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SIMULTANEOUS ITERATIVE ALGORITHMS FOR THE SPLIT COMMON FIXED-POINT PROBLEM GOVERNED BY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. Recently, Moudafi (Alternating CQ-algorithms for convex feasibility and split fixed-point problems, J. Nonlinear and Convex Analysis) introduced an alternating iterative algorithms with weak convergence for the following split common fixed-point problem. Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, the split common fixed-point problem introduced by Moudafi is

$$\text{finding } x \in F(U), y \in F(T) \text{ such that } Ax = By, \tag{1}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two firmly quasi-nonexpansive operators with nonempty fixed-point sets $F(U) = \{x \in H_1 : Ux = x\}$ and $F(T) = \{x \in H_2 : Tx = x\}$. Note that, the above problem (1) allows asymmetric and partial relations between the variables x and y. In this paper, we introduce two simultaneous iterative algorithms for the split common fixed-point problem (1) governed by the general class of quasi-nonexpansive operators. We prove the weak convergence of algorithms and weaken the condition of the relaxation parameters $\{\gamma_k\}$. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let I denote the identity operator on H. Let $T : H \to H$ be an operator. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$.

The split feasibility problem (SFP) originally introduced in Censor and Elfving [6] is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where C is a nonempty closed convex subset of real Hilbert space H_1, Q is a nonempty closed convex subset of real Hilbert space H_2, $A : H_1 \to H_2$ is a bounded linear operator. Due to its extraordinary utility and broad applicability in many

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areas of applied mathematics (most notably, fully-discretized models inverse problems which arise from phase retrievals and in medical image reconstruction [2]), algorithms for solving split feasibility problems continue to receive great attention; see for instance [3, 9, 10, 18, 19, 20, 21, 22, 24].

Assuming that the SFP is consistent (i.e has a solution), it is no hard to see that $x \in C$ solves the SFP if and only if it solves the following fixed-point equation

$$x = P_C(I - \gamma A^*(I - P_Q)A)x, \quad x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.

To solve the (1.2), Byrne [2] proposed his CQ algorithm which generates a sequence

$$x_{k+1} = P_C(I - \gamma A^*(I - P_Q)A)x_k, \quad k \geq 1,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Recently, Moudafi [13] introduced a new split feasibility problem. Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The split feasibility problem in [13] is to find

$$x \in C, y \in Q \ \text{ such that } \ Ax = By, \tag{1.3}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situation, for instance in decomposition methods for PDE's, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [1]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [5]). If $H_2 = H_3$ and $B = I$, then the split feasibility problem (1.3) reduces to the split feasibility problem (1.1).

For solving the SFP (1.3), Moudafi [13] introduced the following alternating CQ algorithm

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.3}$$

where $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B are the spectral radius of A^*A and B^*B, respectively. Then he proved the weak convergence of the sequence (x_k, y_k) to a solution of (1.3) provided that the solution set $\Gamma = \{x \in C, y \in Q; Ax = By\}$ is nonempty and some conditions on the sequence of positive parameters $\{\gamma_k\}$ are satisfied.

Censor and Segal [7] considered the following split common fixed-point problem (SCFP):

$$\text{finding } \ x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{1.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear operators with nonempty fixed-point sets $F(U) = C$ and $F(T) = Q$. The SCFP is a generalization of the SFP and the convex feasibility problem (CFP); see [7]. The SCFP is in itself at the core of the modeling of many inverse problems in various areas of mathematics and physical sciences and has been used to model significant real-world inverse problems in sensor net-works, in radiation therapy treatment planning, in resolution enhancement, in wavelet-based denoising, in antenna design, in computerized tomography, in materials science, in watermarking, in data compression, in magnetic resonance imaging, in holography, in color imaging, in optics and neural networks, in graph matching...(see [8]). Some iterative algorithms have been proposed to solve the SCFP (for example [16, 23]).

To solve the SCFP (1.4) of directed operators, Censor and Segal [7] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k), \quad k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix $A^TA(A^T$ stands for matrix transposition). For solving the SCFP of quasi-nonexpansive operators, Moudafi [14] introduced the following relaxed algorithm:

$$x_{k+1} = \alpha_k u_k + (1 - \alpha_k)U(u_k), \quad k \in N, \tag{1.5}$$

where $u_k = x_k + \gamma\beta A^*(T - I)Ax_k$, $\beta \in (0, 1)$, $\alpha_k \in (0, 1)$ and $\gamma \in (0, \frac{1}{\lambda\beta})$ with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the above algorithm (1.5) in Hilbert spaces.

Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. In [13], Moudafi introduced the following general SCFP:

$$\text{finding } \ x \in F(U), y \in F(T), \text{such that } \ Ax = By, \tag{1.6}$$

and proposed the following alternating iterative algorithm:

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases} \tag{1.7}$$

for firmly quasi-nonexpansive operators $U : H_1 \to H_1$ and $T : H_2 \to H_2$. If $B = I$, SCFP (1.6) reduces to the classical SCFP (1.4). Moudafi [13] obtained the following result.

Theorem 1.1 *Let H_1, H_2, H_3 be real Hilbert spaces, let $U : H_1 \to H_1$, $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators such $I - U$, $I - T$ are demiclosed at 0. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set Γ is nonempty, (γ_k) is a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively. Then, the sequence (x_k, y_k) generated by (1.7) weakly converges to a solution $(\bar{x}, \bar{y})$ of (1.6). Moreover $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

Recently, in [4], Byrne and Moudafi considered and studied the algorithms to solve the split equality problem (ASEP), which can be regarded as obtaining the consistent case and the inconsistent case of the split equality problem (SEP)(i.e., SFP (1.3)). There they proposed a simultaneous iterative algorithm (SSEA) to solve the SFP (1.3):

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^T(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^T(Ax_k - By_k)), \end{cases}$$

where $\varepsilon \le \gamma_k \le \frac{2}{\lambda_G} - \varepsilon$, λ_G stands for the spectral radius of G^TG and $G = [A \quad -B]$.

In what follows, we will focus our attention on SCFP (1.6) for the general quasi-nonexpansive operators with nonempty fixed-point sets $F(U)$ and $F(T)$ and denote the solution set of the two-operator SCFP (1.6) by Γ. Inspired and motivated by the works mentioned above, we introduce and study the convergence properties of the following simultaneous iterative algorithms for solving the SCFP (1.6) for the general class of quasi-nonexpansive operators:

$$\begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1-\alpha_k)U(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1-\beta_k)T(v_k), \end{cases} \tag{1.8}$$

and

$$\begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k)U(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k)T(v_k). \end{cases} \tag{1.9}$$

This paper establishes the weak convergence of the sequence given by (1.8) and (1.9) to the solution of SCFP (1.6) and the relaxation parameters $\{\gamma_k\}$ are allowed to be in the interval $(\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, instead of non-decreasing sequence in the interval $(\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$ when ε is small enough in alternative algorithm.

Now let us first recall the definition of quasi-nonexpansive operators which appear naturally when using subgradient projection operator techniques in solving some feasibility problems, and also some definitions of classes of operators often used in fixed-point theory and which are commonly encountered in the literature.

- An operator $T : H \to H$ belongs to the set Φ_N of nonexpansive operators if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall (x, y) \in H \times H.$$

- An operator $T : H \to H$ belongs to the set Φ_{FN} of firmly nonexpansive operators (or directed operators) if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \quad \forall (x, y) \in H \times H.$$

- An operator $T : H \to H$ belongs in the general class Φ_Q of (possibly discontinuous) quasi-nonexpansive operators if

$$\|Tx - q\| \le \|x - q\|, \quad \forall (x, q) \in H \times F(T).$$

- An operator $T : H \to H$ belongs to the set Φ_{FQ} of firmly quasi-nonexpansive operators if

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2, \ \ \forall (x, q) \in H \times F(T).$$

It is easily observed that $\Phi_{FN} \subset \Phi_N \subset \Phi_Q$ and that $\Phi_{FN} \subset \Phi_{FQ} \subset \Phi_Q$. Furthermore, Φ_{FN} is well known to include resolvents and projection operators, while Φ_{FQ} contains subgradient projection operators (see, for instance, [11], and the reference therein).

An operator $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Now we give a series of preliminary results needed for the convergence analysis of algorithms (1.8) and (1.9). In real Hilbert space, we easily get the following equality:

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \forall x, y \in H. \tag{1.10}$$

In what follows, we give some key properties of the relaxed operator $T_\alpha = \alpha I + (1 - \alpha)T$ which will be needed in the convergence analysis of our algorithms.

