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FIXED POINT RESULTS FOR ORBITAL CONTRACTIONS IN COMPLETE GAUGE SPACES WITH APPLICATIONS

ADRIAN PETRUŞEL, GABRIELA PETRUŞEL, AND MU-MING WONG*

Abstract. In this paper, we will present fixed point results for orbital contractions in complete gauge spaces. We discuss existence and approximation of the fixed point (in the local and the global case), data dependence, well-posedness, Ostrowski stability property and Ulam-Hyers stability for the fixed point equation. An application to an initial value problem associated to a first order differential equation is given.

1. Introduction

Let E be a nonempty set. The functional $d : E \times E \to \mathbb{R}_+$ is said to be a gauge (or a pseudo-metric) on E if the following axioms hold:

a) for $x, y \in E$, if $x = y$ then $d(x, y) = 0$;

b) $d(x, y) = d(y, x)$, for all $x, y \in E$;

c) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in E$.

If d is a gauge on E and $r > 0$, then the open ball centered at $x_0 \in E$ of radius r is

$$B_d(x_0; r) := \{x \in E : d(x_0, x) < r\}.$$

Let $\mathcal{D} := \{d_\alpha\}_{\alpha\in\Lambda}$ be a family of gauges on E. Then, the family $\{d_\alpha\}_{\alpha\in\Lambda}$ is said to be separating if, for all $(x, y) \in E \times E$, $x \neq y$, there exists $d_\alpha \in \mathcal{D}$ such that $d_\alpha(x, y) \neq 0$. Moreover, if E is a nonempty set and $\mathcal{D} := \{d_\alpha\}_{\alpha\in\Lambda}$ is a family of gauges on E (where Λ is a directed set), then the topology $\mathcal{T}(\mathcal{D})$ having as a subbasis the family of open balls $\mathcal{B}(\mathcal{D}) := \{B_{d_\alpha}(x; r) : x \in E, r > 0, d_\alpha \in \mathbb{D}\}$ is called the topology on E induced by the family $\mathcal{D}$. A pair $(E, \mathcal{T}(\mathcal{D}))$ of a nonempty set E and a separating gauge structure $\mathcal{D}$ on E is called a gauge space. It was proved that any family $\mathcal{D}$ of gauges on a set E induces on E a uniform structure $\mathcal{U}$ and conversely, any uniform structure $\mathcal{U}$ on E is induced by a family of gauges on E. Notice that the space $(\mathbb{E}, \mathcal{T}(\mathcal{D}))$ is Hausdorff if $\mathcal{D}$ is separating.

Throughout this paper $\mathbb{E} := (E, \mathcal{T}(\mathcal{D}))$ will denote a nonempty set E endowed with a separating gauge structure $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$. Let $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We also denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+ := [0, +\infty)$.

A sequence (x_n) of elements in $\mathbb{E}$ is said to be Cauchy if for every $\varepsilon > 0$ and all $\alpha \in \Lambda$, there exists $N \in \mathbb{N}$ with $d_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}^*$. The sequence (x_n) is called convergent if there exists an $x^* \in \mathbb{E}$ such that for every $\varepsilon > 0$ and all $\alpha \in \Lambda$, there exists $N \in \mathbb{N}$ with $d_\alpha(x^*, x_n) \leq \varepsilon$, for all $n \geq N$. A

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*Corresponding author.

gauge space $\mathbb{E}$ is called complete (or sequentially complete) if any Cauchy sequence is convergent. A subset X of $\mathbb{E}$ is said to be closed if it contains the limit of any convergent sequence of its elements. See [8] for details, [5, 7, 13] and [16] for several nice results in this context.

If $f : \mathbb{E} \to \mathbb{E}$ is an operator, then $x \in \mathbb{E}$ is called fixed point for f if and only if $x = f(x)$. The set $F_f := \{x \in \mathbb{E} | x = f(x)\}$ denotes the fixed point set of f, while $Graph(f) := \{(x, y) \in X \times X : y = f(x)\}$ is the graph of the operator f.

On the other hand, Ran and Reurings [33] proved the following Banach-Caccioppoli type principle for contractions in ordered and complete metric spaces.

Theorem 1.1 (Ran and Reurings [33]). *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing) operator. Suppose that the following two assertions hold:*

1) *there exists $a \in]0, 1[$ such that $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$ with $x \geq y$*
2) *there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$.*

Then f has an unique fixed point $x^* \in X$, i. e. $f(x^*) = x^*$, and for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f starting from x converges to $x^* \in X$.

Several extensions or generalizations of the above results were proved in the last decades, see [1, 2, 4, 14, 15, 18–22, 24, 26, 27, 29, 30] and others.

In the case of an ordered complete gauge space Chifu and G. Petruşel in [6] proved fixed point results for the case of nonlinear contractions (also called φ-contractions), see Remark 3.3 of this paper. The case of orbital contractions (also called graphic contractions, since the contraction condition is imposed only for elements $(x, y) \in Graph(f)$) was treated in Bota et all. [3], where the Ulam-Hyers stability of the fixed point equation was discussed.

