.4. Let (X, ω_λ) be a modular metric space. A_1, A_2,...,$A_r (r \in \mathbb{N})$ be ω-closed nonempty subsets of X, $Y = \cup_{i=1}^{r} A_i$, and $\alpha : Y \times Y \to [0, \infty)$ be a mapping. Let $A_{r+1} = A_1$. We say that $F : Y \to Y$ is a cyclic $\mathbb{Z}_\alpha$-contractive mapping if

(1)

$$F(A_j) \subseteq A_{j+1}, i = 1, 2, \ldots, r; \tag{3.2}$$

(2) for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \ldots, r$ and $\alpha(x, Fx)\alpha(y, Fy) \geq 1$, there exists $\xi \in \mathbb{Z}$ such that

$$\xi\Big(\rho(\omega_\lambda(Fx, Fy) + \varphi(\omega_\lambda(Fx, Fy))), \rho(N_\omega(x, y) + \varphi(N_\omega(x, y)))\Big) \geq 0, \tag{3.3}$$

where ρ is an altering distance function, $\varphi \in \Phi$ and

$$N_\omega(x, y) = \max\left\{\omega_\lambda(x, y), \omega_\lambda(x, Fx), \omega_\lambda(y, Fy), \frac{\omega_\lambda(x, Fy) + \omega_\lambda(y, Fx)}{2}\right\}.$$

Remark 3.5. If we take $X = A_i$, $i = 1, 2, \ldots, r$ in Definition 3.4, then we say that F is a $\mathbb{Z}_\alpha$-contractive mapping. We denote the set of all fixed points of F by $Fix(F)$, i.e. $Fix(F) = \{x \in X : Fx = x\}$.

Now, we want to state and prove our mainly result of this section.

Theorem 3.6. *Let (X, ω_λ) be a ω-complete modular space. Let r be a positive integer, A_1, A_2,...,A_r be ω-closed nonempty subsets of X, $Y = \cup_{i=1}^{r} A_i$ and $\alpha : Y \times Y \to [0, \infty)$ be a mapping. Assume that $F : Y \to Y$ is a cyclic $\mathbb{Z}_\alpha$-contractive mapping satisfying the following conditions:*

(i) *F is an α-admissible mapping;*
(ii) *there exists an element x_0 in Y such that, $\alpha(x_0, Fx_0) \geq 1$;*
(iii) (a) *F is α-continuous, or;*
(b) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x, Fx) \geq 1$;*

then F has a fixed point $x \in \cap_{i=1}^{p} A_i$. Moreover, if

(iv) *for all $x \in Fix(F)$ we have $\alpha(x, x) \geq 1$;*

then F has a unique fixed point $x \in \cap_{i=1}^{r} A_i$.

Proof. By the condition (ii), let x_0 be an arbitrary point of Y such that $\alpha(x_0, Fx_0) \geq 1$. Then there exists some i_0 such that $x_0 \in A_{i_0}$. Now by (3.2), $F(A_{i_0}) \subseteq A_{i_0+1}$ implies that $Fx_0 \in A_{i_0+1}$. Thus there exists x_1 in A_{i_0+1} such that $Fx_0 = x_1$. Similarly, $Fx_n = x_{n+1}$, where $x_n \in A_{i_n}$. Hence for $n \geq 0$, there exists $i_n \in \{1, 2, \ldots, r\}$ such that $x_n \in A_{i_n}$ and $Fx_n = x_{n+1}$. On the other hand, F is an α-admissible mapping, hence by Definition 3.1, we get

$$\alpha(x_0, Fx_0) \geq 1 \Rightarrow \alpha(x_1, Fx_1) = \alpha(Fx_0, F(Fx_0)) \geq 1.$$

Again, since F is an α-admissible mapping, we have

$$\alpha(x_1, Fx_1) \geq 1 \Rightarrow \alpha(x_2, Fx_2) = \alpha(Fx_1, F(Fx_1)) \geq 1.$$

Continuing this process, we can get construct a sequence $\{x_n\}$ in A_{i_n} such that

$$\alpha(x_{n-1}, Fx_{n-1}) \geq 1 \Rightarrow \alpha(x_n, Fx_n) \geq 1 \text{ for all } n \in \mathbb{N}_0,$$

and so

$$\alpha(x_n, Fx_n)\alpha(x_{n-1}, Fx_{n-1}) \geq 1 \text{ for all } n \in \mathbb{N}_0. \tag{3.4}$$

