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APPROXIMATING COMMON FIXED POINT OF A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we study some implicit algorithms to approximate common fixed point of a countable family of nonexpansive mappings in a Hilbert space. We obtain certain strong convergence results using hybrid method in mathematical programming. Some new algorithms are suggested to find the zeros of maximal monotone operators.

1. Introduction

Throughout this paper $\mathcal{H}$ is a Hilbert space equipped with norm $\|.\|$ induced by inner product $\langle .,. \rangle$ and $\mathcal{K}$ is a nonempty subset of $\mathcal{H}$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ for all } x, y \in \mathcal{K}.$$

A point $z \in \mathcal{K}$ is a fixed point of T if $T(z) = z$. We write $F(T)$ as a fixed point set of T, that is, $F(T) := \{x \in \mathcal{K} : T(x) = x\}$. A variety of real world problems can be transformed into equivalent fixed point problems. This approach has many fruitful applications in different areas, such as, convex minimization, image recovery, signal processing, split feasibility, equilibrium and inverse problems (cf. [5, 7, 13] and references therein). The theoretical framework of finding a fixed point of nonexpansive mappings is useful to approximate solutions of above listed problems, see also [5–7, 10, 11, 13, 19, 22, 23].

Xu and Ori [22] considered the following implicit method to obtain common fixed points of a finite family of nonexpansive mappings $\{T_n\}_{n=1}^m$ in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n(x_n) \tag{1.1}$$

for all $n \in \mathbb{N}$, where $m \in \mathbb{N}$ (m is finite), $x_0 \in \mathcal{K}$, $T_n = T_{n \bmod m}$ and $\alpha_n \in (0, 1)$. They obtained the weak convergence of the sequence $\{x_n\}$ under certain assumptions.

Motivated by Solodov and Svaiter [16], Nakajo and Takahashi [10] proposed the following method to obtain strong convergence of the sequence $\{x_n\}$:

$$\begin{cases} x_0 = x \in \mathcal{K} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T(x_n) \\ \mathcal{C}_n := \{z \in \mathcal{K} : \|y_n - z\| \leq \|x_n - z\|\} \\ \mathcal{Q}_n := \{z \in \mathcal{K} : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_0) \end{cases}$$

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for all $n \in \mathbb{N} \cup \{0\}$, where $\alpha_n \in [0, \alpha]$, for some $\alpha \in [0, 1)$.

Using the idea in [10], Zhang and Su [23] modified method (1.1) and obtained strong convergent results to approximate common fixed points of a finite family of nonexpansive mappings $\{T_n\}_{n=1}^m$ in Hilbert spaces:

$$\begin{cases} x_1 \in \mathcal{K} \\ y_n = \alpha_n x_n + (1-\alpha_n) T_n(z_n) \\ z_n = \beta_n y_n + (1-\beta_n) T_n(y_n) \\ \mathcal{C}_n := \{z \in \mathcal{K} : \|y_n - z\| \le \|x_n - z\|\} \\ \mathcal{Q}_n := \{z \in \mathcal{K} : \langle x_n - z, x_1 - x_n \rangle \ge 0\} \\ x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_1) \end{cases}$$

for all $n \in \mathbb{N}$, where $T_n = T_{n \bmod m}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

On the other hand, the θ-method is a fundamental method and has been extensively used for solving differential equations, see [2, 4, 17]. Consider the following initial value problem for the first-order ordinary differential equation:

$$y' = f(y);\ y(0) = y_0$$

by the following recursive formula, θ-method generates a sequence $\{y_n\}$ as follows:

$$y_{n+1} = y_n + h[\theta f(y_n) + (1-\theta) f(y_{n+1})], \tag{1.2}$$

where $h > 0$ is a step size, y_0 is an initial guess and $\theta \in [0, 1]$ is a parameter. If the function f is in the form $f(y) = y - g(y)$, then the method (1.2) reduces to:

$$y_{n+1} = (1-\tau) y_n + \tau[\theta g(y_n) + (1-\theta) g(y_{n+1})],$$

where $\tau = -\frac{h}{1-h+h\theta}$.
Motivated by this recursive formula, Xu *et al.* [21] introduced the following powerful implicit method for a nonexpansive mapping and obtained weak convergence results:

$$x_1 \in \mathcal{H},\ x_{n+1} = (1-\tau_n) x_n + \tau_n[\theta T(x_n) + (1-\theta) T(x_{n+1})] \tag{1.3}$$

where $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping, $\tau_n \in [0, 1]$ and $\theta \in [0, 1]$ such that $(1-\theta)\tau_n < 1$ for all $n \in \mathbb{N}$, see also [15].

