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PEROV TYPE THEOREMS FOR ORBITAL CONTRACTIONS

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Abstract. In this paper, we will present a study of the fixed point equation $x = f(x), x \in X$ (where (X, d) is a generalized metric space in the sense that $d(x, y) \in \mathbb{R}^m_+$ and $f : X \to X$ is an orbital contraction) by the following perspectives: existence, uniqueness, approximation, data dependence of the operator perturbation, well-posedness, and Ulam-Hyers stability. The non-self case is also discussed and some applications are given.

1. Introduction and preliminary results

There are many generalizations of Banach's Contraction Principle. One of these generalizations is based on the concept of vector-valued metric. We consider the following notations.

If $x, y \in \mathbb{R}^m$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, then, by definition

$$x \leq y \text{ if and only if } x_i \leq y_i, \text{ for each } i \in \{1, 2, \ldots, m\}.$$

Through this section, we will make an identification between row and column vectors in $\mathbb{R}^m$.

We can now recall the concept of vector-valued metric, see [7]. (X, d) is a vector-valued metric space if X is a nonempty set and $d : X \times X \to \mathbb{R}^m_+$ satisfies all the axioms of the usual metric, where the inequalities from the axioms of the metric are in the sense mentioned above.

We may suppose that

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_m(x, y) \end{pmatrix}, \text{ for } x, y \in X.$$

We denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by I_m the identity $m \times m$ matrix and by O_m the null $m \times m$ matrix.

By definition, $K \in M_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if $K^n \to O_m$ as $n \to \infty$. The following result will be important for our next considerations (see, e.g., [1, 18]).

Theorem 1.1. *Let $K \in M_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *$K^n \to O_m$ as $n \to \infty$;*

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(ii) *The spectral radius $\rho(K)$ of K is strictly less than 1, i.e., the eigenvalues of K are in the open unit disc;*

(iii) *The matrix $(I_m - K)$ is nonsingular and*

$$(I_m - K)^{-1} = I_m + K + \cdots + K^n + \cdots ; \tag{1.1}$$

(iv) *The matrix $(I_m - K)$ is nonsingular and $(I_m - K)^{-1}$ has nonnegative elements.*

Using the above properties, Perov (see e.g. [7]) proved the following result.

Theorem 1.2 (Perov)**.** *Let (X, d) be a complete vector-valued metric space and let $f : X \to X$ be an K-contraction, i.e., $K \in M_{m,m}(\mathbb{R}_+)$ converges towards zero and*

$$d(f(x), f(y)) \leq K d(x, y), \text{ for all } x, y \in X.$$

Then:

(1) *$Fix(f) = \{x^*\}$, i.e., there exists a unique solution $x^* \in X$ of the fixed point equation $x = f(x)$;*

(2) *the sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := f^n(x_0)$ of successive approximations for f starting from any $x_0 \in X$ is convergent to x^*;*

(3) *the following estimation holds*

$$d(x_n, x^*) \leq K^n (I_m - K)^{-1} d(x_0, x_1), \text{ for every } n \in \mathbb{N}; \tag{1.2}$$

(4) *if $g : X \to X$ is an operator for which there exists $\eta := (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for each $i \in \{1, 2, \ldots, m\}$, such that $d(f(x), g(x)) \leq \eta$ for each $x \in X$, then*

$$d(x^*, y^*) \leq (I_m - K)^{-1} \eta, \text{for every } y^* \in Fix(g).$$

The purpose of this paper is to study the fixed point equation $x = f(x), x \in X$, where (X, d) is a vector-valued metric space (in the sense that $d(x, y) \in \mathbb{R}^m_+$) and $f : X \to X$ is an orbital K-contraction, i.e., $K \in M_{m,m}(\mathbb{R}_+)$ converges towards zero and

$$d\left(f(x), f^2(x)\right) \leq K d(x, f(x)), \text{ for all } x \in X.$$

The following problems will be considered: existence, uniqueness, approximation, data dependence of the operator perturbation, well-posedness and Ulam-Hyers stability. The non-self case is also discussed and some applications are given. The results of this paper extend some recent theorems given in [14]. See also [6] for the case of orbital contractions in metric spaces. It is also of interest to extend the above study to some other contractive conditions, see [17].

