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EXISTENCE RESULTS FOR A WIDE CLASS OF EQUILIBRIUM PROBLEMS: A GENERAL SCHEME

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Abstract. We take into account the general scheme of equilibrium problems proposed by Joly and Mosco [8] which encompasses several classical problems: fixed points, complementarity problems, Nash solutions, variational and quasi variational inequalities and so on. Using Michael selection theorem and Brouwer fixed point theorem we get a theorem ensuring existence of equilibria. Particular interesting cases are analysed.

1. Introduction

A wide number of equilibrium problems (fixed points, Nash games, variational inequalities, complementarity problems, optimization problems and so on), which are apparently different, have a common structure and they share a unified format: the *Ky Fan inequality* (also called *equilibrium problem*) asks to

$$\text{(1.1)} \qquad \text{find } x \in C \text{ such that } f(x,y) \geq 0, \text{ for all } y \in C,$$

where $C \subseteq \mathbb{R}^n$ is a nonempty set and $f : C \times C \to \mathbb{R}$ is a bifunction (see [3] for a recent tutorial on this subject). In this format the constraint set is fixed and hence the model can not be used in cases where the constraint depends on the current analysed point. This more general setting, commonly called *quasiequilibrium problem*, was studied for the first time in [2] to investigate impulse control problems and it was subsequently used by several authors for describing a lot of problems that arise in different fields: telecommunication systems, spatial oligopolistic electricity models, dynamic traffic assignment model (see [4] and references therein). Formally speaking, a quasiequilibrium problem is a Ky Fan inequality whose feasible region is subject to modifications according to the point considered as a candidate solution, i.e. it asks to

$$\text{(1.2)} \qquad \text{find } x \in K(x) \text{ such that } f(x,y) \geq 0, \text{ for all } y \in K(x),$$

where $K : C \rightrightarrows C$ is a set-valued map.

Joly and Mosco [8] described a unifying framework for studying both Ky Fan inequalities and quasiequilibrium problems at the same time. Let $\varphi : C \times C \to (-\infty, +\infty]$ be an extended valued bifunction such that the domain of $\varphi(x, \cdot)$

$$D_\varphi(x) = \{y \in C : \varphi(x,y) < +\infty\}$$

is nonempty for every $x \in C$; the problem considered in [8] asks to

$$\text{(1.3)} \qquad \text{find } x \in C \text{ such that } f(x,y) + \varphi(x,y) \geq \varphi(x,x), \text{ for all } y \in C.$$

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Clearly problem (1.3) encompasses (1.1) choosing $\varphi \equiv 0$ and (1.2) choosing $\varphi(x, y) = \delta(y, K(x))$ where δ is the indicator function defined as follows

$$\delta(x, A) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A, \end{cases}$$

for all $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. In order to establish existence results for (1.3), Joly and Mosco introduced the so-called *variational selection* associated to (1.3) which is the set-valued map $S : C \rightrightarrows C$ defined as

$$S(x) = \{z \in C : f(z, y) + \varphi(x, y) \geq \varphi(x, z),\ \forall y \in C\}$$

and whose set of fixed points coincides with the set of solutions of the problem (1.3). The proof of the existence of a fixed point of S [8, Théorème 1] is based on the application of the Kakutani fixed point theorem.

In this short note we use a different approach to describe the solution set of (1.3). Our approach follows the line of the analogous one presented in [6] and [5]. Associated to problem (1.3) we define the set-valued map $F : C \rightrightarrows C$ as

$$F(x) = \{y \in C : f(x, y) + \varphi(x, y) < \varphi(x, x)\}.$$

Clearly if x solves (1.3) then x is a maximal element of F, i.e. $F(x) = \emptyset$, while the reverse implication holds since $D_\varphi(x) \neq \emptyset$ for every x. The key tool in our analysis is a famous selection result due to Michael [9] which exploits the finite dimension of the decision space. By using this result together with the Brouwer fixed point theorem, we get the existence of equilibria for (1.3). Our result subsumes several similar statements existing in literature, which can be obtained by means of a suitable choice of f and φ.