Motivated by Xu and Ori [22], Zhang and Su [23], Xu *et al.* [21] and others, we study three implicit algorithms for approximating common fixed point of a countable family of nonexpansive mappings. More precisely, Section 3 deals with the weak convergence of algorithm to find common fixed point of a countable family of nonexpansive mappings. In Section 4, we present two modifications of (1.3) by two different approaches to obtain strong convergence results. These results are used to approximate common fixed point for one-parameter semigroup of nonexpansive mappings. Some new algorithms are suggested to find the zeros of maximal monotone operators. This way results in [10, 11, 19, 21–23] are complemented, extended and generalized.

2. Preliminaries

We shall use the following notation:

- $\to$ for strong convergence and $\rightharpoonup$ for weak convergence;
- $\omega_w(x_n)$ denotes cluster point (ω-limit) set of a sequence $\{x_n\}$, that is, $\omega_w(x_n) := \{x : \exists\, x_{n_k} \rightharpoonup x\}$.

Let $\mathcal{K}$ be a closed convex subset of a Hilbert space $\mathcal{H}$. Then the nearest point projection or metric projection of $\mathcal{H}$ onto $\mathcal{K}$, denoted by $P_{\mathcal{K}}$, which assigns each point $x \in \mathcal{H}$ with its nearest point in $\mathcal{K}$. In other words, $P_{\mathcal{K}}$ is the unique point in $\mathcal{K}$ such that

$$\|x - P_{\mathcal{K}}(x)\| = \inf\{\|x - y\| : y \in \mathcal{K}\}.$$

Further, for $x \in \mathcal{H}$ and $z \in \mathcal{K}$,

$$z = P_{\mathcal{K}}(x) \text{ if and only if } \langle x - z, z - y\rangle \geq 0$$

for all $y \in \mathcal{K}$.

Lemma 2.1 ([20])**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

The following is Browder's demiclosedness principle in the setting of Hilbert space.

Lemma 2.2 ([3])**.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $T : \mathcal{K} \to \mathcal{K}$ a nonexpansive mapping with a fixed point. Suppose $\{x_n\}$ is a sequence in $\mathcal{K}$ such that $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. Then $T(x) = x$. That is, $I - T$ is demiclosed at zero.*

All Hilbert spaces have the Opial property [12], that is, if for every weakly convergent sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$ with weak limit $x \in \mathcal{H}$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{H}$ with $x \neq y$.

The following is a consequence of Opial property.

Lemma 2.3. *Let $\mathcal{K}$ be a nonempty subset of a Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that $\{\|x_n - z\|\}$ converges for some $z \in \mathcal{K}$. Suppose $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to $p, q \in \mathcal{K}$, respectively. Then $p = q$.*

Proof. Let $z \in \mathcal{K}$. Since $\{\|x_n - z\|\}$ converges to a nonnegative real number, any subsequence of $\{\|x_n - z\|\}$ converges to the same nonnegative real number. Arguing by contradiction, assume that $p \neq q$. Then, by $p, q \in \mathcal{K}$ and the Opial property, we get the following:

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{j\to\infty} \|x_{n_j} - p\| \\ &< \lim_{j\to\infty} \|x_{n_j} - q\| = \lim_{k\to\infty} \|x_{n_k} - q\|\end{aligned}$$

$$< \lim_{k\to\infty} \|x_{n_k} - p\| = \lim_{n\to\infty} \|x_n - p\|,$$

a contradiction. □

The following definitions are useful in dealing with countable family of mappings. Let $\{T_n\}$ and $\mathfrak{T}$ be families of self-mappings on $\mathcal{K}$ with $\emptyset \neq F(\mathfrak{T}) = \cap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathfrak{T})$ is the set of all common fixed points of all mappings in $\mathfrak{T}$.

(i) A family of mappings $\{T_n\}$ is said to satisfy AKTT-condition [1] if for every bounded subset $\mathcal{B}$ of $\mathcal{K}$, $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}(x) - T_n(x)\| : x \in \mathcal{B}\} < \infty$;

(ii) A family of mappings $\{T_n\}$ is said to satisfy NST-condition (I) with $\mathfrak{T}$ [9] if for every bounded sequence $\{x_n\}$ in $\mathcal{K}$

$$\lim_{n\to\infty} \|x_n - T_n(x_n)\| = 0 \text{ implies } \lim_{n\to\infty} \|x_n - T(x_n)\| = 0$$

for all $T \in \mathfrak{T}$;

(iii) A family of mappings $\{T_n\}$ is said to satisfy NST*-condition with $\mathfrak{T}$ [11] if for every bounded sequence $\{x_n\}$ in $\mathcal{K}$

$$\lim_{n\to\infty} \|x_n - T_n(x_n)\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$$

imply that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$ for all $T \in \mathfrak{T}$.

Remark 2.4. The NST-condition (I) with $\mathfrak{T}$ implies to NST*-condition with $\mathfrak{T}$ but the reverse implication does not hold (for more details see [11]).