In Jleli et all. [16], using a fixed point result for Caristi type mappings given in Frigon [12], several fixed point results in complete gauge spaces endowed with an ordered or an oriented graph structure are given. In particular, the following theorem was obtained.

Theorem 1.2 ([16]). *Let $(X, \mathcal{T}(\mathcal{F}))$ be a complete gauge space, where $\mathcal{F} := \{d_n : n \in \mathbb{N}\}$ is a family of gauges satisfying the following condition*

$$d_1(x, y) \leq d_2(x, y) \leq \cdots \leq d_n(x, y) \leq \cdots \text{ for all } x, y \in X.$$

Let $f : X \to X$ be a continuous mapping for which there exist $k_n \in [0, 1[$ such that

$$d_n(f(x), f^2(x)) \leq k_n d_n(x, f(x)), \text{ for each } x \in X.$$

Then, f has at least one fixed point in X.

The purpose of this paper is to extend the above mentioned results in gauge spaces, pursuing with a complete study of the fixed point problem. Since the orbital contraction condition

$$d_\alpha(f(x), f^2(x)) \leq k_\alpha d_\alpha(x, f(x)), \text{ for all } x \in X \text{ and } \alpha \in \Lambda$$

includes several generalized contraction assumption (such as the Banach contraction condition, Ćirić-Reich-Rus contraction condition, Ćirić type contraction condition and others) our results extend and complement some well-known theorems in the recent literature. For related results see [1, 3, 5, 6, 10, 11, 13, 16, 25, 34] and others. The multi-valued case is treated in [31, 32].

2. Preliminaries

Let X be a nonempty set and let $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [9]; see, for example, [35]) if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.
(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence, $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$.

Let X be a nonempty set and $f : X \to X$ be an operator. Then,

$$f^0 := 1_X,\ f^1 := f, \ldots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$$

denote the iterate operators of f. In this setting, if $U \subset X \times X$, then an operator $f : X \to X$ is called orbitally U-continuous (see [20]) if: $[x \in X$ and $f^{n(i)}(x) \to a \in X$, as $i \to +\infty$ and $(f^{n(i)}(x), a) \in U$ for any $i \in \mathbb{N}]$ imply $[f^{n(i)+1}(x) \to f(a)$, as $i \to +\infty]$. In particular, if $U = X \times X$, then f is called orbitally continuous.

Let $(E, \preceq)$ be a partially ordered set and $f : E \to E$. Then,

$$(LF)_f := \{x \in E : x \preceq f(x)\}$$

is the lower fixed point set of f, while

$$(UF)_f := \{x \in E : x \succeq f(x)\}$$

is the upper fixed point set of f. We also denote

$$E^f_{\preceq} := (LF)_f \cup (UF)_f.$$

If X, Y are two sets, $f : X \to X$ and $g : Y \to Y$ are two given operators, then the Cartesian product of f and g is denoted by $f \times g$ and it is defined by:

$$f \times g : X \times Y \to X \times Y, (f \times g)(x, y) := (f(x), g(y)).$$

Definition 2.1. *Let X be a nonempty set. By definition $(X, \to, \preceq)$ is an ordered L-space if and only if:*

(i) $(X, \to)$ *is an L-space;*
(ii) $(X, \preceq)$ *is a partially ordered set;*
(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ *and* $x_n \preceq y_n$, *for each* $n \in \mathbb{N} \Rightarrow x \preceq y$.

If $\mathbb{E} := (E, \mathcal{D})$ is a gauge space, then the convergence structure is given by the family of gauges $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$. Hence, $(E, \mathcal{D}, \preceq)$ is an ordered L-space and it will be called an ordered gauge space, see also [31], [32], [6]. We will denote by $(\mathbb{E}, \preceq)$ the ordered gauge space $(E, \mathcal{D}, \preceq)$. Moreover, for $r := \{r_\alpha\}_{\alpha\in\Lambda} \in]0, \infty[^\Lambda$ and $x_0 \in E$, we will denote by $\overline{\mathbb{B}}_d(x_0; r)$ the closure of $\mathbb{B}_d(x_0; r)$ in $\mathbb{E}$, where

$$\mathbb{B}_d(x_0; r) := \{x \in E \ : d_\alpha(x_0, x) < r_\alpha, \text{ for all } \alpha \in \Lambda\}.$$

3. Fixed point results

Our first main result is the following existence and approximation fixed point theorem.

Theorem 3.1. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space and $f : \mathbb{E} \to \mathbb{E}$ be an orbitally continuous operator. We suppose:*

(i) $E^f_\preceq \neq \emptyset$;

(ii) $E^f_\preceq$ *is invariant with respect to f, i.e., $f(E^f_\preceq) \subseteq E^f_\preceq$;*

(iii) *there exists a family of constants $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0, 1[^\Lambda$ such that*

$$d_\alpha(f(x), f^2(x)) \leq k_\alpha d_\alpha(x, f(x)), \text{ for all } x \in E^f_\preceq \text{ and } \alpha \in \Lambda.$$

Then f has at least one fixed point and the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to $x^*(x) \in F_f$, for every $x \in E^f_\preceq$.

