Journal of Nonlinear and Convex Analysis
Volume 21, Number 4, 2020, 943–951

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

PICARD TYPE ITERATIVE METHOD WITH APPLICATIONS TO MINIMIZATION PROBLEMS AND SPLIT FEASIBILITY PROBLEMS

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Abstract. In this paper, we study convergence analysis of a Picard type iterative algorithm for finding fixed points of nonexpansive mappings in the setting of Hilbert spaces. As an application of our result, we propose a new gradient projection algorithm for solving convex minimization problems and derive weak convergence result of such algorithm. An example is presented to illustrate our algorithm and result. As another application of our result, we present an algorithm for solving split feasibility problems and discuss the weak convergence of the algorithm.

1. Introduction

Consider convex minimization problem:

$$\min_{x\in C} f(x), \tag{1.1}$$

where C is a nonempty closed convex subset of a Hilbert space H and $f : C \to \mathbb{R}$ is a convex mapping. Let P_C be the projection from H onto C and $f : C \to \mathbb{R}$ be Fréchet differentiable. It is known that a minimum x^* of $f(x)$ in C is a fixed point of the mapping $P_C(I-\lambda\nabla f)$, that is $x^* = P_C(x^*-\lambda\nabla f(x^*))$, where ∇f is the gradient of f and $\lambda > 0$. This has motivated to develop some iterative algorithms for finding the solutions of minimization problem (1.1); See, for example, [8,13–15] and the references therein. The most commonly used algorithms are: Picard iterative algorithm to find the fixed point of the mapping $T : C \to C$:

$$x_{n+1} = T(x_n), \tag{1.2}$$

and gradient projection algorithm (in short, GPA) to find the fixed point of the mapping $P_C(I - \lambda\nabla f)$ which is a solution of the minimization problem (1.1):

$$x_{n+1} = P_C(x_n - \lambda_n\nabla f(x_n)), \tag{1.3}$$

where $\lambda_n > 0$ is a stepsize. The convergence of algorithm (1.3) depends on the behavior of the gradient ∇f. In a more expressive way, $P_C(I - \lambda_n\nabla f)$ is a contraction mapping for $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{L^2}$ when ∇f is strongly monotone, that is, there exists $\alpha > 0$ such that

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \alpha \|x - y\|^2, \quad \text{for all } x, y \in C, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 47J25, 54H25, 90C25, 49M37, 65K10.
Key words and phrases. Convex minimization problems, fixed point problems, split feasibility problems, gradient projection method, picard type iterative methods, convergence analysis.
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and Lipschtiz continuous, that is, there exists $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \quad \text{for all } x, y \in C. \tag{1.5}$$

In this case, GPA reduces to the Picard iteration algorithm, and hence by Banach contraction theorem the sequence $\{x_n\}_{n=0}^{\infty}$ generated by algorithm (1.3) converges to a unique fixed point of $P_C(I - \lambda_n \nabla f)$, which is the unique solution of the minimization problem (1.1).

Levitin and Polyak [12] also proved the weak convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ generated by algorithm (1.3) to a solution of minimization problem (1.1) when ∇f is Lipschitz continuous (but not necessarily strongly monotone) and

$$0 < \lim_{n\to\infty} \inf \lambda_n \leq \lim_{n\to\infty} \sup \lambda_n < \frac{2}{L}.$$

It is further studied by Xu [16, 17] by using averaged mapping approach.

On the other hand, Gürsoy and Karakaya [11] proposed the following extension of Picard iterative algorithm for finding a fixed point of a contraction mapping $T : C \to C$.

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= Ty_n \\ y_n &= (1 - a_n)Tx_n + a_n T z_n \\ z_n &= (1 - b_n)x_n + b_n T x_n, \end{aligned} \tag{1.6}$$

where C is a nonempty closed convex subset of a Banach space B and a_n, $b_n \in [0, 1]$ for all $n = 0, 1, 2, \ldots$. They proved the strong convergence of the sequence generated by algorithm (1.6) to a fixed point of T under the condition that T is contraction and $\sum_{i=0}^{\infty} a_i b_i = \infty$. It is well-known that every contraction mapping is nonexpansive but the converse need not be true. This motivates us to study the weak convergence of the sequence generated by algorithm (1.6) to a fixed point of T under the condition that T is nonexpansive.