Lemma 2.5 ([1]). *Let $\mathcal{K}$ be a nonempty closed subset of a Hilbert space and $\{T_n\}$ a family of self mappings on $\mathcal{K}$ which satisfies the AKTT-condition. Then, there is self mapping T on $\mathcal{K}$ satisfying the following:*

(a) *$\{T_n(x)\}$ converges strongly to $T(x)$ for each $x \in \mathcal{K}$: $\lim_{n\to\infty} \|T(x) - T_n(x)\| = 0$ for $x \in \mathcal{K}$*

(b) *On each bounded subset $\mathcal{D}$ of $\mathcal{K}$, $\{T_n\}$ converges uniformly to T:*

$$\lim_{n\to\infty} \sup\{\|T(y) - T_n(y)\| : y \in \mathcal{D}\} = 0.$$

Remark 2.6. It can be easily seen that T is nonexpansive if each T_n is nonexpansive.

Lemma 2.7 ([18]). *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $\{x_n\}$ a sequence in $\mathcal{H}$ such that*

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

for all $p \in \mathcal{K}$ and $n \in \mathbb{N}$. Then, the sequence $\{P_{\mathcal{K}}(x_n)\}$ converges strongly to a point $z \in \mathcal{K}$.

The following is a well-known assertion and can be seen in [8].

Lemma 2.8. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. For given $x, y \in \mathcal{H}$, the set*

$$\mathcal{K}_1 := \{z \in \mathcal{K} : \|y - z\| \leq \|x - z\|\}$$

is closed and convex.

Lemma 2.9 ([8]). *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$, $x \in \mathcal{H}$ and $z = P_{\mathcal{K}}(x)$. If the sequence $\{x_n\}$ is such that $\omega_w(x_n) \subset \mathcal{K}$ and satisfies the condition*

$$\|x_n - x\| \leq \|x - z\|$$

for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to z.

Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. A family $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$ of self-mappings on $\mathcal{K}$ is said to be one-parameter nonexpansive semigroup on $\mathcal{K}$ if the following assumptions hold:

(i) $T(0)(x) = x$ for all $x \in \mathcal{K}$;
(ii) $T(s+t) = T(t)T(s)$ for all $s, t \geq 0$;
(iii) $\|T(t)(x) - T(t)(y)\| \leq \|x - y\|$ for all $t \geq 0$ and $x, y \in \mathcal{K}$;
(iv) A mapping $t \mapsto T(t)(x)$ is continuous for all $x \in \mathcal{K}$.

Lemma 2.10 ([9,14]). *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$ one-parameter nonexpansive semigroup on $\mathcal{K}$ with $F(\mathfrak{S}) \neq \emptyset$. Let $\{s_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} s_n = \infty$. For $n \in \mathbb{N}$, consider a self-mapping T_n on $\mathcal{K}$ by*

$$T_n(x) = \frac{1}{s_n} \int_0^{s_n} T(t)(x)dt$$

for all $x \in \mathcal{K}$. Then $\{T_n\}$ satisfies the NST-condition (I) with $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$.

A multifunction $A : \mathcal{H} \to 2^{\mathcal{H}}$ is called a monotone operator if and only if for all $x, x', y, y' \in \mathcal{H}$

$$\langle x - x', y - y' \rangle \geq 0 \text{ such that } y \in A(x) \text{ and } y' \in A(x').$$

It is called maximal monotone if the graph

$$\{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in A(x)\}$$

is not properly contained in the graph of any other monotone operator on $\mathcal{H}$. Let $D(A)$ be the domain and $R(A)$ the range of an operator A. A point $z \in D(A)$ is said to be zero of A if $0 \in A(z)$. Denote by J_r the resolvent of a maximal monotone operator A for $r > 0$, that is,

$$J_r = (I + rA)^{-1}$$

where $I : \mathcal{H} \to \mathcal{H}$ is an identity operator. It is known that J_r is a nonexpansive operator.

Lemma 2.11 ([9]). *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone operator having resolvent J_r for $r > 0$ such that $A^{-1}(0) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Define $T_n = J_{t_n}$ for all $n \in \mathbb{N}$. Then $\{T_n\}$ satisfies the* NST**-condition with $J_1 = (I + A)^{-1}$.*

3. Weak convergence results

Now we present some lemmas which will be further utilized to prove our main result of this section.

Lemma 3.1. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$, S and T self-mappings on $\mathcal{K}$. Let $a, b \in [0,1]$ and $u \in \mathcal{K}$. For all $x \in \mathcal{K}$, define a self-mapping M on $\mathcal{K}$ by*

$$M(x) = (1-a)u + a(bS(u) + (1-b)T(x))$$

Assume that T is a nonexpansive mapping and $a(1-b) < 1$. Then, there is a unique $v \in F(M)$.