Proof. Let $x \in E^f_\preceq$ be arbitrary. Then, by (ii), we get that $(f^n(x))_{n\in\mathbb{N}} \subset E^f_\preceq$, for every $n \in \mathbb{N}^*$. Then, by (iii), there exists a family of constants $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0, 1[^\Lambda$ such that, for every $n \in \mathbb{N}$, we have

$$d_\alpha(f^n(x), f^{n+1}(x)) \leq k^n_\alpha d_\alpha(x, f(x)), \text{ for all } x \in E_\preceq \text{ and } \alpha \in \Lambda.$$

Letting $n \to \infty$, we obtain that $d_\alpha(f^n(x), f^{n+1}(x)) \to 0$, for every $\alpha \in \Lambda$. Then, for an arbitrary $\varepsilon := \{\varepsilon_\alpha\}_{\alpha\in\Lambda} \in]0, +\infty[^\Lambda$, we can choose $N \in \mathbb{N}^*$ such that

$$d_\alpha(f^n(x), f^{n+1}(x)) < \varepsilon_\alpha(1 - k_\alpha), \text{ for each } n \geq N \text{ and } \alpha \in \Lambda.$$

Then,

$$\begin{aligned} d_\alpha(f^n(x), f^{n+2}(x)) &\leq d_\alpha(f^n(x), f^{n+1}(x)) + d_\alpha(f^{n+1}(x), f^{n+2}(x)) \\ &\leq \varepsilon_\alpha(1 - k_\alpha) + k_\alpha d_\alpha(f^n(x), f^{n+1}(x)) \\ &\leq \varepsilon_\alpha(1 - k_\alpha) + k_\alpha \varepsilon_\alpha(1 - k_\alpha) \\ &= \varepsilon_\alpha(1 - k^2_\alpha) < \varepsilon_\alpha. \end{aligned}$$

By mathematical induction we obtain

$$d_\alpha(f^n(x), f^{n+p}(x)) \leq \varepsilon_\alpha(1 - k^p_\alpha) < \varepsilon_\alpha, \tag{3.1}$$

for every $n \in \mathbb{N}, p \in \mathbb{N}^*$ and for each $\alpha \in \Lambda$. The relation (3.1) shows that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{E}$. Thus, $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $\mathbb{E}$ and we denote by $x^*(x) \in E$ its limit. Then, by the orbital continuity of f, we obtain that $x^*(x) \in F_f$. □

A local fixed point theorem in the same context is the following result, which is useful for applications.

Theorem 3.2. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space, $x_0 \in E$ and the family of constants $r := \{r_\alpha\}_{\alpha\in\Lambda} \in]0, \infty[^\Lambda$. Let $f : \overline{\mathbb{B}}_d(x_0; r) \to \mathbb{E}$ be an orbitally continuous operator. We suppose that:*

(i) $f : (\overline{\mathbb{B}}_d(x_0; r), \preceq) \to (\mathbb{E}, \preceq)$ *is increasing;*

(ii) $x_0 \in (LF)_f \cup (UF)_f$*;*

(iii) *there exists a family of constants* $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0,1[^\Lambda$ *such that if the elements* $x, f(x) \in \overline{\mathbb{B}}_d(x_0; r)$ *are such that* $x \preceq f(x)$ *or* $f(x) \preceq x$*, then*

$$d_\alpha(f(x), f^2(x)) \le k_\alpha d_\alpha(x, f(x)), \text{ for each } \alpha \in \Lambda;$$

(iv) $d_\alpha(x_0, f(x_0)) \le (1 - k_\alpha) r_\alpha$*, for each* $\alpha \in \Lambda$*;*

Then f *has at least one fixed point and* $(f^n(x_0))_{n\in\mathbb{N}} \to x^*(x_0) \in \overline{\mathbb{B}}_d(x_0; r) \cap F_f$*.*

Proof. Starting from x_0 we construct the sequence of Picard iterations $(f^n(x_0))_{n\in\mathbb{N}}$ for f. By (i) and (ii) we observe that the successive elements of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ are comparable with respect to $\preceq$. Moreover, $(f^n(x_0))_{n\in\mathbb{N}} \subset \overline{\mathbb{B}}_d(x_0; r)$, for every $n \in \mathbb{N}^*$. Indeed, by (iii), we obtain that there exists a family of constants $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0,1[^\Lambda$ such that, for every $n \in \mathbb{N}$, we have

$$d_\alpha(f^n(x_0), f^{n+1}(x_0)) \le k_\alpha^n d_\alpha(x_0, f(x_0)), \text{ for all } \alpha \in \Lambda. \tag{3.2}$$