The present paper is organized as follows: In the next section, we gather some known results and definitions which will be used in the sequel. Section 3 is devoted to the study of weak convergence of the sequence generated by algorithm (1.6) to a fixed point of T under the condition that T is nonexpansive. In Section 4, we give an application of our result developed in Section 3 to convex minimization problem (1.1). By considering $T := P_C(I - \lambda_n \nabla f)$, we propose a gradient projection type algorithm for computing the solutions of convex minimization problem (1.1). Since $T := P_C(I - \lambda_n \nabla f)$ is nonexpansive, the result of Section 3 can be used to show the weak convergence of the proposed algorithm for the convex minimization problem (1.1). An example in support of our algorithm and result is also presented. In the last section, we propose an algorithm for finding the solutions of the split feasibility problems. By using the result of Section 3, we derive the weak convergence of such algorithm.

2. Preliminaries

Throughout this paper, we shall use the following notations:

- $x_n \rightharpoonup x$ means that x_n converges weakly to x.

- $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ is the set of weak limit of the sequence $\{x_n\}_{n=0}^{\infty}$.
- F_T is the set of fixed points of T.

The following definitions and lemmas will be used in the sequel.

Definition 2.1. Let H be a Hilbert space and C be a nonempty closed convex subset of H. The mapping $P_C : H \to C$ is called a projection map if it assigns the unique point $P_C x \in C$ to each $x \in H$ with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

That is, $P_C x$ is nearest point of C to x.

Lemma 2.2. *Let H be a Hilbert space. Then the following inequality holds:*

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad \textit{for all } x, y \in H,\ \lambda \in [0,1].$$

Lemma 2.3 (Demiclosedness Principle)**.** [3] *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F_T \neq \emptyset$. If $\{x_n\}_{n=0}^{\infty}$ is a sequence in C converges weakly to x and if $\{(I-T)x_n\}_{n=0}^{\infty}$ converges strongly to y, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in F_T$.*

Lemma 2.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in H such that the following conditions hold:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each $x \in C$;*

(ii) $\omega_w(x_n) \subset C$.

Then $\{x_n\}_{n=0}^{\infty}$ converges weakly to a point in C.

3. A weak convergence result for Picard type iterative algorithm

In this section we establish the weak convergence of the sequence generated by algorithm (1.6) under the nonexpansiveness of the mapping T.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping. Let*

$$\begin{aligned} &0 < \liminf a_n \le \limsup a_n < 1, \\ &0 < \liminf b_n \le \limsup b_n < 1. \end{aligned} \tag{3.1}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by algorithm (1.6) *converges weakly to a fixed point of T.*

Proof. Let $x^* \in F_T = \{x \in C : Tx = x\}$. By assumption (3.1), we have $0 < a \le a_n,\ b_n \le b < 1$ for all $n = 0, 1, 2, \ldots$, where a, b are constants. By using (1.6) and nonexpansiveness of T, we have

$$\|x_{n+1} - x^*\| = \|Ty_n - Tx^*\| \le \|y_n - x^*\|, \tag{3.2}$$

$$\begin{aligned} \|y_n - x^*\| &= \|(1-a_n)Tx_n + a_n Tz_n - x^*\| \\ &\le (1-a_n)\|Tx_n - x^*\| + a_n\|Tz_n - x^*\| \\ &\le (1-a_n)\|x_n - x^*\| + a_n\|z_n - x^*\|, \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
\|z_n - x^*\| &= \|(1-b_n)x_n + b_n T x_n - x^*\| \\
&= \|(1-b_n)(x_n - x^*) + b_n(Tx_n - x^*)\| \\
&\leq (1-b_n)\|x_n - x^*\| + b_n\|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned} \tag{3.4}
$$