Lemma 1.1([14]) *Let H be a real Hilbert space and $T : H \to H$ a quasi-nonexpansive operator. Set $T_\alpha = \alpha I + (1 - \alpha)T$ for $\alpha \in [0, 1)$. Then, the following properties are reached for all $(x, q) \in H \times F(T)$:*

(i) $\langle x - Tx, x - q\rangle \geq \frac{1}{2}\|x - Tx\|^2$ and $\langle x - Tx, q - Tx\rangle \leq \frac{1}{2}\|x - Tx\|^2$,

(ii) $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2$;

(iii) $\langle x - T_\alpha x, x - q\rangle \geq \frac{1-\alpha}{2}\|x - Tx\|^2$.

Remark 1.2 *Let $T_\alpha = \alpha I + (1-\alpha)T$, where $T : H \to H$ is a quasi-nonexpansive operator and $\alpha \in [0, 1)$. We have $F(T_\alpha) = F(T)$ and $\|T_\alpha x - x\|^2 = (1-\alpha)^2\|Tx - x\|^2$. It follows from (ii) of Lemma 1.1 that $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \frac{\alpha}{1-\alpha}\|T_\alpha x - x\|^2$, which implies that T_α is firmly quasi-nonexpansive when $\alpha = \frac{1}{2}$. On the other hand, if $\hat{T}$ is a firmly quasi-nonexpansive operator, we can easily obtain $\hat{T} = \frac{1}{2}I + \frac{1}{2}T$, where T is quasi-nonexpansive.*

Lemma 1.3([12]) *Let H be a real Hilbert space. Then for all $t \in [0, 1]$ and x, $y \in H$,*

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

2. Main results.

Theorem 2.1 *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive operators with nonempty fixed point set $F(U)$ and $F(T)$. Assume that $U - I$, $T - I$ are demiclosed at origin and the solution set Γ of (1.6) is nonempty. Let $\{\gamma_k\}$ is a positive sequence such that $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, where λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively and ε is small enough. Then, the sequence $\{(x_k, y_k)\}$ generated by (1.8) weakly converges to a solution (x^*, y^*) of*

(1.6), provided that $\{\alpha_k\} \subset (\delta, 1-\delta)$ *and* $\{\beta_k\} \subset (\sigma, 1-\sigma)$ *for small enough* δ, $\sigma > 0$. *Moreover* $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ *and* $\|y_k - y_{k+1}\| \to 0$ *as* $k \to \infty$.

Proof. Taking $(x, y) \in \Gamma$, i.e., $x \in F(U)$; $y \in F(T)$ and $Ax = By$. We have

$$\begin{aligned} &\|u_k - x\|^2 \\ =&\|x_k - \gamma_k A^*(Ax_k - By_k) - x\|^2 \qquad (2.1)\\ =&\|x_k - x\|^2 - 2\gamma_k\langle x_k - x, A^*(Ax_k - By_k)\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2. \end{aligned}$$

From the definition of λ_A it follows that

$$\begin{aligned} &\gamma_k^2\|A^*(Ax_k - By_k)\|^2 \\ =&\gamma_k^2\langle A^*(Ax_k - By_k), A^*(Ax_k - By_k)\rangle \\ =&\gamma_k^2\langle Ax_k - By_k, AA^*(Ax_k - By_k)\rangle \qquad (2.2)\\ \le&\lambda_A\gamma_k^2\langle Ax_k - By_k, Ax_k - By_k\rangle \\ =&\lambda_A\gamma_k^2\|Ax_k - By_k\|^2. \end{aligned}$$

Using the equality (1.10), we have

$$\begin{aligned} &-2\langle x_k - x, A^*(Ax_k - By_k)\rangle \\ =&-2\langle Ax_k - Ax, Ax_k - By_k\rangle \qquad (2.3)\\ =&-\|Ax_k - Ax\|^2 - \|Ax_k - By_k\|^2 + \|By_k - Ax\|^2. \end{aligned}$$

By (2.1)-(2.3) we obtain

$$\begin{aligned} \|u_k - x\|^2 \le&\|x_k - x\|^2 - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2 \qquad (2.4)\\ &- \gamma_k\|Ax_k - Ax\|^2 + \gamma_k\|By_k - Ax\|^2. \end{aligned}$$

Similarly, by (1.8) we have

$$\begin{aligned} \|v_k - y\|^2 \le&\|y_k - y\|^2 - \gamma_k(1 - \lambda_B\gamma_k)\|Ax_k - By_k\|^2 \qquad (2.5)\\ &- \gamma_k\|By_k - By\|^2 + \gamma_k\|Ax_k - By\|^2. \end{aligned}$$