2. Result

Let $\Phi : X \rightrightarrows Y$ be a set-valued map with $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. The graph of Φ is the set $\{(x, y) \in X \times Y : y \in \Phi(x)\}$ and the lower section at $y \in Y$ is $\{x \in X : y \in \Phi(x)\}$. The neighborhood system at x, that is the collection of all neighborhoods of a point x is denoted $\mathcal{N}_x$. The map Φ is said to be lower semicontinuous at x if for each open set $\Omega \subseteq \mathbb{R}^m$ such that $\Phi(x) \cap \Omega \neq \emptyset$ there exists a neighborhood $U_x \in \mathcal{N}_x$ such that

$$\Phi(x') \cap \Omega \neq \emptyset, \qquad \forall x' \in U_x \cap X.$$

The map Φ is lower semicontinuous if it is lower semicontinuous at each point $x \in X$. If $A \subseteq X$, then the restriction of Φ to A is the map $\Phi_{|A} : A \rightrightarrows Y$ given by $\Phi_{|A}(x) = \Phi(x)$, for $x \in A$. The map Φ is lower semicontinuous on A if its restriction to A, $\Phi_{|A}$, is lower semicontinuous.

Let $g : X \times Y \to (-\infty, +\infty]$ be an extended valued bifunction with the domain of $g(x, \cdot)$ nonempty for each $x \in X$. A sufficient condition for the lower semicontinuity of the set-valued map $\Phi : X \rightrightarrows Y$ defined as $\Phi(x) = \{y \in Y : g(x, y) < 0\}$ is given by the inequality

$$\limsup_{x' \to x} \inf_{y' \to y} g(x', y') = \sup_{U_y \in \mathcal{N}_y} \inf_{U_x \in \mathcal{N}_x} \sup_{x' \in U_x} \inf_{y' \in U_y} g(x', y') \leq g(x, y), \tag{2.1}$$

for all (x, y). Indeed let $y \in \Phi(x) \cap \Omega$ be fixed. Then $g(x, y) < 0$ and, choosing $U_y = \Omega$ there exists U_x such that $\inf_{y' \in U_y} g(x', y') \leq g(x, y)/2 < 0$ for all $x' \in U_x \cap X$, that is there is $y' \in \Omega$ such that $g(x', y') < 0$.

A set-valued map with open graph has open lower sections and, in turn, if it has open lower sections then it is lower semicontinuous. Moreover the domain of a lower semicontinuous set-valued map is open.

A selection of Φ is a function $\phi : X \to Y$ that satisfies $\phi(x) \in \Phi(x)$ for each $x \in X$. Given a set A, $\operatorname{co} A$ is the convex hull of A and, if Φ is a set-valued map, $\operatorname{co}\Phi$ is the set-valued map defined by $(\operatorname{co}\Phi)(x) = \operatorname{co}\Phi(x)$. If $X = Y$ a fixed point of Φ is a point $x \in X$ satisfying $x \in \Phi(x)$. The set of the fixed points of Φ is denoted by $\operatorname{fix}\Phi$.

Theorem 2.1. *Let C be compact convex and assume that*

(i) $\operatorname{fix} \operatorname{co} F = \emptyset$,
(ii) $\operatorname{fix} D_\varphi$ *is closed,*
(iii) D_φ *is lower semicontinuous,*
(iv) F *is lower semicontinuous on* $\operatorname{fix} D_\varphi$.

Then the problem (1.3) has a solution.

Proof. Firstly we show that F is lower semicontinuous on the whole set C. The set-valued map F is a submap of D_φ, that is $F(x) \subseteq D_\varphi(x)$ for all $x \in C$. Moreover $F(x) = D_\varphi(x)$ if $x \notin \operatorname{fix} D_\varphi$. Since D_φ is lower semicontinuous on the open set $C \setminus \operatorname{fix} D_\varphi$, for proving the lower semicontinuity of F it is enough to show its lower semicontinuity at any $x \in \operatorname{fix} D_\varphi$. Take an open set $\Omega \subseteq \mathbb{R}^n$ such that $F(x) \cap \Omega \neq \emptyset$. From (iii) and (iv) there are two neighborhoods $U_x^1, U_x^2 \in \mathcal{N}_x$ such that

$$F(x') \cap \Omega \neq \emptyset, \qquad \forall x' \in U_x^1 \cap \operatorname{fix} D_\varphi$$
$$D_\varphi(x') \cap \Omega \neq \emptyset, \qquad \forall x' \in U_x^2 \cap C.$$

Hence, choosing $U_x = U_x^1 \cap U_x^2$ we have $F(x') \cap \Omega \neq \emptyset$ for every $x' \in U_x \cap C$.