Substituting (3.4) in (3.3), we get

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{3.5}$$

By using (3.5) in (3.2), we obtain

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|, \tag{3.6}$$

which implies that $\{\|x_n - x^*\|\}_{n=0}^{\infty}$ is nonincreasing. Thus

$$\lim_{n\to\infty} \|x_n - x^*\| \text{ exists for all } x^* \in F_T. \tag{3.7}$$

Now we examine the limit of $\|Tx_n - x_n\|$. By using Lemma 2.2, we get

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|Ty_n - x^*\|^2 \\
&\leq \|y_n - x^*\|^2 \\
&= \|(1-a_n)Tx_n + a_n T z_n - x^*\|^2 \\
&= (1-a_n)\|Tx_n - x^*\|^2 + a_n\|Tz_n - x^*\|^2 - (1-a_n)a_n\|Tx_n - Tz_n\|^2 \\
&\leq (1-a_n)\|x_n - x^*\|^2 + a_n\|z_n - x^*\|^2 \\
&\leq (1-a_n)\|x_n - x^*\|^2 + a_n\|(1-b_n)x_n + b_n T x_n - x^*\|^2 \\
&= (1-a_n)\|x_n - x^*\|^2 \\
&\quad + a_n\Big((1-b_n)\|x_n - x^*\|^2 + b_n\|Tx_n - x^*\|^2 - b_n(1-b_n)\|x_n - Tx_n\|^2\Big) \\
&\leq (1-a_n)\|x_n - x^*\|^2 \\
&\quad + a_n\Big((1-b_n)\|x_n - x^*\|^2 + b_n\|x_n - x^*\|^2 - b_n(1-b_n)\|x_n - Tx_n\|^2\Big) \\
&= (1-a_n)\|x_n - x^*\|^2 + a_n\|x_n - x^*\|^2 - a_n b_n(1-b_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - x^*\|^2 - a_n b_n (1-b_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{3.8}
$$

Rearranging (3.8), we have

$$
\begin{aligned}
\|Tx_n - x_n\|^2 &\leq \frac{1}{(1-b_n)a_n b_n}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big) \\
&\leq \frac{1}{a^2(1-b)}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big).
\end{aligned} \tag{3.9}
$$

By taking the limit both sides of inequality (3.9), we get

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.10}$$

Now we prove that $\omega_w(x_n) \subset F_T$. Suppose that $x' \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ of the sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n_j} \rightharpoonup x'$. From (3.10), we have $\lim_{j\to\infty} \left\|Tx_{n_j} - x_{n_j}\right\| = 0$. From Lemma 2.3, we observe that $x' \in F_T$. So, $\omega_w(x_n) \subset F_T$. Since T is nonexpansive, by Lemma 3.4 in [10], F_T is closed and convex. Thus by Lemma 2.4, we conclude that $\{x_n\}_{n=0}^{\infty}$ converges weakly to a point in F_T. □

4. A new gradient projection algorithm for convex minimization problems

Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \to \mathbb{R}$ be a convex Fréchet differentiable function. We assume that the minimization problem (1.1) is solvable, and we denote by S the set of solutions of minimization problem (1.1). We propose the following new gradient projection algorithm for computing the solutions of convex minimization problems.

$$\begin{array}{rcl} x_0 & \in & C \\ x_{n+1} & = & P_C(I - \lambda_n \nabla f) y_n \\ y_n & = & (1-a_n) P_C(I - \lambda_n \nabla f) x_n + a_n P_C(I - \lambda_n \nabla f) z_n \\ z_n & = & (1-b_n) x_n + b_n P_C(I - \lambda_n \nabla f) x_n, \end{array} \tag{4.1}$$

where the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ satisfy (3.1) and λ_n is a positive real number for all $n = 0, 1, 2, \ldots$.

As a corollary of Theorem 3.1, we derive the weak convergence of the sequence generated by algorithm (4.1) to a solution of convex minimization problem (1.1).