Then, using (iv), we have

$$\begin{aligned} d_\alpha(x_0, f^n(x_0)) &\le d_\alpha(x_0, f(x_0)) + d_\alpha(f(x_0), f^2(x_0)) + \cdots + d_\alpha(f^{n-1}(x_0), f^n(x_0)) \\ &\le (1 - k_\alpha) r_\alpha + k_\alpha (1 - k_\alpha) r_\alpha + \cdots + k_\alpha^{n-1} (1 - k_\alpha) r_\alpha \\ &= (1 - k_\alpha^n) r_\alpha \le r_\alpha, \text{ for each } \alpha \in \Lambda. \end{aligned}$$

By (3.2), following the same argument as before, we can prove that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent in $\overline{\mathbb{B}}_d(x_0; r)$ to an element $x^*(x_0)$, which is a fixed point of f (by the orbital continuity of f) □

Remark 3.1. *Instead of the orbital continuity of* f*, some similar results can be obtained working with the following regularity property of the space:*

(Ri) *if an increasing sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* x *in* $\overline{\mathbb{B}}_d(x_0; r)$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*.*

In particular, we have the following consequences of the above result for the case of a complete metric space without a partial order relation.

Corollary 3.1. *Let* $\mathbb{E}$ *be a complete gauge space,* $x_0 \in E$ *and* $r := \{r_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$ *be a family of constants. Let* $f : \overline{\mathbb{B}}_d(x_0; r) \to \mathbb{E}$ *be an orbitally continuous operator. We suppose that:*

(i) *there exists a family of constants* $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0,1[^\Lambda$ *such that, if the elements* $x, f(x) \in \overline{\mathbb{B}}_d(x_0; r)$*, then*

$$d_\alpha(f(x), f^2(x)) \le k_\alpha d_\alpha(x, f(x)), \text{ for each } \alpha \in \Lambda;$$

(ii) $d_\alpha(x_0, f(x_0)) \le (1 - k_\alpha) r_\alpha$*, for each* $\alpha \in \Lambda$*;*

Then f *has at least one fixed point and* $(f^n(x_0))_{n\in\mathbb{N}} \to x^*(x_0) \in \overline{\mathbb{B}}_d(x_0; r) \cap F_f$*.*

Corollary 3.2. *Let* $\mathbb{E}$ *be a complete gauge space and* $f : \mathbb{E} \to \mathbb{E}$ *be an orbitally continuous operator. We suppose that there exists a family of constants* $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0,1[^\Lambda$ *such that*

$$d_\alpha(f(x), f^2(x)) \le k_\alpha d_\alpha(x, f(x)), \text{ for each } x \in \mathbb{E} \text{ and } \alpha \in \Lambda.$$

Then f *has at least one fixed point and* $(f^n(x))_{n\in\mathbb{N}} \to x^*(x) \in F_f$*, for every* $x \in \mathbb{E}$*.*

A more general local fixed point result is the following.

Theorem 3.3. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space, $x_0 \in E$ and the family of constants $r := \{r_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$. Let $f : \overline{\mathbb{B}}_d(x_0; r) \to \mathbb{E}$ be an orbitally continuous operator. Suppose that:*

(i) *$B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r) \neq \emptyset$, where $B^f_\preceq := \{x \in E | x \preceq f(x) \text{ or } f(x) \preceq x\}$;*
(ii) *$B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r)$ is invariant with respect to f;*
(iii) *there exists a family of constants $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0, 1[^\Lambda$ such that, for each $\alpha \in \Lambda$ we have*

$$d_\alpha(f(x), f^2(x)) \le k_\alpha d_\alpha(x, f(x)), \text{ for each } x \in B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r).$$

Then f has at least one fixed point in $\overline{\mathbb{B}}_d(x_0; r)$ and, for each $x \in B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r)$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to $x^*(x) \in F_f$.

Proof. Let $x \in B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r)$. If $x = f(x)$ we are done. Suppose that $x \neq f(x)$. From (ii) we obtain $(f^n(x))_{n\in\mathbb{N}} \subset B^f_\preceq \cap \overline{\mathbb{B}}_d(x_0; r)$. By (iii), using mathematical induction, we get, that for each $\alpha \in \Lambda$, we have

$$d_\alpha(f^n(x), f^{n+1}(x)) \le k^n_\alpha d_\alpha(x, f(x)), \text{ for each } n \in \mathbb{N}.$$

Thus, as before, we can prove that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{\mathbb{B}}_d(x_0; r)$. From the completeness of the gauge space $\mathbb{E}$ we have $(f^n(x))_{n\in\mathbb{N}} \to x^*(x) \in \overline{\mathbb{B}}_d(x_0; r)$ as $n \to +\infty$. By the orbital continuity of f we get that $x^*(x) \in F_f$. □

Remark 3.2. *Condition (ii) of the above theorem is implied by each of the following assertions:*

(ii)' *$f : (\overline{\mathbb{B}}_d(x_0; r), \preceq) \to (\mathbb{E}, \preceq)$ is increasing*
(ii)" *$f : (\overline{\mathbb{B}}_d(x_0; r), \preceq) \to (\mathbb{E}, \preceq)$ is decreasing.*

As a consequence of the above results a continuation result can be given now. For a nice survey on this topic see Frigon [11].