2. Main results

We start this section by recalling the notion of weakly Picard operator, see [15].

Let (X, d) be a vector-valued metric space. Then, by definition, $f : X \to X$ is called a weakly Picard operator if

$$f^n(x) \to x^*(x) \in Fix(f) \text{ as } n \to \infty, \text{ for all } x \in X.$$

The above definition induces a set retraction on $Fix(f)$ given by

$$f^\infty : X \to Fix(f),\ f^\infty(x) = \lim_{n\to\infty} f^n(x).$$

A weakly Picard operator $f : X \to X$ for which there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that

$$d(x, f^{\infty}(x)) \leq Cd(x, f(x)), \text{ for all } x \in X,$$

is called a weakly C - Picard operator. We denote by $Graph(f)$ the graph of the operator f and by $Fix(f)$ the fixed point set of f. If x^* is a fixed point of f, then we denote the attraction basin of the fixed point x^* of f by

$$(AB)_f(x^*) := \{x \in X : f^n(x) \to x^* \text{ as } n \to \infty\}.$$

We start this section with an existence and approximation result.

Theorem 2.1. *Let (X, d) be a complete vector-valued metric space and let $f : X \to X$ be an orbital K-contraction with closed graph. Then, the following conclusions hold:*

(1) *$Fix(f) \neq \emptyset$;*
(2) *for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := f^n(x_0)$ of successive approximations for f starting from x_0 is convergent to $f^{\infty}(x_0) \in Fix(f)$;*
(3) *for each $x_0 \in X$, the following estimation holds*

$$d(x_n, f^{\infty}(x_0)) \leq K^n (I_m - K)^{-1} d(x_0, f(x_0)), \text{ for every } n \in \mathbb{N}. \tag{2.1}$$

In particular

$$d(x_0, f^{\infty}(x_0)) \leq (I_m - K)^{-1} d(x_0, f(x_0)), \text{ for every } x_0 \in X, \tag{2.2}$$

showing that f is a weakly $(I_m - K)^{-1}$ - Picard operator.

Proof. Let $x \in X$ be arbitrary and $x_n := f^n(x)$, $n \in \mathbb{N}$. By the orbital contraction condition, we get, for all $x \in X$, that

$$d\left(f^n(x), f^{n+1}(x)\right) \leq Kd\left(f^{n-1}(x), f^n(x)\right) \leq \cdots \leq K^n d(x, f(x)). \tag{2.3}$$

Then, for all $x \in X$, we have

$$d(x_n, x_{n+p}) \leq \left(K^n + K^{n+1} + \cdots\right) d(x, f(x)) = K^n (I_m - K)^{-1} d(x, f(x)). \tag{2.4}$$

Since the matrix K converges to zero, we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Thus there exists $x^*(x) \in X$ such that $x_n \to x^*(x)$ as $n \to \infty$. Since $Graph(f)$ is a closed set, by the recursive representation $x_{n+1} = f(x_n)$, we obtain that $x^*(x) \in Fix(f)$. Conclusion (3) follows by (2.4) letting $p \to \infty$, while (2.2) is a consequence of (2.1) for $n = 0$. □

Some stability results are given in the next theorem.

Theorem 2.2. *Let (X, d) be a complete vector-valued metric space and let $f : X \to X$ be an orbital K-contraction with closed graph. Then, we have the following conclusions:*