Proof. By our assumptions, it can be easily seen that M is a self-mapping on $\mathcal{K}$. Using the fact that T is a nonexpansive mapping and $a(1-b) < 1$, we get

$$\|M(x) - M(y)\| = a(1-b)\|T(x) - T(y)\| \leq a(1-b)\|x - y\|$$

for all $x, y \in \mathcal{K}$. Thus M is a contraction and by Banach contraction principle, we have the desired conclusion. □

Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a family of nonexpansive self-mappings of $\mathcal{K}$. Let $\theta \in [0,1]$ and $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying $\alpha_n(1-\theta) < 1$ for $n \in \mathbb{N}$. Let $x_1 \in \mathcal{K}$. Then, by Lemma 3.1, we can generate a sequence $\{x_n\}$ in $\mathcal{K}$ as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\theta T_n(x_n) + (1-\theta)T_{n+1}(x_{n+1})\}, \tag{3.1}$$

for all $n \in \mathbb{N}$.

Lemma 3.2. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$, U and V self-mappings on $\mathcal{K}$. Let $a, b \in [0,1]$ satisfying $a(1-b) < 1$ and $x, y \in \mathcal{K}$. Suppose $x = (1-a)y + a\{bU(y) + (1-b)V(x)\}$ and V is nonexpansive. Then the following inequality holds:*

$$\|V(x) - x\| \leq \|U(y) - y\| + \|V(y) - U(y)\|.$$

Proof. Since $x = (1-a)y + a\{bU(y) + (1-b)V(x)\}$, we immediately get the followings:

$$\begin{aligned} x - y &= -ay + a\{bU(y) + (1-b)V(x)\} \\ &= ab(U(y) - y) + a(1-b)(V(x) - y). \end{aligned} \tag{3.2}$$

And

$$\begin{aligned} x - V(x) &= (1-a)(y - U(y)) + (1-a)U(y) + abU(y) + a(1-b)V(x) - V(x) \\ &= (1-a)(y - U(y)) + (1-a+ab)(U(y) - V(x)). \end{aligned} \tag{3.3}$$

Then, by (3.2), we have

$$\begin{aligned} \|x - y\| &\leq ab\|U(y) - y\| + a(1-b)\|V(x) - y\| \\ &\leq ab\|U(y) - y\| + a(1-b)(\|V(x) - x\| + \|x - y\|). \end{aligned}$$

Thus, we get the following:

$$(1-a+ab)\|x - y\| \leq ab\|U(y) - y\| + a(1-b)\|V(x) - x\|. \tag{3.4}$$

By (3.3), (3.4) and using the fact that V is nonexpansive, we have

$$\begin{aligned}\|x - V(x)\| &\le (1-a)\|U(y) - y\| + (1-a+ab)(\|U(y) - V(y)\| + \|V(y) - V(x)\|)\\ &\le (1-a)\|U(y) - y\| + (1-a+ab)(\|U(y) - V(y)\| + \|y - x\|)\\ &\le (1-a)\|U(y) - y\| + (1-a+ab)\|U(y) - V(y)\| + ab\|U(y) - y\|\\ &\quad + a(1-b)\|V(x) - x\|,\end{aligned}$$

this implies

$$(1-a+ab)\|V(x) - x\| \le (1-a+ab)\|U(y) - y\| + (1-a+ab)\|V(y) - U(y)\|.$$

Since $a(1-b) < 1$, $1-a+ab > 0$. Thus we have

$$\|V(x) - x\| \le \|U(y) - y\| + \|V(y) - U(y)\|.$$

□

Lemma 3.3. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$, U and V nonexpansive self-mappings on $\mathcal{K}$. Let $a, b \in [0,1]$ satisfying $a(1-b) < 1$. Let $x, y, z \in \mathcal{K}$ and set $z_y^x = bU(y) + (1-b)V(x)$. Assume $x = (1-a)y + az_y^x$ holds and $v \in F(U) \cap F(V)$. Then, we have the followings:*

(1). $b\|U(y) - y\|^2 \le \|z_y^x - y\|^2 + b(1-b)\|U(y) - V(x)\|^2$.

(2). $\|x - v\|^2 \le \|y - v\|^2 - a(1-a)b(1-b)\|U(y) - V(x)\|^2 - a(1-a)\|y - z_y^x\|^2$.

Proof. By $z_y^x = bU(y) + (1-b)V(x)$, we get

$$\begin{aligned}\|z_y^x - z\|^2 &= \|bU(y) + (1-b)V(x) - z\|^2\\ &= b\|U(y) - z\|^2 + (1-b)\|V(x) - z\|^2 - b(1-b)\|U(y) - V(x)\|^2.\end{aligned} \tag{3.5}$$

Take $z = y$ and considering $(1-b)\|V(x) - y\|^2 \ge 0$, (1) is immediate.
Again by $x = (1-a)y + az_y^x$ and (3.5), we have

$$\begin{aligned}\|x - z\|^2 &= \|(1-a)y + az_y^x - z\|^2\\ &= (1-a)\|y - z\|^2 + a\|z_y^x - z\|^2 - a(1-a)\|y - z_y^x\|^2\\ &= (1-a)\|y - z\|^2 + a\left\{b\|U(y) - z\|^2 + (1-b)\|V(x) - z\|^2\right.\\ &\quad \left.-b(1-b)\|U(y) - V(x)\|^2\right\} - a(1-a)\|y - z_y^x\|^2.\end{aligned} \tag{3.6}$$