Theorem 3.4. *Let $\mathbb{E}$ be a complete gauge space and U be an open subset of $\mathbb{E}$. Let $G : \overline{U} \times [0, 1] \to \mathbb{E}$ be a continuous operator. Suppose the following conditions are satisfied:*

(i) *$x \neq G(x, t)$, for each $x \in \partial U$ (the boundary of U) and each $t \in [0, 1]$;*
(ii) *there exists a continuous function $\phi : [0, 1] \to \mathbb{R}$ such that, for each $x \in \overline{U}$, we have*

$$d_\alpha(G(x, t), G(x, s)) \le |\phi(t) - \phi(s)|, \text{ for all } t, s \in [0, 1] \text{ and each } \alpha \in \Lambda;$$

(iv) *there exists $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0, +\infty[^\Lambda$ such that $k_\alpha < 1$ and, for $t \in [0, 1]$, we have*

$$d_\alpha(G_t(x), G_t(y)) \le k_\alpha \cdot d_\alpha(x, y), \text{ for each } (x, y) \in Graph(G_t) \text{ and } \alpha \in \Lambda.$$

Then $G(\cdot, 0)$ has a fixed point if and only if $G(\cdot, 1)$ has a fixed point.

Proof. Suppose that $z \in F_{G(\cdot,0)}$. From (i) we have that $z \in U$. Consider the set

$$Q := \{(t, x) \in [0, 1] \times U : x = G(x, t)\},$$

which is nonempty since $(0,z)\in Q$. We introduce on Q the order relation defined by the formula

$$(t,x)\preceq(s,y) \text{ if and only if } t\le s \text{ and } d_\alpha(x,y)\le \frac{2}{1-k_\alpha}[\phi(s)-\phi(t)].$$

Let P be a totally ordered subset of Q, $t^*:=\sup\{t\ :\ (t,x)\in P\}$ and let $(t_n,x_n)_{n\in\mathbb{N}^*}\subset P$ be a sequence such that $(t_n,x_n)\preceq(t_{n+1},x_{n+1})$ for each $n\in\mathbb{N}^*$ and let $t_n\to t^*$ as $n\to\infty$. Then

$$d_\alpha(x_m,x_n)\le\frac{2}{1-k_\alpha}[\phi(t_m)-\phi(t_n)], \text{ for each } m,n\in\mathbb{N}^*,\ m>n.$$

Letting $m,n\to+\infty$ we obtain that $d_\alpha(x_m,x_n)\to 0$, proving that $(x_n)_{n\in\mathbb{N}^*}$ is Cauchy. Denote by $x^*\in\mathbb{E}$ the limit of this sequence. Since $x_n=G(x_n,t_n)$, $n\in\mathbb{N}^*$, using (iv), we get that $x^*=G(x^*,t^*)$. From (i) we note that $x^*\in U$. Thus $(t^*,x^*)\in Q$. Since P is totally ordered we have that $(t,x)\preceq(t^*,x^*)$, for each $(t,x)\in P$. Thus (t^*,x^*) is an upper bound of P. By Zorn's Lemma, the set Q admits a maximal element $(t_0,x_0)\in Q$. We will prove that $t_0=1$.

Suppose that $t_0<1$. Let $r=\{r_\alpha\}_{\alpha\in\Lambda}\in]0,\infty[^\Lambda$ and $t\in]t_0,1]$ be such that $\mathbb{B}_d(x_0;r_\alpha)\subset U$ and $r_\alpha:=\frac{2}{1-k_\alpha}[\phi(t)-\phi(t_0)]$ for every $\alpha\in\Lambda$. Then for each $\alpha\in\Lambda$ we have that

$$\begin{aligned} d_\alpha(x_0,G(x_0,t)) &\le d_\alpha(x_0,G(x_0,t_0))+d_\alpha(G(x_0,t_0),G(x_0,t))\\ &\le \phi(t)-\phi(t_0)=\frac{r_\alpha(1-k_\alpha)}{2}<(1-k_\alpha)r_\alpha. \end{aligned}$$

Since $\overline{\mathbb{B}_d}(x_0;r_\alpha)\subset\overline{U}$, the operator $G_t:\overline{\mathbb{B}_d}(x_0;r)\to\mathbb{E}$ verifies, for all $t\in[0,1]$, all the conditions of Corollary 3.1. Hence, there exists $x\in\overline{\mathbb{B}_d}(x_0;r_\alpha)$ such that $x=G(x,t)$. Thus $(t,x)\in Q$. Since we have that

$$d_\alpha(x_0,x)\le r_\alpha=\frac{2}{1-k_\alpha}[\phi(t)-\phi(t_0)],$$

thus we have that

$$(t_0,x_0)\prec(t,x),$$

which contradicts the maximality of (t_0,x_0). Thus $t_0=1$ and the proof is complete. □

We will give now some data dependence results for the fixed point equation.