If for some $n_0 = 0, 1, 2, \ldots$, we have $x_{n_0} = x_{n_0+1}$, then $Fx_{n_0} = x_{n_0}$, that is x_{n_0} is a fixed point of F. So from now on, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Hence we have $\omega_\lambda(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Now, we want to show that $\omega_\lambda(x_n, x_{n+1}) < \omega_\lambda(x_{n-1}, x_n)$. If $\omega_\lambda(x_n, x_{n+1}) \geq \omega_\lambda(x_{n-1}, x_n)$, then by (3.3) and (3.4), we have

$$0 \leq \xi\big(\rho(\omega_\lambda(x_n, x_{n+1}) + \varphi(\omega_\lambda(x_n, x_{n+1}))), \rho(N_\omega(x_{n-1}, x_n) + \varphi(N_\omega(x_{n-1}, x_n)))\big), \tag{3.5}$$

where

$$\begin{aligned} N_\omega(x_{n-1}, x_n) &= \max\Big\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_{n-1}, Fx_{n-1}), \omega_\lambda(x_n, Fx_n), \\ &\qquad \frac{\omega_\lambda(x_{n-1}, Fx_n) + \omega_\lambda(x_n, Fx_{n-1})}{2}\Big\} \\ &= \max\Big\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1}), \\ &\qquad \frac{\omega_\lambda(x_{n-1}, x_{n+1}) + \omega_\lambda(x_n, x_n)}{2}\Big\} \\ &= \max\Big\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1}), \frac{\omega_\lambda(x_{n-1}, x_{n+1})}{2}\Big\} \\ &= \max\big\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1})\big\}. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned} \xi\Big(&\rho(\omega_\lambda(x_n, x_{n+1}) + \varphi(\omega_\lambda(x_n, x_{n+1}))), \\ &\qquad \rho(\max\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1})\} \\ &\qquad\quad + \varphi(\max\{\omega_\lambda(x_{n-1}, x_n), \omega_\lambda(x_n, x_{n+1})\}))\Big) \\ &\geq 0. \end{aligned} \tag{3.7}$$

If $\max\{\omega_\lambda(x_{n-1},x_n),\omega_\lambda(x_n,x_{n+1})\}=\omega_\lambda(x_n,x_{n+1})$, then by (3.7) and $\omega_\lambda(x_n,x_{n+1})>0$, we have

$$\begin{aligned}0&\le \xi\Big(\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1}))),\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1})))\Big)\\&<\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1})))-\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1})))\\&=0,\end{aligned}$$

which is contradiction. Hence

$$\omega_\lambda(x_n,x_{n+1})<\omega_\lambda(x_{n-1},x_n) \tag{3.8}$$

holds for all $n\in\mathbb{N}$ and there exists $\delta\ge 0$ such that

$$\lim_{n\to+\infty}\omega_\lambda(x_n,x_{n+1})=\delta \ \text{ and } \lim_{n\to+\infty}\omega_\lambda(x_{n-1},x_n)=\delta.$$

Next, we shall show that $\delta=0$. In order to prove $\delta=0$, we assume that $\delta>0$. By (3.8), and the fact that ρ is an increasing function, φ is a nondecreasing function, we have

$$\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1})))<\rho(\omega_\lambda(x_{n-1},x_n)+\varphi(\omega_\lambda(x_{n-1},x_n))).$$

By Definition 1.8 and

$$\begin{aligned}0\le \limsup_{n\to\infty}\xi\Big(&\rho(\omega_\lambda(x_n,x_{n+1})+\varphi(\omega_\lambda(x_n,x_{n+1}))),\\&\rho(\omega_\lambda(x_{n-1},x_n)+\varphi(\omega_\lambda(x_{n-1},x_n)))\Big),\\<0,&\end{aligned}$$

which is contradiction. Thus we get $\lim_{n\to+\infty}\omega_\lambda(x_n,x_{n+1})=0$.

Next, we want to prove that $\lim_{n\to+\infty}\omega_\lambda(x_n,x_m)=0$, If $\lim_{n\to\infty}\omega_\lambda(x_n,x_m)\neq 0$, then there exists $\epsilon>0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k)>n(k)>k$ and

$$\omega_\lambda(x_{m(k)},x_{n(k)})\ge\epsilon. \tag{3.9}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k)>n(k)$ and satisfying (3.9) and

$$\omega_\lambda(x_{m(k)-1},x_{n(k)})<\epsilon. \tag{3.10}$$

Combining (3.9) with (3.10), we have

$$\begin{aligned}\epsilon\le\omega_\lambda(x_{m(k)},x_{n(k)})&\le\omega_\lambda(x_{m(k)},x_{m(k)-1})+\omega_\lambda(x_{m(k)-1},x_{n(k)})\\&<\omega_\lambda(x_{m(k)},x_{m(k)-1})+\epsilon.\end{aligned} \tag{3.11}$$