By adding the two last inequalities and by taking into account assumptions on $\{\gamma_k\}$ and the fact that $Ax = By$, we obtain

$$\|u_k - x\|^2 + \|v_k - y\|^2 \le \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k[2 - (\lambda_A + \lambda_B)\gamma_k]\|Ax_k - By_k\|^2. \qquad (2.6)$$

Using the fact that U and T are quasi-nonexpansive operators, it follows from the property (ii) of Lemma 1.1 that

$$\|x_{k+1} - x\|^2 \le \|u_k - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - u_k\|^2$$

and

$$\|y_{k+1} - y\|^2 \le \|v_k - y\|^2 - \beta_k(1 - \beta_k)\|T(v_k) - v_k\|^2.$$

So, by (2.6) we have

$$\begin{aligned}(2.7)\qquad &\|x_{k+1}-x\|^2+\|y_{k+1}-y\|^2\\ \le&\|x_k-x\|^2+\|y_k-y\|^2-\gamma_k[2-(\lambda_A+\lambda_B)\gamma_k]\|Ax_k-By_k\|^2\\ &-\alpha_k(1-\alpha_k)\|U(u_k)-u_k\|^2-\beta_k(1-\beta_k)\|T(v_k)-v_k\|^2.\end{aligned}$$

Now, by setting $\rho_k(x,y) := \|x_k-x\|^2+\|y_k-y\|^2$, we obtain the following inequality

$$\begin{aligned}(2.8)\qquad \rho_{k+1}(x,y)\le&\rho_k(x,y)-\gamma_k[2-(\lambda_A+\lambda_B)\gamma_k]\|Ax_k-By_k\|^2\\ &-\alpha_k(1-\alpha_k)\|U(u_k)-u_k\|^2-\beta_k(1-\beta_k)\|T(v_k)-v_k\|^2.\end{aligned}$$

We see the sequence $\{\rho_k(x,y)\}$ being decreasing and lower bounded by 0, consequently it converges to some finite limit, says $\rho(x,y)$. So the sequences $\{x_k\}$ and $\{y_k\}$ are bounded. Again from (2.8) we have $\rho_{k+1}(x,y)\le\rho_k(x,y)-\gamma_k[2-(\lambda_A+\lambda_B)\gamma_k]\|Ax_k-By_k\|^2$ and hence

$$\lim_{k\to\infty}\|Ax_k-By_k\|=0$$

by the assumption on $\{\gamma_k\}$. Similarly, by the conditions on $\{\alpha_k\}$ and $\{\beta_k\}$ we obtain

$$\lim_{k\to\infty}\|U(u_k)-u_k\|=\lim_{k\to\infty}\|Tv_k-v_k\|=0.$$

Since

$$\|u_k-x_k\|=\gamma_k\|A^*(Ax_k-By_k)\|$$

and the fact that $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty}\|u_k-x_k\|=0$. It follows from $\lim_{k\to\infty}\|U(u_k)-u_k\|=0$ that $\lim_{k\to\infty}\|U(u_k)-x_k\|=0$. So

$$\|x_{k+1}-x_k\|\le\alpha_k\|u_k-x_k\|+(1-\alpha_k)\|U(u_k)-x_k\|\to 0$$

as $k\to\infty$, which infers that $\{x_k\}$ is asymptotically regular, namely $\lim_{k\to\infty}\|x_{k+1}-x_k\|=0$. Similarly, $\lim_{k\to\infty}\|v_k-y_k\|=0$ and $\{y_k\}$ is asymptotically regular too.

Taking $x^*\in\omega_w(x_k)$ and $y^*\in\omega_w(y_k)$, from $\lim_{k\to\infty}\|u_k-x_k\|=0$ and $\lim_{k\to\infty}\|v_k-y_k\|=0$ we have $x^*\in\omega_w(u_k)$ and $y^*\in\omega_w(v_k)$. Combined with the demiclosednesses of $U-I$ and $T-I$ at 0,

$$\lim_{k\to\infty}\|U(u_k)-u_k\|=\lim_{k\to\infty}\|Tv_k-v_k\|=0$$

yields $Ux^*=x^*$ and $Ty^*=y^*$. So $x\in F(U)$ and $y\in F(T)$. On the other hand, $Ax^*-By^*\in\omega_w(Ax_k-By_k)$ and weakly lower semicontinuity of the norm imply

$$\|Ax^*-By^*\|\le\liminf_{k\to\infty}\|Ax_k-By_k\|=0,$$

hence $(x^*,y^*)\in\Gamma$.