Theorem 3.5. *Let* $(\mathbb{E},\preceq)$ *be a partially ordered complete gauge space and* $f,g:\mathbb{E}\to\mathbb{E}$ *be two operators. We suppose:*

(i) *f satisfies all the assumptions of Theorem 3.1;*
(ii) *there exists a family of constants* $\eta:=\{\eta_\alpha\}_{\alpha\in\Lambda}\in]0,\infty[^\Lambda$ *such that, for each* $\alpha\in\Lambda$ *we have*

$$d_\alpha(f(x),g(x))\le\eta_\alpha, \text{ for each } x\in E.$$

Then, for each $y^*\in Fix(g)\cap E^f_\preceq$, *there exists* $x^*(y^*)\in F_f$ *such that*

$$d(x^*(y^*),y^*)\le\frac{\eta_\alpha}{1-k_\alpha}, \text{ for every } \alpha\in\Lambda.$$

Proof. Let $y^* \in Fix(g) \cap E^f_{\preceq}$. By Theorem 3.1, the sequence $(f^n(y^*))_{n\in\mathbb{N}}$ converges to $x^*(y^*) \in F_f$. On the other hand, we have

$$\begin{aligned} d_\alpha(x^*(y^*), y^*) &\le d_\alpha(x^*(y^*), f(y^*)) + d_\alpha(f(y^*), y^*) \\ &\le d_\alpha(f(y^*), f^2(y^*)) + \cdots + d_\alpha(f^{n-1}(y^*), f^n(y^*)) \\ &\quad + d_\alpha(f^n(y^*), x^*(y^*)) + d_\alpha(f(y^*), y^*) \\ &\le (k_\alpha + \cdots + k_\alpha^{n-1}) d_\alpha(f(y^*), y^*) \\ &\quad + d_\alpha(f^n(y^*), x^*(y^*)) + d_\alpha(f(y^*), y^*) \\ &\le \frac{k_\alpha}{1-k_\alpha}\eta_\alpha + d_\alpha(f^n(y^*), x^*(y^*)) + \eta_\alpha \\ &= \frac{1}{1-k_\alpha}\eta_\alpha + d_\alpha(f^n(y^*), x^*(y^*)). \end{aligned}$$

The conclusion follows letting $n \to \infty$. □

Our next results prove that, under suitable conditions, the fixed point equation has certain stability properties.

Theorem 3.6. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space and $f : \mathbb{E} \to \mathbb{E}$ be an operator. We suppose that f satisfies all the assumptions of Theorem 3.1.*

Then the fixed point problem for f is well-posed, in the sense that for each $x_0 \in E^f_{\preceq}$ and each sequence $(y_n)_{n\in\mathbb{N}}$ from $(AB)_f(x^*(x_0)) \cap E^f_{\preceq}$ which satisfies, for every $\alpha \in \Lambda$, the condition $d_\alpha(y_n, f(y_n)) \to 0$ as $n \to \infty$, we have that $y_n \to x^*(x_0) \in F_f$ as $n \to \infty$.

Proof. Let $x_0 \in E^f_{\preceq}$ be arbitrary. Then, by Theorem 3.1, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $x^*(x_0) \in F_f$. On the other hand, given $(y_n)_{n\in\mathbb{N}} \subset (AB)_f(x^*(x_0)) \cap E^f_{\preceq}$ we observe that $y_n, f(y_n), \ldots, f^n(y_n), \ldots \in E^f_{\preceq}$, for all $n \in \mathbb{N}$. Thus, we get

$$\begin{aligned} d_\alpha(x^*(x_0), y_n) &\le d_\alpha(y_n, f(y_n)) + \cdots + d_\alpha(f^{n-1}(y_n), f^n(y_n)) + d_\alpha(f^n(y_n), x^*(x_0)) \\ &\le (1 + k_\alpha + \cdots + k_\alpha^{n-1}) d_\alpha(y_n, f(y_n)) + d_\alpha(f^n(y_n), x^*(x_0)) \\ &\le \frac{1}{1-k_\alpha} d_\alpha(y_n, f(y_n)) + d_\alpha(f^n(y_n), x^*(x_0)). \end{aligned}$$

The conclusion follows letting $n \to \infty$. □

The following result is know as Cauchy-Toeplitz Lemma.

Lemma 3.1 (Cauchy-Toeplitz Lemma, see, for example, [23] or [35])**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n\ge 0} a_n$ is convergent and $(b_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ be a sequence such that $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k\right) = 0.$$

Theorem 3.7. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space and $f : \mathbb{E} \to \mathbb{E}$ be an operators. We suppose that f satisfies all the assumptions of Theorem 3.1 with $k := \{k_\alpha\}_{\alpha\in\Lambda} \in]0, \frac{1}{3}[^\Lambda$.*

Then the fixed point problem for f has the Ostrowski property, in the sense that for each $x_0 \in E^f_{\preceq}$ and each sequence $(z_n)_{n\in\mathbb{N}}$ from $(AB)_f(x^*(x_0)) \cap E^f_{\preceq}$ which satisfies, for every $\alpha \in \Lambda$, the condition $d_\alpha(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$, we have that $z_n \to x^*(x_0) \in F_f$ as $n \to \infty$.