By contradiction assume that $F(x) \neq \emptyset$ for all $x \in C$. One of the famous Michael's selection results affirms that every lower semicontinuous set-valued map from a metric space to $\mathbb{R}^n$ with nonempty convex values admits a continuous selection [9, Theorem 3.1‴ (b)]. Since $\operatorname{co} F$ is lower semicontinuous [1, Theorem 17.36], it admits a continuous selection $g : C \to C$. The Brouwer fixed point theorem provides the existence of a fixed point of g which in turn is a fixed point of $\operatorname{co} F$ and it contradicts (i). □

Remark 2.2. Assumption (iv) of Theorem 2.1 requires that the restriction of F to $\operatorname{fix} D_\varphi$ is lower semicontinuous, and taking into account that $\varphi(x, x) < +\infty$, for each $x \in \operatorname{fix} D_\varphi$, the values of this map are

$$F_{|\operatorname{fix} D_\varphi}(x) = \{y \in C : f(x, y) + \varphi(x, y) - \varphi(x, x) < 0\}.$$

Thus, letting $g : \operatorname{fix} D_\varphi \times C \to (-\infty, +\infty]$ the extended valued bifunction defined as $g(x, y) = f(x, y) + \varphi(x, y) - \varphi(x, x)$, a sufficient condition for (iv) is the validity of condition (2.1), for all pairs $(x, y) \in \operatorname{fix} D_\varphi \times C$.

Remark 2.3. The fact that $x \in \operatorname{fix} F$ if and only if $x \in \operatorname{fix} D_\varphi$ and $f(x, x) < 0$ follows directly from the definition of F. Thus $f(x, x) \geq 0$, for all $x \in \operatorname{fix} D_\varphi$, is a necessary condition for (i), but it is not sufficient. Instead a sufficient condition for (i) is the convex valuedness of F which is guaranteed by the quasiconvexity of $f(x,\cdot)+\varphi(x,\cdot)$ for all $x \in C$.

3. Particular cases

This section deals with the application of Theorem 2.1 when f and φ are properly chosen.

3.1. **The case $\varphi \equiv 0$.** In this case the problem (1.3) collapses to the Ky Fan inequality (1.1) with

$$F(x) = \{y \in C : f(x,y) < 0\} \quad \text{and} \quad D_\varphi(x) = C,$$

for each $x \in C$. Assumptions (ii) and (iii) of Theorem 2.1 hold trivially. Furthermore assumption (i) is guaranteed from the quasiconvexity of $f(x,\cdot)$ together with $f(x,x) \geq 0$ for all x, as noticed in Remark 2.3. The upper semicontinuity of $f(\cdot,y)$ implies that F has open lower sections and hence F is lower semicontinuous. Therefore Theorem 2.1 contains the well-known Fan minimax inequality [7] as a special case while being more general as the following two examples show.

Example 3.1. The Ky Fan inequality problem (1.1) associated to $f(x,y) = (x - 3y + 6)(2x - 3y + 3)(x - y)$ and $C = [0,3]$ has a unique solution at $x = 3$. The sublevel set $F(x) = \{y \in [0,3] : f(x,y) < 0\}$ is not convex for all $x \in [0,3)$ and the Fan minimax inequality does not apply. Anyway the map $\operatorname{co} F(x) = (x,3]$ has not fixed points and all the assumptions of Theorem 2.1 hold.

Example 3.2. Consider the Ky Fan inequality associated to $C = [0,1]$ and

$$f(x,y) = \begin{cases} -1 & \text{if } 0 < y = x^2 < 1 \\ 0 & \text{otherwise,} \end{cases}$$

Clearly $F(x)$ is convex but $f(\cdot,y)$ is not upper semicontinuous for all $y \in (0,1)$. Anyway

$$\limsup_{x' \to x} \inf_{y' \to y} f(x',y') = f(x,y)$$

and this fact ensures that F is lower semicontinuous and the applicability of Theorem 2.1. The points $x = 0$ and $x = 1$ are two equilibria.

3.2. **The case $\varphi(x,y) = \delta(y,K(x))$.** In this case the problem (1.3) collapses to the quasiequilibrium problem (1.2) and $D_\varphi(x) = K(x)$ which is nonempty for every x. Hence assumptions (ii) and (iii) of Theorem 2.1 become $\operatorname{fix} K$ closed and K lower semicontinuous, respectively. Invoking Remark 2.3, assumption (i) holds true if K is convex valued, $f(x,\cdot)$ is quasiconvex and $f(x,x) \geq 0$ for each $x \in \operatorname{fix} K$. Assumption (iv) requires the lower semicontinuity of the intersection map $K(x) \cap \{y \in C : f(x,y) < 0\}$ on $\operatorname{fix} K$. Since the intersection of a lower semicontinuous set-valued map with an open graph set-valued map is lower semicontinuous, a sufficient condition for (iv) is the upper semicontinuity of f on $\operatorname{fix} K \times C$. Therefore Theorem 2.1 contains [6, Theorem 2.1] as a special case while being more general as the following example shows.