(A) *if $g : X \to X$ is an operator for which there exists $\eta := (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for each $i \in \{1, 2, \ldots m\}$, such that $d(f(x), g(x)) \leq \eta$ for each $x \in X$, then, for each $y^* \in Fix(g)$, there exists $x^* \in Fix(f)$, such that*

$$d(x^*, y^*) \leq (I_m - K)^{-1} \eta. \tag{2.5}$$

(B) *the fixed point equation* $x = f(x)$ *has the following well-posedness type property: for each* $x^* \in Fix(f)$ *and any sequence* $(y_n)_{n\in\mathbb{N}}$ *in* $(AB)_f(x^*)$ *for which*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(C) *the fixed point equation* $x = f(x)$ *is Ulam-Hyers stable, in the sense that there exists a matrix* $C \in M_{m,m}(\mathbb{R}_+)$ *such that for every* $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{R}^m_+$ *with* $\varepsilon_i > 0$ *for each* $i \in \{1, 2, \dots, m\}$*, and every* ε*-solution* z *of the fixed point equation* $x = f(x)$*, i.e.,*

$$d(z, f(z)) \le \varepsilon,$$

there exists a solution $x^* \in X$ *of the fixed point equation* $x = f(x)$ *such that*

$$d(x^*, z) \le C\varepsilon.$$

Proof. (A) By Theorem 2.1, for each $x \in X$, we have that

$$d(x, f^\infty(x)) \le (I_m - K)^{-1} d(x, f(x)).$$

Choosing $x := y^* \in Fix(g)$ and denoting $x^* := f^\infty(y^*) \in Fix(f)$, we get that

$$d(y^*, x^*) \le (I_m - K)^{-1} d(y^*, f(y^*)) \le (I_m - K)^{-1} \eta.$$

(B) Let $x^* \in Fix(f)$ and take any sequence $(y_n)_{n\in\mathbb{N}}$ in $(AB)_f(x^*)$ such that

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

Then we have

$$\begin{aligned} d(y_n, x^*) &\le d(y_n, f(y_n)) + \dots + d(f^{n-1}(y_n), f^n(y_n)) + d(f^n(y_n), x^*) \\ &\le (I_m + K + K^2 + \cdots) d(y_n, f(y_n)) + d(f^n(y_n), x^*) \\ &\le (I_m - K)^{-1} d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

The conclusion follows letting $n \to \infty$.

(C) Take $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{R}^m_+$ with $\varepsilon_i > 0$ for each $i \in \{1, 2, \dots m\}$, and take any ε-solution z of the fixed point equation, i.e., with the property

$$d(z, f(z)) \le \varepsilon.$$

By Theorem 2.1, for each $x \in X$, we have that

$$d(x, f^\infty(x)) \le (I_m - K)^{-1} d(x, f(x)).$$

Choosing $x := z$ and denoting $x^* := f^\infty(z) \in Fix(f)$, we obtain

$$d(z, x^*) \le (I_m - K)^{-1} d(z, f(z)) \le (I_m - K)^{-1} \varepsilon.$$

□

Remark 2.3. It is an open question to prove Ostrowski's stability property in the above framework. Another open question is to extend the results to more general spaces. For related results in fixed point theory see [3, 8, 12, 13, 16].

3. The non-self case

In this section we will consider the case of a non-self orbital K-contractions. Let (X, d) be a vector-valued metric space, $x_0 \in X$ and $R := (R_1, \ldots, R_m)$, with $R_i > 0$ for each $i \in \{1, \ldots, m\}$. Then we denote by $\tilde{B}(x_0, R) := \{x \in X : d(x_0; x) \leq R\}$ the closed ball centered at x_0 with radius R. In this section, the attraction basin of a fixed point x^* of f is

$$(AB)_f(x^*) := \{x \in \tilde{B}(x_0, R) : \ f^n(x) \to x^* \text{ as } n \to \infty\}.$$

Then, we have the following fixed point result.

Theorem 3.1. *Let (X, d) be a complete vector-valued metric space, $R := (R_1, \ldots, R_m)$, with $R_i > 0$ for each $i \in \{1, \ldots, m\}$, $x_0 \in X$ and $f : \tilde{B}(x_0; R) \to X$ be an operator with closed graph. We suppose that there exists a matrix $K \in M_{m,m}(\mathbb{R}_+)$ convergent towards zero, such that the following assumptions take place:*

(i) *if $x, f(x) \in \tilde{B}(x_0; R)$, then $d(f(x), f^2(x)) \leq K d(x, f(x))$;*
(ii) *$R^0 := (I_m - K)^{-1} d(x_0, f(x_0)) \leq R$.*