Letting $k\to+\infty$ in (3.11), we have

$$\lim_{k\to\infty}\omega_\lambda(x_{m(k)},x_{n(k)})=\epsilon. \tag{3.12}$$

There exists some i_0 such that $x_{m(k)}\in A_{i_0}$, and some $s\in\{1,2,...,r\}$ such that $x_{n(k)+s}\in A_{i_0+1}$. Using the triangle inequality we can easily get that

$$\lim_{k\to\infty}\omega_\lambda(x_{m(k)},x_{n(k)+s})=\epsilon, \tag{3.13}$$

and

$$\lim_{k\to\infty}\omega_\lambda(x_{m(k)+1},x_{n(k)+s+1})=\epsilon. \tag{3.14}$$

Therefore, using the contractive condition (3.3) with $x = x_{m(k)}$, $y = x_{n(k)+s}$, we obtain

$$\begin{aligned}\xi\Big(&\rho(\omega_\lambda(x_{m(k)+1},x_{n(k)+s+1})+\varphi(\omega_\lambda(x_{m(k)+1},x_{n(k)+s+1}))),\\ &\rho(N_\omega(x_{m(k)},x_{n(k)+s})+\varphi(N_\omega(x_{m(k)},x_{n(k)+s})))\Big)\geq 0,\end{aligned} \tag{3.15}$$

where

(3.16)

$$\begin{aligned}&N_\omega(x_{m(k)},x_{n(k)+s})\\ &=\max\Big\{\omega_\lambda(x_{m(k)},x_{n(k)+s}),\omega_\lambda(x_{m(k)},x_{m(k)+1}),\\ &\qquad\omega_\lambda(x_{n(k)+s},x_{n(k)+s+1}),\frac{\omega_\lambda(x_{m(k)},x_{n(k)+s+1})+\omega_\lambda(x_{n(k)+s},x_{m(k)+1})}{2}\Big\}\\ &=\max\Big\{\omega_\lambda(x_{m(k)},x_{n(k)+s}),\omega_\lambda(x_{m(k)},x_{m(k)+1}),\omega_\lambda(x_{n(k)+s},x_{n(k)+s+1}),\\ &\qquad\frac{2\omega_\lambda(x_{m(k)},x_{n(k)+s})+\omega_\lambda(x_{n(k)+s},x_{n(k)+s+1})+\omega_\lambda(x_{m(k)},x_{m(k)+1})}{2}\Big\}.\end{aligned}$$

Letting $k\to\infty$ in (3.16), by Lemma 1.9 we have $\epsilon = 0$ which is a contradiction, that is, the sequence $\{x_n\}$ is a ω-Cauchy sequence. Since A_i is ω-closed subset of X, there exists $x\in\bigcap_{i=1}^r A_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y,ω), equivalently,

$$\omega_\lambda(x,x)=\lim_{n\to\infty}\omega_\lambda(x,x_n)=\lim_{n\to\infty}\omega_\lambda(x_n,x_m)=0.$$

On the one hand, we assume that (a) of (iii) holds, that is, F is α-continuous. Then it is obviously that

$$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}Fx_n=Fx.$$

On the other hand, we assume that (b) of (iii) holds and $\alpha(x_{n(k)},x_{n(k)+1})\geq 1$. Then we have $\alpha(x,Fx)\geq 1$ and so $\alpha(x,Fx)\alpha(x_{n(k)},Fx_{n(k)})\geq 1$.

Next, we prove that x is a fixed point of F . Since $\lim_{n\to\infty} x_n = x$ and $F(A_j)\subseteq A_{j+1}$, $j = 1,2,\dots,r$, where $A_{n+1} = A_1$, the sequence $\{x_n\}$ has infinitely many terms in each A_i for $i\in\{1,2,\dots,r\}$. Suppose that $x\in A_i$, then $Fx\in A_{i+1}$ and we take a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $x_{n(k)}\in A_{i-1}$ (the existence of this subsequence is guaranteed by the above mentioned comment). By using the contractive condition (3.3), we obtain

$$\xi\big(\rho(\omega_\lambda(Fx,Fx_{n(k)})+\varphi(\omega_\lambda(Fx,Fx_{n(k)}))),\rho(N_\omega(x,x_{n(k)})+\varphi(N_\omega(x,x_{n(k)})))\big)\geq 0, \tag{3.17}$$

where

$$N_\omega(x, x_{n(k)}) = \max\Big\{\omega_\lambda(x, x_{n(k)}), \omega_\lambda(x, Fx), \omega_\lambda(x_{n(k)}, Fx_{n(k)}), \frac{\omega_\lambda(x, Fx_{n(k)}) + \omega_\lambda(x_{n(k)}, Fx)}{2}\Big\}$$
$$= \max\Big\{\omega_\lambda(x, x_{n(k)}), \omega_\lambda(x, Fx), \omega_\lambda(x_{n(k)}, x_{n(k)+1}), \frac{\omega_\lambda(x, x_{n(k)+1}) + \omega_\lambda(x_{n(k)}, Fx)}{2}\Big\}. \tag{3.18}$$