Corollary 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \to \mathbb{R}$ be a convex Fréchet differentiable function such that its gradient ∇f is L-Lipschitzian with constant $L > 0$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by algorithm* (4.1) *converges weakly to a minimizer of* (1.1) *provided that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ satisfies the following condition:*

$$0 < \liminf \lambda_n \le \limsup \lambda_n < 2/L.$$

Proof. Since $P_C(I - \lambda_n \nabla f)$ is a projection mapping, by taking $T := P_C(I - \lambda_n \nabla f)$ and using Theorem 3.1, we obtain the desired result. □

We now illustrate algorithm (4.1) and Corollary 4.1 by the following example.

Example 4.2. Let $H = l_2 = \{x = (x_0, x_1, x_2, \ldots) : \sum_{n=0}^{\infty} |x_n|^2 < \infty$ and $x_n \in \mathbb{R}$ for all $n = 0, 1, 2, \ldots\}$. Then it is well-known that H is a Hilbert space with norm induced by inner product $\|x\| = \sqrt{\langle x, x\rangle} = (\sum_{n=0}^{\infty} |x_n|^2)^{\frac{1}{2}}$. Let $C = \{x = (x_0, x_1, x_2, \ldots) : \|x\| = (\sum_{n=0}^{\infty} |x_n|^2)^{\frac{1}{2}} \le \frac{\pi}{2}\}$. Then C is a closed convex subset of H. Define $f : C \to \mathbb{R}$ by $f((x_0, x_1, \ldots)) = -\cos x_0$. It can be easily seen that f is a convex function since $-\cos x_0$ is convex for all $x_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Also, $S = \{x = (x_0, x_1, x_2, \ldots) : x_0 = 0\}$ is the set of minimizers of f since $x_0 = 0$ minimizes $-\cos x_0$.

It can be easily seen that f is Fréchet differentiable and the Fréchet derivative of f at x is $\nabla f x = (\sin x_0, 0, 0, \ldots)$ which is not strongly monotone.

No. of Iterates	Iterative Algorithm (4.1)
0	$\left(\frac{\pi}{2}, 0, 0, 0, \ldots\right)$
1	$(0.620992, 0, 0, 0, \ldots)$
2	$(0.167308, 0, 0, 0, \ldots)$
3	$\left(0.413786 \times 10^{-1}, 0, 0, 0, \ldots\right)$
4	$\left(0.100765 \times 10^{-1}, 0, 0, 0, \ldots\right)$
$\vdots$	$\vdots$
50	$\left(0.107622 \times 10^{-30}, 0, 0, 0, \ldots\right)$
$\vdots$	$\vdots$
100	$\left(0.218063 \times 10^{-62}, 0, 0, 0, \ldots\right)$
$\vdots$	$\vdots$
150	$\left(0.409455 \times 10^{-94}, 0, 0, 0, \ldots\right)$
$\vdots$	$\vdots$

TABLE 1. Convergence behavior of iterative algorithm (4.1) with initial point $x_0 = \left(\frac{\pi}{2}, 0, 0, \ldots\right) \in C$ to $(0, 0, 0, \ldots) \in S$.

Indeed, let $x = (0, \frac{1}{2}, 0, 0, \ldots)$ and $y = \left(0, -\frac{1}{2}, , 0, 0, \ldots\right)$. Then we have $\langle \nabla f x - \nabla f y, x - y \rangle = 0$. On the other hand, $\|x - y\|^2 = 1$.

Now we show that ∇f is a $1-$Lipschitzian. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}
\|\nabla f x - \nabla f y\| &= \|(\sin x_0, 0, 0, \ldots) - (\sin y_0, 0, 0, \ldots)\| \\
&= \|(\sin x_0 - \sin y_0, 0, 0, \ldots)\| \\
&= |\sin x_0 - \sin y_0| \\
&= \left| 2 \cos \frac{(x_0 + y_0)}{2} \sin \frac{(x_0 - y_0)}{2} \right| \\
&\leq 2 \left| \sin \frac{(x_0 - y_0)}{2} \right| \\
&\leq |x_0 - y_0| = \left(|x_0 - y_0|^2 \right)^{\frac{1}{2}} \\
&\leq \left(|x_0 - y_0|^2 + |x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots \right)^{\frac{1}{2}} \\
&= \|x - y\| .
\end{aligned}$$

For the initial point $x_0 = \left(\frac{\pi}{2}, 0, 0, \ldots\right) \in C$, $a_n = \frac{2n-1}{3n+5}$, $b_n = \frac{n+1}{2n+3}$ and $\lambda_n = \frac{1}{2}$ for all $n = 0, 1, 2, \ldots$, the convergence result for the iterative algorithm (4.1) to $(0, 0, 0, \ldots) \in S$ is given in Table 1.