Then, the following conclusions take place:

(a) *for every $n \in \mathbb{N}$, we have that $f^n(x_0) \in \tilde{B}(x_0; R^0)$ and $f^n(x_0) \to x^*(x_0) \in Fix(f)$ as $n \to \infty$;*
(b) *if for some $y \in \tilde{B}(x_0; R)$ we have that $f^n(y) \in \tilde{B}(x_0; R)$ for every $n \in \mathbb{N}^*$, then $f^n(y) \to x^*(y) \in Fix(f)$ as $n \to \infty$.*
(c) *if $x^* \in Fix(f)$ and $x \in (AB)_f(x^*)$, then $d(x, x^*) \leq (I_m - K)^{-1} d(x, f(x))$.*
(d) *the fixed point equation $x = f(x)$ has the following well-posedness type property: if $x^* \in Fix(f)$, $y_n \in (AB)_f(x^*)$ for every $n \in \mathbb{N}$ is such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$, then $y_n \to x^*$ as $n \to \infty$.*
(e) *the fixed point equation $x = f(x)$ has the following Ulam-Hyers stability property: for every $\varepsilon > 0$ and every ε-solution $y^* \in (AB)_f(x^*(x_0))$ of the fixed point equation, i.e.,*

$$d(y^*, f(y^*)) \leq \varepsilon,$$

we have

$$d(x^*(x_0), y^*) \leq (I_m - K)^{-1} \varepsilon.$$

Proof. (a) Since $R^0 := (I_m - K)^{-1} d(x_0, f(x_0))$, we have that $f(x_0) \in \tilde{B}(x_0; R^0)$. Let us show, inductively, that

$$\begin{aligned} d(x_0, f^n(x_0)) &\leq (I_m + K + \cdots + K^n) d(x_0, f(x_0)) \\ &\leq (I_m - K)^{-1} d(x_0, f(x_0)) = R^0, \text{ for every } n \geq 2. \end{aligned}$$

Indeed, we have

$$d(x_0, f^2(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) \leq (I_m + K) d(x_0, f(x_0)) \leq R^0.$$

If we suppose that the relation takes place for every $k \in \{2, \ldots, n\}$, then for $n + 1$ we have

$$\begin{aligned} d(x_0, f^{n+1}(x_0)) &\leq d(x_0, f^n(x_0)) + d(f^n(x_0), f^{n+1}(x_0)) \\ &\leq (I_m + K + \cdots + K^n) d(x_0, f(x_0)) \end{aligned}$$

$$\leq (I_m - K)^{-1} d(x_0, f(x_0)) = R^0.$$

Hence, $f^n(x_0) \in \tilde{B}(x_0; R^0)$, for every $n \in \mathbb{N}^*$. The fact that $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy and hence it is convergent to an element $x^*(x_0) \in \tilde{B}(x_0, ; R^0)$ follows now in a similar manner to the self operators case. Since f has closed graph, we obtain that $x^*(x_0) \in Fix(f)$.

(b) Let $y \in \tilde{B}(x_0; R)$ such that $f^n(y) \in \tilde{B}(x_0; R)$, for every $n \in \mathbb{N}^*$. Then, in a similar way to the above proof, we get that the sequence $(f^n(y))_{n\in\mathbb{N}}$ is Cauchy and it converges to a fixed point $f^\infty(y) \in \tilde{B}(x_0; R)$ of f.

(c) Let $x^* \in Fix(f)$ and $x \in (AB)_f(x^*)$. Then

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), x^*) \\ &\leq (I_m + K + \cdots + K^{n-1}) d(x, f(x)) + d(f^n(x), x^*) \\ &\leq (I_m - K)^{-1} d(x, f(x)) + d(f^n(x), x^*). \end{aligned}$$

Letting $n \to \infty$ we obtain our third conclusion.