Example 3.3. The quasiequilibrium problem (1.2) associated to $C = [0,5]$, $f(x,y) = (x-y)(y-3)$ and

$$K(x) = \begin{cases} (2,3) \cup (4,5) & \text{if } x \in [0,1] \\ (2,5) & \text{if } x \in (1,2) \\ [3,4] & \text{if } x \in [2,5], \end{cases}$$

has a unique solution at $x = 4$. The set $\operatorname{fix} K = [3,4]$ is closed, K has open lower sections and f is continuous. For any $x \in [0,1]$ the value $F(x) = K(x)$ is not convex. Moreover the sublevel set $\{y \in [0,5] : f(x,y) < 0\}$ is not convex for all $x \in (0,5)$ and [6, Theorem 2.1] does not apply. Anyway the map

$$\operatorname{co} F(x) = \begin{cases} (2,5) & \text{if } x \in [0,2) \\ [3,4] & \text{if } x \in [2,3) \cup (4,5] \\ (x,4] & \text{if } x \in [3,4], \end{cases}$$

has not fixed points and all the assumptions of Theorem 2.1 hold.

3.3. **The case $f \equiv 0$ and $\varphi(x,y) = \delta(y, K(x))$.** In this case the problem (1.3) collapses to find a fixed point of the set-valued map $K = D_\varphi$ which has nonempty values and Theorem 2.1 is a fixed point result. Assumption (ii) becomes $\operatorname{fix} K$ closed: anyway, this assumption becomes redundant since $\operatorname{fix} K$ not closed implies directly $\operatorname{fix} K \neq \emptyset$. Assumption (iii) is equivalent to require the lower semicontinuity of K. Moreover

$$F(x) = \begin{cases} \emptyset & \text{if } x \in \operatorname{fix} K \\ K(x) & \text{if } x \notin \operatorname{fix} K, \end{cases}$$

hence, assumption (iv) is trivially satisfied and assumption (i) coincides with $\operatorname{fix} K = \operatorname{fix} \operatorname{co} K$. Therefore Theorem 2.1 collapses in the following fixed point result.

Corollary 3.4. *Every lower semicontinuous and nonempty valued map K from a convex compact subset of $\mathbb{R}^n$ to itself with $\operatorname{fix} K = \operatorname{fix} \operatorname{co} K$ has a fixed point.*

Corollary 3.4 can be viewed as a consequence of the selection result [9, Theorem 3.1$'''$ (b)] and the Brouwer fixed point theorem.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis. A hitchhikers guide*, Springer-Verlag, Berlin, 2006.

[2] A. Bensoussan, M. Goursat and J. L. Lions, *Contrôle impulsionnel et inéquations quasi-variationnelles stationnaires*, C.R. Acad. Sci. Paris Sér. A **276** (1973), 1279–1284.

[3] G. Bigi, M. Castellani, M. Pappalardo and M. Passacantando, *Nonlinear Programming Techniques for Equilibria*, EURO Advanced Tutorials on Operational research, Springer, 2019.

[4] M. Castellani and M. Giuli, *A coercivity condition for nonmonotone quasiequilibria on finite-dimensional spaces*, J. Global Optim. **75** (2019), 163–176.

[5] M. Castellani, M. Giuli and M. Pappalardo, *A Ky Fan minimax inequality for quasiequilibria on finite dimensional spaces*, J. Optim. Theory Appl. **179** (2018), 53–64.

[6] P. Cubiotti, *Existence of Nash equilibria for generalized games without upper semicontinuity*, Internat. J. Game Theory **26** (1997), 267–273.

[7] K. Fan, *A minimax inequality and applications*, in Inequalities III, Academic Press, New York (1972), 103–113.

[8] J.-L. Joly and U. Mosco, *À propos de l'existence et de la régularité des solutions de certaines inéquations quasi-variationnelles*, J. Functional Analysis **34** (1979), 107–137.

[9] E. Michael, *Continuous selections. I*, Ann. of Math. **63** (1956), 361–382.

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