In (3.17) and (3.18), letting $k \to \infty$ and using the lower semi-continuity of φ, we can get

$$0 \le \xi\Big(\rho(\omega_\lambda(Fx, x) + \varphi(\omega_\lambda(Fx, x))), \rho(\omega_\lambda(x, Fx) + \varphi(\omega_\lambda(x, Fx)))\Big) < 0,$$

which is a contradiction. Hence, $\omega_\lambda(Fx, x) = 0$, that is, $Fx = x$. Therefore, x is a fixed point of F . The cyclic character of F and the fact that $x \in X$ is a fixed point of F, imply that $x \in \bigcap_{i=1}^r A_i$.

At last, we shall prove the uniqueness of the fixed point point. Now, we assume that $x, y \in \bigcap_{i=1}^r A_i$ are two fixed points of F and $x \neq y$. Suppose that condition (iv) holds, then we have $\omega_\lambda(x, y) > 0$, $\alpha(x, x) \ge 1$, $\alpha(y, y) \ge 1$ and then by (3.3), we obtain

$$0 \le \xi\Big(\rho(\omega_\lambda(x, y) + \varphi(\omega_\lambda(x, y))), \rho(N_\omega(Fx, Fy) + \varphi(N_\omega(Fx, Fy)))\Big), \tag{3.19}$$

where

$$N_\omega(x, y) = \max\Big\{\omega_\lambda(x, y), \omega_\lambda(x, Fx), \omega_\lambda(y, Fy), \frac{\omega_\lambda(x, Fy) + \omega_\lambda(y, Fx)}{2}\Big\}$$
$$= \max\Big\{\omega_\lambda(x, y), \omega_\lambda(x, x), \omega_\lambda(y, y), \frac{\omega_\lambda(x, y) + \omega_\lambda(y, x)}{2}\Big\}. \tag{3.20}$$

Combining (3.19), (3.20) with (ξ_2) in Definition 1.8, we can get

$$0 \le \xi\Big(\rho(\omega_\lambda(x, y) + \varphi(\omega_\lambda(x, y))), \rho(\omega_\lambda(x, y) + \varphi(\omega_\lambda(x, y)))\Big) < 0,$$

which is a contradiction. Hence, $\omega_\lambda(x, y) = 0$, that is to say that $x = y$. The proof is completed. □

Corollary 3.7. *Let (X, ω_λ) be a ω-complete modular space. Let r be a positive integer, A_1, A_2,...,A_r be ω-closed nonempty subsets of X, $Y = \cup_{i=1}^r A_i$ and $\alpha : Y \times Y \to [0, \infty)$ be a mapping. Assume that $F : Y \to Y$ is a cyclic $\mathbb{Z}_\alpha$-contractive mapping satisfying the following conditions:*

(i) *F is an α-admissible mapping;*
(ii) *there exists an element x_0 in Y such that, $\alpha(x_0, Fx_0) \ge 1$;*
(iii) (a) *F is α-continuous, or;*
(b) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x, Fx) \ge 1$;*

(iv) *for any* $x \in A_i$ *and* $y \in A_{i+1}$, *and* $i = 1, 2, \ldots, r$, *there exists* $k \in [0, 1)$ *such that*

$$\alpha(x, Fx)\alpha(y, Fy) \geq 1 \Longrightarrow$$

$$\omega_\lambda(Fx, Fy) \leq \max\left\{\omega_\lambda(x, y), \omega_\lambda(x, Fx), \omega_\lambda(y, Fy), \frac{\omega_\lambda(x, Fy) + \omega_\lambda(y, Fx)}{2}\right\},$$

then F *has a fixed point* $x \in \cap_{i=1}^{p} A_i$. *Moreover, if*

(v) *for all* $x \in Fix(F)$ *we have* $\alpha(x, x) \geq 1$;
then F *has a unique fixed point* $x \in \cap_{i=1}^{r} A_i$.

Remark 3.8. Based on Example 1.11-1.16, we can get some corollaries. Once again to avoid repetition, we don't list the corollaries.

4. Example

In this section, we want to give an example to illustrate our results.