(d) Let $x^* \in Fix(f)$ and $y_n \in (AB)_f(x^*)$ for every $n \in \mathbb{N}$ be such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. Then

$$\begin{aligned} d(y_n, x^*) &\leq d(y_n, f(y_n)) + \cdots + d(f^{n-1}(y_n), f^n(y_n)) + d(f^n(y_n), x^*) \\ &\leq (I_m + K + \cdots + K^{n-1}) d(y_n, f(y_n)) + d(f^n(y_n), x^*) \\ &\leq (I_m - K)^{-1} d(y_n, f(y_n)) + d(f^n(y_n), x^*). \end{aligned}$$

The conclusion follows again by letting $n \to \infty$.

(e) Let $\varepsilon > 0$ and $y^* \in (AB)_f(x^*(x_0))$ such that $d(y^*, f(y^*)) \leq \varepsilon$. Then, we have

$$\begin{aligned} d(x^*(x_0), y^*) &\leq d(x^*, f^n(y^*)) + d(f^n(y^*), f^{n-1}(y^*)) + \cdots + d(f(y^*), y^*) \\ &\leq d(x^*(x_0), f^n(y^*)) + K^{n-1} d(f(y^*), y^*) + \cdots + K d(f(y^*), y^*) \\ &\quad + d(f(y^*), y^*) \\ &\leq d(x^*(x_0), f^n(y^*)) + (K^{n-1} + \cdots + K + I_m) d(f(y^*), y^*) \\ &\leq d(x^*(x_0), f^n(y^*)) + (I_m - K)^{-1} \varepsilon. \end{aligned}$$

The conclusion follows letting $n \to \infty$. □

4. An application

Let (X_1, d_1), (X_2, d_2) and (Y, ρ) be three metric spaces and let $f : X_1 \to Y$, $g : X_2 \to Y$ and $h : X_1 \times X_2 \to Y$ be three operators. We consider the following coincidence problem for f, g and h : find $(x_1, x_2) \in X_1 \times X_2$ such that

$$f(x_1) = g(x_2) = h(x_1, x_2). \tag{4.1}$$

Using the main fixed point theorem for orbital K-contractions in the setting of a vector-valued metric space, we have the following result.

Theorem 4.1. *(X_1, d_1), (X_2, d_2) and (Y, ρ) be three complete metric spaces and $f : X_1 \to Y$, $g : X_2 \to Y$ and $h : X_1 \times X_2 \to Y$ be three operators. Assume that the following assertions are satisfied:*

(i) *f is a continuous l_1-dilatation, i.e., $l_1 > 0$ and*

$$\rho(f(x_1), f(u_1)) \geq l_1 d_1(x_1, u_1), \text{ for every } x_1, u_1 \in X_1;$$

(ii) *g is a continuous l_2-dilatation, i.e., $l_2 > 0$ and*

$$\rho(g(x_2), g(u_2)) \geq l_2 d_2(x_2, u_2), \text{ for every } x_2, u_2 \in X_2;$$

(iii) *h is k-Lipschitz, i.e., $k > 0$ and*

$$\rho(h(x_1, x_2), h(u_1, u_2)) \leq k\left(d_1(x_1, u_1) + d_2(x_2, u_2)\right) \quad \text{for every } (x_1, x_2), (u_1, u_2) \in X_1 \times X_2;$$

(iv) $k(l_1 + l_2) \in]0, 1[$;

(v) *$f(X_1) \cap g(X_2) \neq \emptyset$, $h(X_1 \times X_2) \subset f(X_1)$ and $h(X_1 \times X_2) \subset g(X_2)$.*

Then, the following conclusions hold:

(a) *the coincidence problem (4.1) has at least one solution and, for every pair of elements $x = (x_1, x_2) \in X_1 \times X_2$, the sequence*

$$((f^{-1} \circ h)^n(x), (g^{-1} \circ h)^n(x))_{n \in \mathbb{N}}$$

converges to a solution (x_1^*, x_2^*) of the coincidence problem (4.1);