Take $z = v$ in (3.6), Since V and U are nonexpansive, by $v \in F(V) \cap F(U)$, we have

$$\begin{aligned}\|x - v\|^2 &= (1-a)\|y - v\|^2 + a\left\{b\|U(y) - U(v)\|^2 + (1-b)\|V(x) - V(v)\|^2\right.\\ &\quad \left.-b(1-b)\|U(y) - V(x)\|^2\right\} - a(1-a)\|y - z_y^x\|^2\\ &\le (1-a)\|y - v\|^2 + a\left\{b\|y - v\|^2 + (1-b)\|x - v\|^2\right.\\ &\quad \left.-b(1-b)\|U(y) - V(x)\|^2\right\} - a(1-a)\|y - z_y^x\|^2\\ &= (1-a+ab)\|y - v\|^2 + a(1-b)\|x - v\|^2\\ &\quad -ab(1-b)\|U(y) - V(x)\|^2 - a(1-a)\|y - z_y^x\|^2,\end{aligned}$$

this implies that

$$\begin{aligned}(1-a+ab)\|x - v\|^2 &\le (1-a+ab)\|y - v\|^2 - ab(1-b)\|U(y) - V(x)\|^2\\ &\quad -a(1-a)\|y - z_y^x\|^2.\end{aligned}$$

Since $a(1-b)<1$, $0<1-a+ab\leq 1$. Thus, we have

$$\begin{aligned}\|x-v\|^2 &\leq \|y-v\|^2-\frac{ab(1-b)}{1-a+ab}\|U(y)-V(x)\|^2-\frac{a(1-a)}{1-a+ab}\|y-z_y^x\|^2\\ &\leq \|y-v\|^2-ab(1-b)\|U(y)-V(x)\|^2-a(1-a)\|y-z_y^x\|^2.\end{aligned}$$

By $a(1-a)b(1-b)\leq ab(1-b)$, (2) follows from above inequality. □

Now we present our main result of this section.

Theorem 3.4. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a family of nonexpansive self-mappings of $\mathcal{K}$ which satisfies the AKTT-condition and $\emptyset\neq\cap_{n=1}^{\infty}F(T_n)=F(T)$, where T is the mapping as in Lemma 2.5. Let $\theta\in[0,1]$, $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $(1-\theta)\alpha_n<1$ for $n\in\mathbb{N}$ and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$. Let $x_1\in\mathcal{K}$ and generate a sequence $\{x_n\}$ by (3.1), that is,*

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n\{\theta T_n(x_n)+(1-\theta)T_{n+1}(x_{n+1})\}\text{for all } n\in\mathbb{N}$$

Then $\{x_n\}$ converges weakly to $w\in\cap_{n=1}^{\infty}F(T_n)$.

Proof. Since $\cap_{n=1}^{\infty}F(T_n)\neq\emptyset$, take a point $u\in\cap_{n=1}^{\infty}F(T_n)$. For $n\in\mathbb{N}$, set $U=T_n$, $V=T_{n+1}$, $y=x_n$, $x=x_{n+1}$, $b=\theta$ and $a=\alpha_n$ in Lemma 3.3. Also take $z_y^x=z_n$. Then we have the following inequalities, for each $n\in\mathbb{N}$

$$z_n=\theta T_n(x_n)+(1-\theta)T_{n+1}(x_{n+1}), \tag{3.7}$$

$$\theta\|T_n(x_n)-x_n\|^2\leq\|z_n-x_n\|^2+\theta(1-\theta)\|T_{n+1}(x_{n+1})-T_n(x_n)\|^2 \tag{3.8}$$

and

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq \|x_n-u\|^2-\alpha_n(1-\alpha_n)\theta(1-\theta)\|T_{n+1}(x_{n+1})-T_n(x_n)\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-z_n\|^2. \end{aligned}\tag{3.9}$$

By (3.9), we can easily see that $\{\|x_n-u\|\}$ is a non increasing sequence. Then $\{\|x_n-u\|\}$ converges. Note that $\{\|x_n-v\|\}$ converges for some $v\in\cap_{n=1}^{\infty}F(T_n)$. Set

$$\mathcal{D}=\{x\in\mathcal{K}:\|x-u\|\leq\|x_1-u\|\}.$$

Since $u\in\mathcal{D}$, $\mathcal{D}$ is a nonempty subset of $\mathcal{K}$. By Lemma 2.8 it can be easily seen that $\mathcal{D}$ is closed and convex. Thus $\mathcal{D}$ is a nonempty bounded closed convex subset of $\mathcal{K}$. Also $x_n\in\mathcal{D}$ for each $n\in\mathbb{N}$.