5. AN ALGORITHM FOR SPLIT FEASIBILITY PROBLEMS

In this section, we give another application of Theorem 3.1 to the split feasibility problem (in short, SFP). Let C and Q be nonempty closed convex subset of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator.

The SFP is to find a point x such that

$$x \in C \quad \text{and} \quad Ax \in Q. \tag{5.1}$$

It is clear that if x is a solution of SFP (5.1), then $x \in C$ and $Ax - q = 0$ for some $q \in Q$. This forces us to consider the following minimization problem:

$$\min_{x \in C} f(x) = \frac{1}{2} \|Ax - P_Q Ax\|^2 . \tag{5.2}$$

Xu [17] showed that $x \in H_1$ is a solution of SFP (5.1) if and only if x solves the following fixed point equation

$$P_C(I - \lambda A^*(I - P_Q)A)x = x. \tag{5.3}$$

By using this characterization, Byrne [4] introduced the following algorithm which is referred as CQ algorithm:

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \quad n \geq 0, \tag{5.4}$$

where $0 < \lambda < \frac{2}{\|A\|^2}$ and A^* is adjoint of A. He showed that the CQ algorithm generated by (5.4) converges weakly to a solution of the SFP (5.1). For further details on SFP, we refer to [1, 2, 5–7, 9, 17] and the references therein.

We propose the following algorithm to solve SFP (5.1):

$$\begin{aligned} x_{n+1} &= P_C(I - \lambda_n A^*(I - P_Q)A)y_n \\ y_n &= (1 - a_n)P_C(I - \lambda_n A^*(I - P_Q)A)x_n + a_n P_C(I - \lambda_n A^*(I - P_Q)A)z_n \\ z_n &= (1 - b_n)x_n + b_n P_C(I - \lambda_n A^*(I - P_Q)A)x_n, \end{aligned} \tag{5.5}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty} \subseteq [0, 1]$ and $\lambda_n \in (0, \frac{2}{\|A\|^2})$ for all $n \in \mathbb{N}$.

Theorem 5.1. *Assume that the SFP (5.1) is solvable. Let $\{x_n\}_{n=0}^{\infty}$ be defined by algorithm (5.5) with the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ satisfying* (3.1) *and $\lambda_n \in \left(0, \frac{2}{L}\right)$ for all $n \in \mathbb{N}$ satisfying*

$$0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{\|A\|^2}.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to a solution of SFP (5.1).

Proof. By the definition of f, we have that f is continuously differentiable with a gradient given by

$$\nabla f(x) = A^*(I - PQ)Ax,$$

where A^* is adjoint of A. Since $(I - P_Q)$ is (firmly) nonexpansive,

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||,$$

where $L = ||A||^2$. Using Theorem 3.1, we reach the desired conclusion. $\square$

Conclusions

A Picard type iterative algorithm is considered in [11] and the weak convergence of such algorithm is studied for contraction mappings. In this paper, we established the weak convergence of this algorithm for nonexpansive mappings. As an application of our result, we proposed a new gradient project algorithm for solving convex minimization problems and derived weak convergence result of such algorithm. We illustrated our algorithm and result by an example. As another application of our result, we presented an algorithm for solving split feasibility problems and discussed its weak convergence.

Acknowledgements

Authors are grateful to Prof. Jen-Chih Yao and the referees for their constructive suggestions and comments to improve the previous version of the paper.

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Manuscript received September 5, 2019
revised December 1, 2019

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