(b) *if, additionally, f, g are metrically regular with coefficient $s > 0$ (i.e., for every $x_1 \in X_1, x_2 \in X_2$ and $y \in f(X_1) \cap g(X_2)$, we have*

$$d(x_1, f^{-1}(y)) \leq s\rho(f(x_1), y) \text{ and } d(x_2, g^{-1}(y)) \leq s\rho(g(x_2), y)),$$

then the coincidence problem (4.1) has the following Ulam-Hyers type stability property: there exist $c_1, c_2 > 0$ such that for every $\varepsilon > 0$ and every ε-solution $(\bar{x}_1, \bar{x}_2)$ of the coincidence problem (in the sense that there exists $y \in f(X_1) \cap g(X_2)$ such that

$$d_1(f(\bar{x}_1), y) \leq \varepsilon, d_2(g(\bar{x}_2), y) \leq \varepsilon, \rho(h(\bar{x}_1, \bar{x}_2), y) \leq \varepsilon),$$

there exists a solution (x_1^*, x_2^*) of the coincidence problem (4.1) for which we have

$$d_1\left(\bar{x}_1, x_1^*\right) \leq c_1 \varepsilon \text{ and } d_2\left(\bar{x}_2, x_2^*\right) \leq c_2 \varepsilon,$$

Proof. Consider the space $Z := X_1 \times X_2 \times Y$ endowed with the vector-valued metric

$$\tilde{d}(z, w) := \begin{pmatrix} d_1(x_1, u_1) \\ d_2(x_2, u_2) \\ \rho(y, v) \end{pmatrix}, \text{ for } z := (x_1, x_2, y), w := (u_1, , u_2, v) \in Z.$$

Then $(Z, \tilde{d})$ is a complete vector-valued metric space.

By (i) and (ii) we get that $f^{-1} : f(X_1) \to X_1$ and $g^{-1} : g(X_2) \to X_2$ are well-defined and Lipschitz with constants l_1, and respectively l_2. Moreover, f^{-1}, g^{-1} are also uniformly continuous. Hence, by Lemma 3.9 in [4], the sets $f(X_1), g(X_2)$ are complete in (Y, ρ). Let $p : X_1 \times X_2 \times f(X_1) \cap g(X_2) \to X_1 \times X_2 \times f(X_1) \cap g(X_2)$ be defined by

$$p(x_1, x_2, y) = (f^{-1}(y), g^{-1}(y), h(x_1, x_2)).$$

Notice that any fixed point of p gives, in its first two components, a solution of (4.1). Thus, it is enough to prove that $Fix(p) \neq \emptyset$. We will prove that p is an orbital K-contraction with a matrix $K \in M_{3,3}(\mathbb{R}_+)$ convergent to zero. Indeed, if we denote $a := f^{-1}(y), b := g^{-1}(y), c := h(x_1, x_2)$ and $z := (x_1, x_2, y)$, then we have

$$\begin{aligned}\tilde{d}(p(z), p^2(z)) &= \tilde{d}((f^{-1}(y), g^{-1}(y), h(x_1, x_2)), p(f^{-1}(y), g^{-1}(y), h(x_1, x_2))) \\ &= \tilde{d}((f^{-1}(y), g^{-1}(y), h(x_1, x_2)), (f^{-1}(c), g^{-1}(c), h(a, b))) \\ &= \begin{pmatrix} d_1\left(f^{-1}(y), f^{-1}(c)\right) \\ d_2\left(g^{-1}(y), g^{-1}(c)\right) \\ \rho\left(h(x_1, x_2), h(a, b)\right) \end{pmatrix} \leq \begin{pmatrix} l_1\rho(y, c) \\ l_2\rho(y, c) \\ k\left(d_1(x_1, a) + d_2(x_2, b)\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & l_1 \\ 0 & 0 & l_2 \\ k & k & 0 \end{pmatrix} \begin{pmatrix} d_1(x_1, f^{-1}(y)) \\ d_2(x_2, g^{-1}(y)) \\ \rho(y, h(x_1, x_2)) \end{pmatrix} \\ &= K\tilde{d}(z, p(z)),\end{aligned}$$

where

$$K := \begin{pmatrix} 0 & 0 & l_1 \\ 0 & 0 & l_2 \\ k & k & 0 \end{pmatrix} \text{ converges to } 0. \tag{4.2}$$

By Theorem 2.1 applied to p there exists $(x_1^*, x_2^*, y^*) \in X_1 \times X_2 \times f(X_1) \cap g(X_2)$ such that $(x_1^*, x_2^*, y^*) = p(x_1^*, x_2^*, y^*)$ and the sequence $p^n(x_1, x_2, y)$ converges to (x_1^*, x_2^*, y^*) as $n \to \infty$. As a consequence of the definition of p we get that $y^* = f(x_1^*) = g(x_2^*) = h(x_1^*, x_2^*)$.