Now, we show that $\lim_{n\to\infty}\|T_n(x_n)-x_n\|=0$. For $n\in\mathbb{N}$, set $U=T_n$, $V=T_{n+1}$, $y=x_n$, $x=x_{n+1}$, in Lemma 3.2, we get

$$\|T_{n+1}(x_{n+1})-x_{n+1}\|\leq\|T_n(x_n)-x_n\|+\|T_{n+1}(x_n)-T_n(x_n)\|.$$

Since $\{T_n\}$ satisfies the AKTT-condition, we know

$$\sum_{n=1}^{\infty}\|T_{n+1}(x_n)-T_n(x_n)\|\leq\sum_{n=1}^{\infty}\sup\{\|T_{n+1}(x)-T_n(x)\|:x\in\mathcal{D}\}<\infty.$$

From the above two inequalities, Lemma 2.1 asserts that $\{\|T_n(x_n)-x_n\|\}$ converges. By (3.9), we easily see that, for all $n\in\mathbb{N}$

$$\theta(1-\theta)\sum_{i=1}^{n}\alpha_i(1-\alpha_i)\ \|T_{i+1}(x_{i+1})-T_i(x_i)\|^2+\sum_{i=1}^{n}\alpha_i(1-\alpha_i)\|x_i-z_i\|^2$$

$$\leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \leq \|x_1 - u\|^2.$$

For $a_n, b_n \geq 0$, we know that $a_n \leq a_n + b_n$. Then by letting $n \to \infty$ in the above inequality, we have

$$\theta(1-\theta)\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|T_{n+1}(x_{n+1}) - T_n(x_n)\|^2 < \infty, \tag{3.10}$$

and

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - z_n\|^2 < \infty. \tag{3.11}$$

Furthermore, by (3.8) and $0 \leq \alpha_n(1-\alpha_n)$, we see that, for $n \in \mathbb{N}$,

$$\begin{aligned} \alpha_n(1-\alpha_n)\theta\|T_n(x_n) - x_n\|^2 \;&\leq\; \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &\quad +\alpha_n(1-\alpha_n)\theta(1-\theta)\|T_{n+1}(x_{n+1}) - T_n(x_n)\|^2. \end{aligned}$$

In view of (3.10) and (3.11), we can easily see that

$$\theta\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|T_n(x_n) - x_n\|^2 < \infty. \tag{3.12}$$

If $\theta \neq 0$, by (3.12) and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we obtain $\liminf_{n\to\infty} \|T_n(x_n) - x_n\| = 0$.
Again, if $\theta = 0$, then (3.1) and (3.7) become, for $n \in \mathbb{N}$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{n+1}(x_{n+1}) \text{ and } z_n = T_{n+1}(x_{n+1}).$$

Then, the following is immediate: For $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - T_{n+1}(x_{n+1})\| = (1-\alpha_n)\|x_n - T_{n+1}(x_{n+1})\| \;&=\; (1-\alpha_n)\|x_n - z_n\| \\ &\leq\; \|x_n - z_n\|. \end{aligned} \tag{3.13}$$

By (3.11) and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we obtain $\liminf_{n\to\infty} \|x_n - z_n\| = 0$. Then from (3.13), we have

$$\liminf_{n\to\infty} \|T_n(x_n) - x_n\| = \liminf_{n\to\infty} \|T_{n+1}(x_{n+1}) - x_{n+1}\| \leq \liminf_{n\to\infty} \|x_n - z_n\| = 0.$$

In both cases, we obtained that $\liminf_{n\to\infty} \|T_n(x_n) - x_n\| = 0$. Since $\{\|T_n(x_n) - x_n\|\}$ converges, we see that

$$\lim_{n\to\infty} \|T_n(x_n) - x_n\| = \liminf_{n\to\infty} \|T_n(x_n) - x_n\| = 0.$$

Finally, we claim that $\{x_n\}$ converges weakly to some $w \in \cap_{n=1}^{\infty} F(T_n)$. Since $\{T_n\}$ satisfies the condition AKTT, by Lemma 2.5 there is a nonexpansive self-mapping T on $\mathcal{K}$ satisfying $\lim_{n\to\infty} \sup\{\|T(y) - T_n(y)\| : y \in \mathcal{D}\} = 0$ and $\cap_{n=1}^{\infty} F(T_n) = F(T)$. Note that $\mathcal{D}$ is weakly closed. Since $\{x_n\}$ is bounded, there is a weakly convergent subsequence of $\{x_n\}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $w \in \mathcal{D} \subset \mathcal{C}$. We know that $\lim_{n\to\infty} \sup\{\|T(y) - T_n(y)\| : y \in \mathcal{D}\} = 0$ and $\lim_{n\to\infty} \|T_n(x_n) - x_n\|$. Then by the triangle inequality, we see that, for $j \in \mathbb{N}$.