Next, we will show the uniqueness of the weak cluster points of $\{x_k\}$ and $\{y_k\}$. Indeed, let $\bar{x}$, $\bar{y}$ be other weak cluster points of $\{x_k\}$ and $\{y_k\}$ respectively, then $(\bar{x},\bar{y})\in\Gamma$. From the definition of $\rho_k(x,y)$ we have

$$\begin{aligned}(2.10)\qquad &\rho_k(x^*,y^*)\\ =&\|x_k-\bar{x}\|^2+\|\bar{x}-x^*\|^2+2\langle x_k-\bar{x},\bar{x}-x^*\rangle\\ &+\|y_k-\bar{y}\|^2+\|\bar{y}-y^*\|^2+2\langle y_k-\bar{y},\bar{y}-y^*\rangle\\ =&\rho_k(\bar{x},\bar{y})+\|\bar{x}-x^*\|^2+\|\bar{y}-y^*\|^2+2\langle x_k-\bar{x},\bar{x}-x^*\rangle+2\langle y_k-\bar{y},\bar{y}-y^*\rangle.\end{aligned}$$

Without of generality, we may assume that $x_k \rightharpoonup \bar{x}$ and $y_k \rightharpoonup \bar{y}$. By passing to the limit in the relation (2.10), we obtain

$$\rho(x^*, y^*) = \rho(\bar{x}, \bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2.$$

Reversing the role of (x^*, y^*) and $(\bar{x}, \bar{y})$, we also have

$$\rho(\bar{x}, \bar{y}) = \rho(x^*, y^*) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2.$$

By adding the two last equalities, we obtain $x^* = \bar{x}$ and $y^* = \bar{y}$, this implies that the whole sequence $\{(x_k, y_k)\}$ weakly converges to a solutions of problem (1.6), which completes the proof. □

Theorem 2.2 *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive operators with nonempty fixed point set $F(U)$ and $F(T)$. Assume that $U - I$, $T - I$ are demiclosed at origin and the solution set Γ of (1.6) is nonempty. Let $\{\gamma_k\}$ is a positive sequence such that $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$, where λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively and ε is small enough. Then, the sequence $\{(x_k, y_k)\}$ generated by (1.9) weakly converges to a solution (x^*, y^*) of (1.6), provided that $\{\alpha_k\} \subset (\delta, 1 - \delta)$ for small enough $\delta > 0$. Moreover $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

Proof. Taking $(x, y) \in \Gamma$, i.e., $x \in F(U)$; $y \in F(T)$ and $Ax = By$. By repeating the proof of Theorem 2.1, we have (2.6) is true.

Using the fact that U and T are quasi-nonexpansive operators, it follows from Lemma 1.3 that

$$\begin{aligned}\|x_{k+1} - x\|^2 &= \alpha_k\|x_k - x\|^2 + (1 - \alpha_k)\|U(u_k) - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2 \\ &\le \alpha_k\|x_k - x\|^2 + (1 - \alpha_k)\|u_k - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2\end{aligned}$$

and

$$\|y_{k+1} - y\|^2 \le \alpha_k\|y_k - y\|^2 + (1 - \alpha_k)\|v_k - y\|^2 - \alpha_k(1 - \alpha_k)\|T(v_k) - y_k\|^2.$$

So, by (2.6) we have

$$\begin{aligned}&\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \\ &\le\|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k(1 - \alpha_k)[2 - (\lambda_A + \lambda_B)\gamma_k]\|Ax_k - By_k\|^2 \\ &\quad - \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2 - \alpha_k(1 - \alpha_k)\|T(v_k) - y_k\|^2.\end{aligned} \tag{2.11}$$

Now, by setting $\rho_k(x, y) := \|x_k - x\|^2 + \|y_k - y\|^2$, we obtain the following inequality

$$\begin{aligned}\rho_{k+1}(x, y) \le&\rho_k(x, y) - \gamma_k(1 - \alpha_k)[2 - (\lambda_A + \lambda_B)\gamma_k]\|Ax_k - By_k\|^2 \\ &- \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2 - \alpha_k(1 - \alpha_k)\|T(v_k) - y_k\|^2.\end{aligned} \tag{2.12}$$