Example 4.1. Let $X_\omega = C = [0, \frac{9}{2}]$ and $\omega_\lambda = \frac{|x-y|}{\lambda}$. Define the mappings $A, B, S, T : C \to C$ by

$$Ax = \begin{cases} 0, & x = 0, \\ \frac{1}{2}, & x \neq 0, \end{cases} \qquad Bx = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases} \tag{4.1}$$

$$Sx = \begin{cases} 0, & x = 0, \\ \frac{9}{2}, & x \neq 0, \end{cases} \qquad Tx = \begin{cases} 0, & x = 0, \\ 4, & x \neq 0. \end{cases} \tag{4.2}$$

Consider three functions ξ, ρ, φ given by $\xi(t, s) = \frac{1}{2}s - t$, $\rho(t) = t$ and $\varphi(t) = t$. Next, we will show that all the hypotheses of Theorem 2.5 are satisfied. In fact, for $x, y \in C$, we need to divide the proof into the following cases:

Case I: Assume that $x = y = 0$. It is easy to see that $Ax = By = Sx = Ty = 0$. Thus the contractive condition (2.1) is satisfied.

Case II: Assume that $x = 0$, $y > 0$. Then we have $Ax = 0$, $By = 1$, $Sx = 0$ and $Ty = 4$. Consequently, we obtain

$$\omega_\lambda(Ax, By) = \frac{1}{\lambda},$$

and

$$\begin{aligned} M_\omega(x, y) &= \max\left\{\omega_\lambda(Ax, Sx), \omega_\lambda(By, Ty), \omega_\lambda(Sx, Ty), \frac{\omega_\lambda(Ax, Ty) + \omega_\lambda(By, Sx)}{2}\right\} \\ &= \max\left\{0, \frac{3}{\lambda}, \frac{4}{\lambda}, \frac{5/2}{\lambda}\right\} = \frac{4}{\lambda}. \end{aligned}$$

It follows that

$$\begin{aligned} \xi\left(\rho(\omega_\lambda(A, By)), \rho(M_\omega(x, y))\right) &= \frac{1}{2}\rho(M_\omega(x, y)) - \rho(\omega_\lambda(x, y)) \\ &= \frac{1}{2}M_\omega(x, y) - \omega_\lambda(x, y) \\ &= \frac{1}{\lambda} > 0. \end{aligned} \tag{4.3}$$

Thus the contractive condition (2.1) is satisfied.
Case III: Assume that $x > 0$, $y = 0$. Then we have $Ax = \frac{1}{2}$, $By = 0$, $Sx = \frac{9}{2}$ and $Ty = 0$. Consequently, we obtain

$$\omega_\lambda(Ax, By) = \frac{1/2}{\lambda},$$

and

$$\begin{aligned} M_\omega(x,y) &= \max\left\{\omega_\lambda(Ax,Sx), \omega_\lambda(By,Ty), \omega_\lambda(Sx,Ty), \frac{\omega_\lambda(Ax,Ty)+\omega_\lambda(By,Sx)}{2}\right\} \\ &= \max\left\{\frac{4}{\lambda}, 0, \frac{9/2}{\lambda}, \frac{5/2}{\lambda}\right\} = \frac{4}{\lambda}. \end{aligned}$$

It follows that

$$\begin{aligned} \xi\left(\rho(\omega_\lambda(A,By)), \rho(M_\omega(x,y))\right) &= \frac{1}{2}\rho(M_\omega(x,y)) - \rho(\omega_\lambda(x,y)) \\ &= \frac{1}{2}M_\omega(x,y) - \omega_\lambda(x,y) \\ &= \frac{3/2}{\lambda} > 0. \end{aligned} \tag{4.4}$$

Thus the contractive condition (2.1) is satisfied.
Case IV: Assume that $x > 0$, $y > 0$. Then we have $Ax = \frac{1}{2}$, $By = 1$, $Sx = \frac{9}{2}$ and $Ty = 4$. Consequently, we obtain

$$\omega_\lambda(Ax, By) = \frac{1/2}{\lambda},$$

and

$$\begin{aligned} M_\omega(x,y) &= \max\left\{\omega_\lambda(Ax,Sx), \omega_\lambda(By,Ty), \omega_\lambda(Sx,Ty), \frac{\omega_\lambda(Ax,Ty)+\omega_\lambda(By,Sx)}{2}\right\} \\ &= \max\left\{\frac{4}{\lambda}, \frac{3}{\lambda}, \frac{1/2}{\lambda}, \frac{7/2}{\lambda}\right\} = \frac{4}{\lambda}. \end{aligned}$$

It follows that

$$\begin{aligned} \xi\left(\rho(\omega_\lambda(A,By)), \rho(M_\omega(x,y))\right) &= \frac{1}{2}\rho(M_\omega(x,y)) - \rho(\omega_\lambda(x,y)) \\ &= \frac{1}{2}M_\omega(x,y) - \omega_\lambda(x,y) \\ &= \frac{3/2}{\lambda} > 0. \end{aligned} \tag{4.5}$$

Thus the contractive condition (2.1) is satisfied. Finally, it is easy to see that $\varphi \in \Phi$, ρ is a altering distance function, and the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Therefore, all the hypotheses of Theorem 2.5 are satisfied. In addition, $(0, 0)$ is a unique common fixed point of A, B, S and T.