(b) For the second conclusion, consider $\varepsilon > 0$ and an ε-solution $(\bar{x}_1, \bar{x}_2)$ of the coincidence problem (4.1). Then, there exists $y \in f(X_1) \cap g(X_2)$ such that

$$d_1(f(\bar{x}_1), y) \leq \varepsilon, d_2(g(\bar{x}_2), y) \leq \varepsilon, \rho(h(\bar{x}_1, \bar{x}_2), y) \leq \varepsilon.$$

Denote by (x_1^*, x_2^*, y^*) the fixed point of p obtained, via Theorem 2.1, as the limit of sequence $p^n(\bar{x}_1, \bar{x}_2, y)$ as $n \to \infty$. Then, by (2.2), we have that

$$\tilde{d}((\bar{x}_1, \bar{x}_2, y), (x_1^*, x_2^*, y^*)) \leq (I_3 - K)^{-1}\tilde{d}((\bar{x}_1, \bar{x}_2, y), p(\bar{x}_1, \bar{x}_2, y)).$$

Hence, we get that

$$\begin{aligned}\begin{pmatrix} d_1\left(\bar{x}_1, x_1^*\right) \\ d_2\left(\bar{x}_2, x_2^*\right) \\ \rho\left(y, y^*\right) \end{pmatrix} &\leq \frac{1}{1 - k(l_1 + l_2)} \begin{pmatrix} 1 - kl_2 & kl_1 & l_1 \\ kl_2 & 1 - kl_1 & l_2 \\ k & k & 1 \end{pmatrix} \begin{pmatrix} d_1\left(\bar{x}_1, f^{-1}(y)\right) \\ d_2\left(\bar{x}_2, g^{-1}(y)\right) \\ \rho\left(y, h(\bar{x}_1, \bar{x}_2)\right) \end{pmatrix} \\ &\leq \frac{1}{1 - k(l_1 + l_2)} \begin{pmatrix} 1 - kl_2 & kl_1 & l_1 \\ kl_2 & 1 - kl_1 & l_2 \\ k & k & 1 \end{pmatrix} \begin{pmatrix} d_1\left(\bar{x}_1, f^{-1}(y)\right) \\ d_2\left(\bar{x}_2, g^{-1}(y)\right) \\ \rho\left(y, h(\bar{x}_1, \bar{x}_2)\right) \end{pmatrix}.\end{aligned}$$

If we denote $\alpha_1 := \frac{1-kl_1}{1-k(l_1+l_2)}, \alpha_2 := \frac{1-kl_2}{1-k(l_1+l_2)}$, $\beta_1 := \frac{kl_1}{1-k(l_1+l_2)}, \beta_2 := \frac{kl_2}{1-k(l_1+l_2)}$ and $\gamma_1 := \frac{l_1}{1-k(l_1+l_2)}, \gamma_2 := \frac{l_2}{1-k(l_1+l_2)}$, then we have

$$d_1\left(\bar{x}_1, x_1^*\right) \leq \alpha_2 d_1\left(\bar{x}_1, f^{-1}(y)\right) + \beta_1 d_2\left(\bar{x}_2, g^{-1}(y)\right) + \gamma_1\rho\left(y, h(\bar{x}_1, \bar{x}_2)\right)$$

and

$$d_2\left(\bar{x}_2, x_2^*\right) \leq \beta_2 d_1\left(\bar{x}_1, f^{-1}(y)\right) + \alpha_1 d_2\left(\bar{x}_2, g^{-1}(y)\right) + \gamma_2\rho\left(y, h(\bar{x}_1, \bar{x}_2)\right).$$