Proof. Let $x_0 \in E^f_{\preceq}$ be arbitrary. Then, by Theorem 3.1, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $x^*(x_0) \in F_f$. For simplicity, we denote $x^* := x^*(x_0)$. On the other hand, given $(z_n)_{n\in\mathbb{N}} \subset (AB)_f(x^*(x_0)) \cap E^f_{\preceq}$ we observe that $z_n, f(z_n), \dots, f^n(z_n), \dots \in E^f_{\preceq}$, for all $n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$, we get

$$\begin{aligned}
d_\alpha(x^*, z_{n+1}) &\le d_\alpha(z_{n+1}, f(z_n)) + \dots + d_\alpha(f^{n-1}(z_n), f^n(z_n)) + d_\alpha(f^n(z_n), x^*) \\
&\le d_\alpha(z_{n+1}, f(z_n)) + (k_\alpha + \dots + k_\alpha^{n-1}) d_\alpha(z_n, f(z_n)) + d_\alpha(f^n(z_n), x^*) \\
&\le d_\alpha(z_{n+1}, f(z_n)) + \frac{k_\alpha}{1-k_\alpha} d_\alpha(z_n, f(z_n)) + d_\alpha(f^n(z_n), x^*) \\
&\le d_\alpha(z_{n+1}, f(z_n)) + \frac{k_\alpha}{1-k_\alpha} \left(d_\alpha(z_n, x^*) + d(x^*, z_{n+1}) + d_\alpha(z_{n+1}, f(z_n)) \right) \\
&\quad + d_\alpha(f^n(z_n), x^*).
\end{aligned}$$

If we denote $a_\alpha := \frac{1}{1-2k_\alpha}, b_\alpha := \frac{k_\alpha}{1-2k_\alpha}, c_\alpha := \frac{1-k_\alpha}{1-2k_\alpha}$, then, for each $\alpha \in \Lambda$, we can write

$$d_\alpha(x^*, z_{n+1}) \le a_\alpha d_\alpha(z_{n+1}, f(z_n)) + b_\alpha d_\alpha(x^*(x_0), z_n) + c_\alpha d_\alpha(f^n(z_n), x^*(x_0)).$$

Continuing this approach we obtain

$$\begin{aligned}
d_\alpha(x^*, z_{n+1}) \le\ & a_\alpha \left[d_\alpha(z_{n+1}, f(z_n)) + b_\alpha d_\alpha(z_n, f(z_{n-1})) + \dots + b_\alpha^n d_\alpha(z_1, f(z_0)) \right] \\
& + c_\alpha \left[d_\alpha(f^n(z_n), x^*) + b_\alpha d_\alpha(f^n(z_{n-1}), x^*) + \dots + b_\alpha^n d_\alpha(z_0, x^*) \right] \\
& + b_\alpha^n d_\alpha(z_0, x^*).
\end{aligned}$$

Letting $n \to \infty$, the conclusion follows using the Cauchy-Toeplitz Lemma. □

Finally, we will discuss the concept of Ulam-Hyers stability for the fixed point equation in a partially ordered and complete gauge space. For a similar result see [3]. For other type of stability results see [17].

Definition 3.1. *Let $\mathbb{E}$ be a complete gauge space and $f : \mathbb{E} \to \mathbb{E}$ be an operator. Then the fixed point equation*

$$x = f(x), x \in \mathbb{E} \tag{3.3}$$

is called Ulam-Hyers stable if there exists a family of constants $c := \{c_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$, such that, for every $\mu := \{\mu_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$ and every $z \in \mathbb{E}^f_{\preceq}$ with

$$d_\alpha(z, f(z)) \le \mu_\alpha, \forall \alpha \in \Lambda, \tag{3.4}$$

there exists a solution $x^* \in \mathbb{E}$ of the fixed point equation (3.3) such that

$$d_\alpha(x^*, z) \le c_\alpha \mu_\alpha, \forall \alpha \in \Lambda.$$

Theorem 3.8. *Let $(\mathbb{E}, \preceq)$ be a partially ordered complete gauge space and $f : \mathbb{E} \to \mathbb{E}$ be an operator. We suppose that f satisfies all the assumptions of Theorem 3.1. Then the fixed point problem for f is Ulam-Hyers stable.*