5. Application

In this section, motivated by [30], we give an application of the existence of solutions of integral equations. At first, we consider

$$u(t) = \int_0^\sigma H(t,s) f(s, u(s)) ds, \quad \text{for all } t \in [0, \sigma], \tag{5.1}$$

where $\sigma > 0$, $f : [0, \sigma] \times \mathbb{R} \to \mathbb{R}$ and $H : [0, \sigma] \times [0, \sigma] \to [0, +\infty)$ are continuous functions. Let $X = C([0, \sigma])$ be the set of real continuous functions on $[0, \sigma]$. We endow X with

$$\omega_\lambda(u, v) = \sup_{s\in[0,\sigma]} \frac{|u(s) - v(s)|}{\lambda}$$

where $a, b \in X$ and all $\lambda > 0$. It is obvious that (X, ω_λ) is a completed modular metric space.

Suppose that $\zeta : X \times X \to \mathbb{R}$ is a function with the following properties:

(1) $\zeta(x, y) \geq 0 \Rightarrow \zeta(Tx, Ty) \geq 0$;

(2) there exists $x_0 \in X$ such that $\zeta(x_0, Tx_0) \geq 0$;

(3) if $\{x_n\}$ is a sequence in X such that $\zeta(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\zeta(x, Tx) \geq 0$, where

$$Tu(t) = \int_0^\sigma H(t, s)f(s, u(s))ds \quad \text{for all} \quad t \in [0, \sigma].$$

Let $(a, b) \in X \times X$, $(a_0, b_0) \in \mathbb{R}^2$ such that

$$a_0 \leq a(t) \leq b(t) \leq b_0, \quad \text{for all} \quad t \in [0, \sigma]. \tag{5.2}$$

Assume that for all $t \in [0, \sigma]$, we have

$$a(t) \leq \int_0^\sigma H(t, s)f(s, b(s))ds \tag{5.3}$$

and

$$b(t) \geq \int_0^\sigma H(t, s)f(s, a(s))ds \tag{5.4}$$

Let for all $s \in [0, \sigma]$, $f(s, \cdot)$ be a decreasing function, that is,

$$x, y \in \mathbb{R}, x \geq y \Longrightarrow f(s, x) \leq f(s, y). \tag{5.5}$$

Assume that

$$\sup_{t\in[0,\sigma]} \int_0^\sigma H(t, s)ds \leq 1. \tag{5.6}$$

Moreover, suppose that for all $s \in [0, \sigma]$, for all $x, y \in \mathbb{R}$ with ($x \leq b_0$ and $y \geq a_0$) or ($x \geq a_0$ and $y \leq b_0$) and $\zeta(y, Ty) \geq 1$ and $\zeta(x, Tx) \geq 1$, we have

$$|f(s, x)-f(s, y)| \leq k \max\left\{|x - y|, |x - Tx|, |y - Ty|, \frac{|x - Ty| + |Tx - y|}{2}\right\}, \tag{5.7}$$

where $k \in [0, 1)$.

Theorem 5.1. *Under assume that (5.2)-(5.7), the integral equation (5.1) has a solution in $\{u \in C([0, \sigma] : a \leq u(t) \leq b\}$ for all $t \in [0, \sigma]$.*

Proof. Define closed subsets of x, A_1 and A_2 by

$$A_1 = \{u \in X : u \leq b\}$$

and

$$A_2 = \{u \in X : u \geq a\}.$$

Define the mapping $T : X \to X$ as follows:

$$Tu(t) = \int_0^\sigma H(t,s)f(s,u(s))ds \;\; \text{for all} \;\; t \in [0,\sigma].$$

Next, we claim that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. In fact, if $u \in A_1$, then $u(s) \le b(s)$ for all $s \in [0,\sigma]$. By (5.5), we can get $f(s,u(s)) \ge f(s,b(s))$ for all $s \in [0,\sigma]$. Since $H(t,s) \ge 0$ for all $s,t \in [0,\sigma]$, one has

$$H(t,s)f(s,u(s)) \ge H(t,s)f(s,b(s)), \;\; \text{for all} \;\; t,s \in [0,\sigma].$$

From (5.3) and the above inequality, we know that

$$Tu(t) = \int_0^\sigma H(t,s)f(s,u(s))ds \ge \int_0^\sigma H(t,s)f(s,b(s))ds \ge a(t), \;\; \text{for all} \;\; t \in [0,\sigma].$$

Thus $Tu \in A_2$. By a similar argument as the above, it follows that $Tu \in A_1$ if $u \in A_2$.