$$\begin{aligned} \|T(x_{n_j}) - x_{n_j}\| \;&\leq\; \|T(x_{n_j}) - T_{n_j}(x_{n_j})\| + \|T_{n_j}(x_{n_j}) - x_{n_j}\| \\ &\leq\; \sup\{\|T(y) - T_{n_j}(y)\| : y \in \mathcal{D}\} + \|T_{n_j}(x_{n_j}) - x_{n_j}\| \end{aligned}$$

From these, we obtain $\lim_{j\to\infty}\|T(x_{n_j}) - x_{n_j}\| = 0$. Since the sequence $\{x_{n_k}\}$ converges weakly to some w in $\mathcal{D}$, Lemma 2.2 asserts that $w \in F(T) = \cap_{n=1}^{\infty} F(T_n)$. Also, we see that every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of $\cap_{n=1}^{\infty} F(T_n)$. For this, by Lemma 2.3, every weakly convergent subsequence of $\{x_n\}$ converges weakly to w. Hence, $\{x_n\}$ itself converges weakly to $w \in \cap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

For $\theta = 0, \frac{1}{2}, 1$ we obtain the following corollaries.

Corollary 3.5. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a family of nonexpansive self-mappings of $\mathcal{K}$ which satisfies the AKTT-condition and $\emptyset \neq \cap_{n=1}^{\infty} F(T_n) = F(T)$, where T is the mapping as in Lemma 2.5. Let $\{\alpha_n\}$ be a sequence in $[0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Let $x_1 \in \mathcal{K}$ and generate a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_{n+1}(x_{n+1}) \text{ for all } n \in \mathbb{N}$$

Then $\{x_n\}$ converges weakly to $w \in \cap_{n=1}^{\infty} F(T_n)$.

Corollary 3.6. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a family of nonexpansive self-mappings of $\mathcal{K}$ which satisfies the AKTT-condition and $\emptyset \neq \cap_{n=1}^{\infty} F(T_n) = F(T)$, where T is the mapping as in Lemma 2.5. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Let $x_1 \in \mathcal{K}$ and generate a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = (1-\alpha_n)x_n + \frac{\alpha_n}{2}\{T_n(x_n) + T_{n+1}(x_{n+1})\} \text{ for all } n \in \mathbb{N}$$

Then $\{x_n\}$ converges weakly to $w \in \cap_{n=1}^{\infty} F(T_n)$.

Corollary 3.7. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ be a family of nonexpansive self-mappings of $\mathcal{K}$ which satisfies the AKTT-condition and $\emptyset \neq \cap_{n=1}^{\infty} F(T_n) = F(T)$, where T is the mapping as in Lemma 2.5. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$. Let $x_1 \in \mathcal{K}$ and generate a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n(x_n) \text{ for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $w \in \cap_{n=1}^{\infty} F(T_n)$.

4. Strong convergence results

In this section, we present some strong convergence results.

Theorem 4.1. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ and $\mathfrak{T}$ be two families of nonexpansive self-mappings on $\mathcal{K}$ with $F(\mathfrak{T}) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, assume that $\{T_n\}$ satisfies the NST^*-condition with $\mathfrak{T}$. Let $\theta \in [0,1]$ and $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$ for $n \in \mathbb{N}$. Let $x_1 \in \mathcal{K}$, then, by Lemma 3.1 we can generate a sequence $\{x_n\}$ as follows:*

$$(4.1)\qquad \begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\{\theta T_n(x_n) + (1-\theta)T_n(y_n)\} \\ \mathcal{C}_n := \{z \in \mathcal{K} : \|y_n - z\| \leq \|x_n - z\|\} \\ \mathcal{Q}_n := \{z \in \mathcal{K} : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_1). \end{cases}$$

Then the sequence defined by (4.1) *converges strongly to* $z = P_{F(\mathfrak{T})}(x_1)$.

Proof. We divide the proof into five steps:

Step 1. $F(\mathfrak{T}) \subseteq \mathcal{C}_n \cap \mathcal{Q}_n$ for all $n \in \mathbb{N}$.

In view of Lemma 2.8, it can be easily seen that $\mathcal{C}_n$ is convex. Now we claim that $F(\mathfrak{T}) \subseteq \mathcal{C}_n$ for all $n \in \mathbb{N}$. Let $p \in F(\mathfrak{T})$, from (4.1), we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n\{\theta T_n(x_n) + (1-\theta)T_n(y_n)\} - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\theta\|T_n(x_n) - p\| + (1-\theta)\|T_n(y_n) - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\theta\|x_n - p\| + (1-\theta)\|y_n - p\|, \end{aligned}$$