Using the metric regularity of f and g, we obtain

$$d_1\left(\bar{x}_1, x_1^*\right) \leq \alpha_2 s\rho(f(\bar{x}_1), y) + \beta_1 s\rho\left(g(\bar{x}_2), y\right) + \gamma_1\rho\left(y, h(\bar{x}_1, \bar{x}_2)\right)$$

and

$$d_2\left(\bar{x}_2, x_2^*\right) \leq \beta_2 s\rho\left(f(\bar{x}_1), y\right) + \alpha_1 s\rho\left(g(\bar{x}_2), y\right) + \gamma_2\rho\left(y, h(\bar{x}_1, \bar{x}_2)\right).$$

Hence

$$d_1\left(\bar{x}_1, x_1^*\right) \leq (\alpha_2 s + \beta_1 s + \gamma_1)\varepsilon \text{ and } d_2\left(\bar{x}_2, x_2^*\right) \leq (\beta_2 s + \alpha_1 s + \gamma_2)\varepsilon,$$

which show that the coincidence problem (4.1) has the Ulam-Hyers stability property. □

We will illustrate the above result, by the following example.

Example 4.2. Let us consider the following system of integro-differential equations:

$$\begin{cases} u'(t) = \displaystyle\int_0^1 K(t, s, u(s), v(s))ds, \ \text{for } t \in [0,1] \\ v''(t) = \displaystyle\int_0^1 K(t, s, u(s), v(s))ds, \ \text{for } t \in [0,1] \\ u(0) = v(0) = v(1) = 0, \end{cases} \tag{4.3}$$

We consider: $X_1 := \{u \in C^1[0,1] : u(0) = 0\}$, $X_2 := \{v \in C^2[0,1] : v(0) = v(1) = 0\}$ and $Y := C[0,1]$. We endow the above spaces with the following complete norms:

$$\|x\|_\infty := \max_{t\in[0,1]} |x(t)|, \ \text{for } x \in Y,$$

$$\|u\|_1 := \max\{\|u\|_\infty, \|u'\|_\infty\}, \ \text{for } u \in X_1,$$

$$\|v\|_2 := \max\{\|v\|_\infty, \|v'\|_\infty, \|v''\|_\infty\}, \ \text{for } u \in X_2.$$

Then, in a similar manner to [4], we have that:

(1) $\|u\|_1 = \|u'\|_0$, for each $u \in X_1$;
(2) $\|v\|_2 = \|v''\|_0$ for each $v \in X_2$.

We consider the operators $T_1 : X_1 \to Y, T_2 : X_2 \to Y$ and $H : X_1 \times X_2 \to Y$, defined by

$$T_1u(t) := u'(t), \ T_2v(t) := v''(t), \ H(u,v)(t) := \int_0^1 K(t, s, u(s), v(s))ds,$$

where $K : [0,1]^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following Lipschitz type condition: there exist $\alpha, \beta > 0$ such that

$$|K(t,s,x_1,x_2) - K(t,s,y_1,y_2)| \leq \alpha|x_1 - y_1| + \beta|x_2 - y_2|, \forall t,s \in [0,1], \forall x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Then, the system (4.3) is equivalent with the following coincidence problem

$$T_1 u = T_2 v = H(u,v). \tag{4.4}$$

Let us observe that T_1, T_2, H are continuous, T_1, T_2 are expansive (i.e., 1-dilatations) and onto and H satisfies the following Lipschitz type condition

$$\|H(x_1,x_2) - H(y_1,y_2)\|_\infty \leq \max\{\alpha, \beta\}\left(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2\right).$$

If, additionally, we assume that $\max\{\alpha, \beta\} < \frac{1}{2}$, then T_1, T_2, H satisfy all the assumptions of Theorem 4.1 and so we obtain the existence of at least one solution for our initial problem.

Remark 4.3. For other coincidence results and related applications see [2, 4, 9, 10]. For the multi-valued case, see [11].

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