Next, if $(u,v) \in A_1 \times A_2$, then for all $t \in [0,\sigma]$, we have

$$u(t) \le b(t), \qquad v(t) \ge a(t).$$

By (5.2), we have $u(t) \le b_0$ and $v(t) \ge a_0$. Let $x \in A_1$ and $y \in A_2$ where $\zeta(x,Tx) \ge 0$ and $\zeta(y,Ty) \ge 0$. Thus by (5.7), one has

$$\begin{aligned}
&\frac{|Tx-Ty|}{\lambda} \\
&= \left| \int_0^\sigma \frac{H(t,s)(f(s,x(s)) - f(s,y(s)))}{\lambda} ds \right| \\
&\le \int_0^\sigma H(t,s) \left| \frac{f(s,x(s)) - f(s,y(s))}{\lambda} \right| ds \\
&\le \int_0^\sigma H(t,s)\frac{k}{\lambda} \max\left\{ |x-y|, |x-Tx|, |y-Ty|, \frac{|x-Ty|+|Tx-y|}{2} \right\} ds \\
&\le \int_0^\sigma kH(t,s)ds \max\left\{ \omega_\lambda(x,y), \omega_\lambda(x,Tx), \omega_\lambda(y,Ty), \frac{\omega_\lambda(x,Ty)+\omega_\lambda(Tx,y)}{2} \right\} \\
&\le k \max\left\{ \omega_\lambda(x,y), \omega_\lambda(x,Tx), \omega_\lambda(y,Ty), \frac{\omega_\lambda(x,Ty)+\omega_\lambda(Tx,y)}{2} \right\},
\end{aligned}$$

which shows that

(5.8)

$$\omega_\lambda(Tx,Ty) \le k \max\left\{ \omega_\lambda(x,y), \omega_\lambda(x,Tx), \omega_\lambda(y,Ty), \frac{\omega_\lambda(x,Ty)+\omega_\lambda(Tx,y)}{2} \right\}.$$

By a similar argument as above, we know that if $(u,v) \in A_2 \times A_1$ where $\zeta(u,Tu) \ge 0$ and $\zeta(v,Tv) \ge 0$, then (5.8) also holds.

Next, if we set

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \zeta(x,y) \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

then $x \in A_i$, $y \in A_{i+1}$, $i = 1,2$ and $\alpha(x,Tx)\alpha(y,Ty) \ge 1$ implies that

$$\omega_\lambda(Tx,Ty) \le k \max\left\{ \omega_\lambda(x,y), \omega_\lambda(x,Tx), \omega_\lambda(y,Ty), \frac{\omega_\lambda(x,Ty)+\omega_\lambda(Tx,y)}{2} \right\}.$$

Thus we know that all conditions of Theorem 3.6 (Corollary 3.7) hold with $\xi(t,s) = ks - t$, $\rho(t) = t$ and $\varphi(t) = 0$. Thus T has a fixed point $\tau \in A_1 \cap A_2 = \{u \in C([0,\sigma]) : a \le u(t) \le b, \text{for all } t \in [0,\sigma]\}$, which implies τ is a solution of (5.1). □

References

[1] A. A. Abdou and M. A. Khamsi, *DeMarr's fixed point theorem in modular vector spaces*, J. Nonl. Convex Anal. **19** (2018), 1083–1091.

[2] A. Amini-Harandi, *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl. **2012(1):204** (2012).

[3] H. Aydi and A. Felhic, *Fixed points in modular spaces via α-admissible mappings and simulation functions*, J. Nonlinear Sci. Appl. **9** (2016), 3686–3701.

[4] J. Ali and M. Imdad, *An implicit function implies several contraction conditions*, Sarajevo J. Math. **17** (2008), 269–285.

[5] H. H. Alsulami, E. Karapínar, and F. Khojasteh, *A proposal to the study of contractions in quasi-metric spaces*, Dis. Dynamics Nature Soc. **2014** (2014), Article ID 269286, 10 pages.

[6] J. H. Chen and X. J. Huang, *Coupled fixed point theorems for compatible mappings in partially ordered G-metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 130–141.

[7] J. H. Chen and X. J. Huang, *Quadruple fixed point theorems under (φ,ψ)-contractive conditions in partially ordered G-metric spaces with mixed g-monotone property*, J. Nonlinear Sci. Appl. **8** (2015), 285–300.

[8] J. H. Chen, X. J. Huang and S. J. Li, *Fixed point theorems for cyclic contractive mappings via altering distance functions in metric-like spaces*, Open Math. **14** (2016), 857–874.

[9] J. H. Chen, and X. H. Tang, *Generalizations of Darbo's fixed point theorem via simulation functions with application to functional integral equations*, J. Comput. Appl. Math. **296** (2016), 564–575.