Proof. Let $\mu := \{\mu_\alpha\}_{\alpha\in\Lambda} \in]0,\infty[^\Lambda$ and take $z \in \mathbb{E}$ with $d_\alpha(z, f(z)) \leq \mu_\alpha, \forall \alpha \in \Lambda$. Define $c_\alpha := \frac{1}{1-k_\alpha}$. Then, since $z \in \mathbb{E}^f_{\preceq}$, by Theorem 3.1, the sequence $(f^n(z))_{n\in\mathbb{N}}$ converges to $x^*(z) \in F_f$. Thus, we have

$$\begin{aligned} d_\alpha(x^*(z), z) &\leq d_\alpha(z, f(z)) + d_\alpha(f(z), f^2(z)) + \cdots + d_\alpha(f^{n-1}(z), f^n(z)) \\ &\quad + d_\alpha(f^n(z), x^*(z)) \\ &\leq \left(1 + k_\alpha + \cdots + k_\alpha^{n-1}\right) d_\alpha(z, f(z)) + d_\alpha(f^n(z), x^*(z)) \\ &\leq \frac{\mu_\alpha}{1-k_\alpha} + d_\alpha(f^n(z), x^*(z)). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$d_\alpha(x^*(z), z) \leq \frac{\mu_\alpha}{1-k_\alpha}.$$

□

Remark 3.3. *It is an open question to prove similar results for the case of operators $f : \mathbb{E} \to \mathbb{E}$ satisfying to a φ-contraction condition:*

$$d_\alpha(f(x), f^2(x)) \leq \varphi(d_\alpha(x, f(x))), \text{ for each } x \in E \text{ and } \alpha \in \Lambda.$$

Notice that, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, then φ is called a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0. For example, $\varphi(t) = kt$ (where $k \in [0, 1[$), $\varphi(t) = \frac{t}{1+t}$ and $\varphi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$ are examples of comparison functions.

4. An application

Let us consider the initial value problem for a first order differential equation

$$\begin{cases} x' = f(t, x(t)) \\ x(0) = 0 \end{cases} \tag{4.1}$$

where $f : [0, \infty[\times\mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping. The following result is well-known.

Lemma 4.1. *Under the above conditions, the initial value problem 4.1 is equivalent to the following Volterra type integral equation*

$$x(t) = \int_0^t f(s, x(s))ds, t \in [0, \infty[. \tag{4.2}$$

We denote $E := C([0, \infty), \mathbb{R}^n)$.

Using the above lemma, we can establish the following existence result.

Theorem 4.1. *Let us consider the initial value problem (4.1). We suppose:*

(i) *$f : [0, \infty[\times\mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping;*
(ii) *for every $n \in \mathbb{N}$ there exists $k_n \in L^1_{loc}([0, \infty[)$ such that, for every $x \in E$, we have*

$$|f(t, x(t)) - f(t, \int_0^t f(s, x(s))ds)| \leq k_n(s)|x(t) - \int_0^t f(s, x(s))ds|, \text{ for every } t \in [0, n].$$

Then, the initial value problem has at least one solution in E.

Proof. We will endow the space $E := C([0,\infty),\mathbb{R}^n)$ with the family of gauges

$$\|x\|_n^B := \max_{t\in[0,n]} |x(t)|e^{-L\int_0^t k_n(s)ds},$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^n$ and $L>1$ is arbitrary. Then $d_n(x,y) := \|x-y\|_n^B$ defines a separating family of gauges on E. For the rest of the proof, we consider the complete gauge space $\mathbb{E} := (C([0,\infty),\mathbb{R}^n), d_n)$ and the operator $A : \mathbb{E}\to\mathbb{E}$, $x\mapsto Ax$ given by

$$Ax(t) := \int_0^t f(s,x(s))ds, t\in[0,\infty[.$$

Then, for $x\in\mathbb{E}, y := Ax$ and for $t\in[0,n]$, we have

$$\begin{aligned}
|Ax(t)-Ay(t)| &\le \int_0^t |f(s,x(s)) - f(s,\int_0^s f(p,x(p))dp)|ds\\
&\le \int_0^t k_n(s)|x(s) - \int_0^s f(p,x(p))dp|ds\\
&= \int_0^t k_n(s)|x(s)-Ax(s)|e^{-L\int_0^s k_n(p)dp}e^{L\int_0^s k_n(p)dp}ds\\
&\le \|x-Ax\|_n^B \int_0^t k_n(s)e^{L\int_0^s k_n(p)dp}ds\\
&\le \frac{1}{L}\|x-Ax\|_n^B e^{L\int_0^t k_n(s)ds}.
\end{aligned}$$

Hence, we get that

$$\|Ax-A^2x\|_n^B \le \frac{1}{L}\|x-Ax\|_n^B.$$

The conclusion follows by Corollary 3.2 and Lemma 4.1. □

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Adrian Petruşel
Babeş-Bolyai University Cluj-Napoca, Department of Mathematics, Kogălniceanu Str., No.1, 400084 Cluj-Napoca and Academy of Romanian Scientists, Bucharest, Romania
E-mail address: petrusel@math.ubbcluj.ro

Gabriela Petruşel
Babeş-Bolyai University Cluj-Napoca, Department of Business, Horea Str., No.7, Cluj-Napoca, Romania
E-mail address: gabi.petrusel@tbs.ubbcluj.ro

Mu-Ming Wong
Department of Applied Mathematics, Chung Yuan Christian University, Chung Li District, Taoyuan City, Taiwan 32023, R.O.C.
E-mail address: mmwong@cycu.edu.tw