Following the lines of the proof of Theorem 2.1, by the conditions on $\{\gamma_k\}$ and $\{\alpha_k\}$ we have that the sequence $\{\rho_k(x, y)\}$ converges to some finite limit, say $\rho(x, y)$. Furthermore, we obtain

$$\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|U(u_k) - x_k\| = \lim_{k\to\infty} \|Tv_k - y_k\| = 0.$$

Since

$$\|u_k - x_k\| = \gamma_k \|A^*(Ax_k - By_k)\|$$

and the fact that $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty} \|u_k - x_k\| = 0$. It follows from

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = \lim_{k\to\infty} (1 - \alpha_k)\|U(u_k) - x_k\| = 0$$

that $\{x_k\}$ is asymptotically regular. Similarly, $\lim_{k\to\infty} \|v_k - y_k\| = 0$ and $\{y_k\}$ is asymptotically regular too.

The rest of the proof is analogous to that of Theorem 2.1. □

3. Applications.

We now turn our attention to provide some applications relying on some convex and nonlinear analysis notions, see for example [17].

3.1 Split feasibility problem.

Taking $U = P_C$ and $T = P_Q$, we have the following simultaneous iterative algorithms for solving the SFP (1.3):

$$\begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1 - \alpha_k)P_C(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1 - \beta_k)P_Q(v_k), \end{cases} \tag{3.1}$$

and

$$\begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k)P_C(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1 - \alpha_k)P_Q(v_k). \end{cases} \tag{3.2}$$

3.2 Null point problem.

Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well-known that its associated resolvent operator, $J_\mu^M(x) := (I + \mu M)^{-1}$, is quasi-nonexpansive and $0 \in M(x) \Leftrightarrow x = J_\mu^M(x)$. In other words, zeroes of M are exactly fixed-points of its resolvent operator. By taking $U = J_\mu^M$, $T = J_\nu^S$, where $S : H_2 \to 2^{H_2}$ is another maximal monotone operator, the problem under consideration is nothing but

$$\text{finding } \ x^* \in M^{-1}(0), y^* \in S^{-1}(0) \ \text{ such that } \ Ax^* = By^*,$$

and the algorithms take the following equivalent form:

$$(3.3)\qquad \begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1-\alpha_k) J_\mu^M(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1-\beta_k) J_\nu^S(v_k), \end{cases}$$

and

$$(3.4)\qquad \begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k) J_\mu^M(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k) J_\nu^S(v_k). \end{cases}$$

3.3 Equilibrium problem.

Let D be a nonempty closed convex subset of H, and $F : D \times D \to R$ a bifunction, where R is the set of real numbers. The equilibrium problem for F is to find $x \in D$ such that

$$F(x, y) \geq 0$$

for all $y \in D$. Noting that the connection between monotone operators and equilibrium functions, we may consider the following problem:

$$(3.5)\quad \text{finding } \; x^* \in C, \; y^* \in Q \text{ such that } F(x^*, u) \geq 0, \; J(y^*, v) \geq 0, \; Ax^* = By^*$$

for all $u \in C$ and $v \in Q$, where C and Q are closed convex sets of H_1 and H_2, respectively, F and J belong in the class of bifunctions G verifying the following usual conditions:
(A1) $G(x, x) = 0$ for all $x \in D$;
(A2) G is monotone, that is, $G(x, y) + G(y, x) \leq 0$ for all $x, y \in D$;
(A3) for each $x, y, z \in D$, $\limsup_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
(A4) for each $x \in D$, the function $y \mapsto G(x, y)$ is convex and lower semicontinuous.
It is well-known, see for example [15]-Lemma 5, that the associated resolvent operator $S_{\lambda G} : H \to D$ defined by

$$S_{\lambda G}(x) = \{z \in D : G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \; \forall y \in D\}$$

is quasi-nonexpansive and its fixed-points are exactly the equilibria of G, that is $G(y^*, v) \geq 0$, $\forall\, v \in D$.

By setting $U = S_{\mu F},\ T = S_{\nu J}$, the problem under consideration is nothing but (3.5) and the algorithms take the following equivalent form

$$(3.6)\qquad \begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1-\alpha_k) S_{\mu F}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \beta_k v_k + (1-\beta_k) S_{\nu H}(v_k), \end{cases}$$

and

$$(3.7)\qquad \begin{cases} \forall x_0 \in\in H_1, \quad y_0 \in H_2 \\ u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k) S_{\mu F}(u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k) S_{\nu H}(v_k). \end{cases}$$

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