[10] V. V. Chistyakov, *Modular metric spaces I: basic concepts*, Nonlinear Anal. **72** (2010), 1–14.

[11] M. Demma, R.Saadati, and P. Vetro, *Fixed Point Results on b-metric space via picard sequences and b-simulation functions*, Iran. J. Math. Sci. Inform. **11** (2016), 123–136.

[12] N. Hussain, A. Latif and I. Iqbal, *Fixed point results for generalized F-contractions in modular metric and fuzzy metric spaces*, Fixed Point Theory Appl. **2015:158** (2015).

[13] M. S. Khan, M. Swalek and S. Sessa, *Fixed point theorems by altering distances between two points*, Bull. Austral. Math. Soc. **30** (1984), 1–9.

[14] E. Karapínar and I. M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1894–1899.

[15] F. Khojasteh, S. Shukla and S. Radenovic, *A new approach to the study of fixed point theory for simulation function*, Filomat **29** (2015), 1189–1194.

[16] W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractinve condition*, Fixed Piont Theory **4** (2003), 79–89.

[17] A. Latif and T. Nazir, *Common fixed point of generalized contraction type mappings in ordered G-metric spaces*, J. Nonl. Convex Anal. **19** (2018), 1275–1285.

[18] M. Liang, C. X. Zhu, C. F. Chen and Z. Wu, *Some new theorems for cyclic contractions in Gb-metric spaces and some applications*, Appl. Math. Comput. **346** (2019), 545–558.

[19] J. Martínez-Moreno, W. Sintunavarat and P. Kumam, *Banach's contraction principle for nonlinear contraction mappings in modular metric spaces*, Bull. Malays. Math. Sci. Soc. **40** (2017), 335–344.

[20] M. Mocanu and V. Popa, *Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces*, Libertas Math. **28** (2008), 1–13.

[21] J. Musielak and W. Orlicz, *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. **7** (1959), 661–668.

[22] J. Musielak and W. Orlicz, *On modular spaces*, Stud. Math. **18** (1959), 49–65.

[23] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, 1950.

[24] A. Nastasi, and P. Vetro, *Fixed point results on metric and partial metric spaces via simulation functions*, J. Nonlinear Sci. Appl. **8** (2015), 1059–1069.

[25] V. Popa and M. Mocanu, *A new viewpoint in the study of fixed point for mapping satisfying a contractive condition of integral type*, Buu. Inst. Poliehn. Iasi, Math. Mech. Theor. Phys. **53** (2007), 269–286.

[26] V. Popa, and M. Mocanu, *Altering distance and common fixed points under implicit relations*, Hacettepe J. Math. Stat. **38** (2009), 329–337.

[27] V. Popa and A. M. Patriciu, *Altering distance and common fixed points for hybrid mappings under implicit relations and application*, Anal. Math. Silesianae **28** (2014), 87–96.

[28] S. Radenović, *A note on fixed point theory for cyclic φ-contractions*, Fixed Point Theory Appl. **2015(1): 189** (2015).

[29] A. F. Roldán-López-de-Hierro, E. Karapínar, C. Roldán-López-de-Hierro and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345–355.

[30] P. Salimi, N. Hussain, S. Shukla, S. Fathollahi and S. Radenavić, *Fixed point results for cyclic α-$\psi\phi$-contractions with application to integral equations*, J. Comput. Appl. Math. **290** (2015), 445–458.

[31] W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Model. **55** (2012), 680–687.

[32] C. Vetro, S. Chauhan and E. Karapínar and W. Shatanawi, *Fixed points of weakly compatible mappings satisfying generalized φ-weak contractions*, Bull. Malays. Math. Sci. Soc. **38** (2015), 1085–1105.

[33] F. Vetro and S. Radenović, *Nonlinear ψ-quasi-contractions of Ćirić-type in partial metric spaces*, Appl. Math. Comput. **219** (2012), 1594–1600.

[34] Z. Wu, C. X. Zhu and C.Yuan, *Fixed point results for cyclic contractions in Menger PM-spaces and generalized Menger PM-spaces*, RACSAM **112** (2018), 449–462.

[35] Q. Zhang and Y. Song, *Fixed point theory for generalized $\varphi-$weak contractions*, Appl. Math. Lett. **22** (2009), 75–78.

[36] C. X. Zhu, C. F. Chen and X. Z. Zhang, *Some results in quasi-b-metric-like spaces*, J. Ineq. Appl. **2014:437** (2014).

[37] L. Zhu, C. X. Zhu, C. F. Chen and Z. Stojanović, *Multidimensional fixed points for generalized ψ-quasi-contractions in quasi-metric-like spaces*, J. Ineq. Appl. **2014(1): 27** (2014).

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