or

$$\|y_n - p\| \leq \|x_n - p\|$$

for all $n \in \mathbb{N}$. Thus $p \in \mathcal{C}_n$ and $F(\mathfrak{T}) \subseteq \mathcal{C}_n$. To show that $F(\mathfrak{T}) \subseteq \mathcal{C}_n \cap \mathcal{Q}_n$ for all $n \in \mathbb{N}$, it remains to show that $F(\mathfrak{T}) \subseteq \mathcal{Q}_n$ for all $n \in \mathbb{N}$. We show this result by induction. For $n = 1$, it can be seen that $F(\mathfrak{T}) \subseteq \mathcal{K} = \mathcal{Q}_1$. Assume that $F(\mathfrak{T}) \subseteq \mathcal{Q}_m$ for any $m \in \mathbb{N}$. Then $F(\mathfrak{T}) \subseteq \mathcal{C}_m \cap \mathcal{Q}_m$ and from (4.1), there is a unique point $x_{m+1} \in \mathcal{C}_m \cap \mathcal{Q}_m$ such that $x_{m+1} = P_{\mathcal{C}_m \cap \mathcal{Q}_m}(x_1)$. By the definition of projection mapping

$$\langle x_{m+1} - z, x_1 - x_{m+1}\rangle \geq 0$$

for all $z \in \mathcal{C}_m \cap \mathcal{Q}_m$. In particular, for each $p \in F(\mathfrak{T})$

$$\langle x_{m+1} - p, x_1 - x_{m+1}\rangle \geq 0.$$

Thus $F(\mathfrak{T}) \subseteq \mathcal{Q}_{m+1}$ and $F(\mathfrak{T}) \subseteq \mathcal{C}_{m+1} \cap \mathcal{Q}_{m+1}$.

Step 2. $\{x_n\}$ is bounded.

By the definition of set $\mathcal{Q}_n$ and projection mapping $P_{\mathcal{Q}_n}$, we have $x_n = P_{\mathcal{Q}_n}(x_1)$ and

$$\|x_n - x_1\| \leq \|z - x_1\| \tag{4.2}$$

for all $z \in \mathcal{Q}_n$, and $n \in \mathbb{N}$. In particular, for each $p \in F(\mathfrak{T})$, and $n \in \mathbb{N}$

$$\|x_n - x_1\| \leq \|p - x_1\|. \tag{4.3}$$

Thus $\{x_n\}$ is bounded.

Step 3. $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Since $x_{n+1} \in \mathcal{C}_n \cap \mathcal{Q}_n \subseteq \mathcal{Q}_n$, from (4.2), we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. Thus the sequence $\{\|x_n - x_1\|\}$ is nondecreasing and by the boundedness of $\{\|x_n - x_1\|\}$, $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Note that $x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_1) \in \mathcal{Q}_n$, thus we have

$$\langle x_n - x_{n+1}, x_1 - x_n\rangle \geq 0. \tag{4.4}$$

We know that, for all $u, v \in \mathcal{H}$

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle.$$

Thus by (4.4) we get

$$\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_1) - (x_n - x_1)\|^2$$

$$\begin{aligned} &= \|x_{n+1}-x_1\|^2 - \|x_n-x_1\|^2 - 2\langle x_{n+1}-x_n, x_n-x_1\rangle \\ &\leq \|x_{n+1}-x_1\|^2 - \|x_n-x_1\|^2. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$.

Step 4. $\lim_{n\to\infty}\|x_n-T_n(x_n)\|=0$.
Since $x_{n+1}\in\mathcal{C}_n$, by the definition of set $\mathcal{C}_n$, we get

$$\|y_n-x_{n+1}\|\leq\|x_n-x_{n+1}\|.$$

By the triangle inequality

$$\begin{aligned} \|y_n-x_n\| &\leq \|y_n-x_{n+1}\|+\|x_{n+1}-x_n\| \\ &\leq 2\|x_{n+1}-x_n\|. \end{aligned}$$

Thus

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{4.5}$$

Now by (4.1)

$$\begin{aligned} \|y_n-T_n(y_n)\| &= \|(1-\alpha_n)x_n+\alpha_n\{\theta T_n(x_n)+(1-\theta)T_n(y_n)\}-T_n(y_n)\| \\ &\leq (1-\alpha_n)\|x_n-T_n(y_n)\|+\alpha_n\theta\|x_n-y_n\|. \end{aligned} \tag{4.6}$$

By the triangle inequality and (4.6), we get

$$\begin{aligned} \|x_n-T_n(y_n)\| &\leq \|x_n-y_n\|+\|y_n-T_n(y_n)\| \\ &\leq \|x_n-y_n\|+(1-\alpha_n)\|x_n-T_n(y_n)\|+\alpha_n\theta\|x_n-y_n\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_n-T_n(y_n)\| &\leq \frac{(1+\alpha_n\theta)}{\alpha_n}\|x_n-y_n\| \\ &\leq \frac{(1+\alpha_n\theta)}{\alpha}\|x_n-y_n\|. \end{aligned} \tag{4.7}$$

By the triangle inequality, (4.5) and (4.7), we have

$$\begin{aligned} \|x_n-T_n(x_n)\| &\leq \|x_n-T_n(y_n)\|+\|T_n(y_n)-T_n(x_n)\| \\ &\leq \frac{(1+\alpha_n\theta)}{\alpha}\|x_n-y_n\|+\|y_n-x_n\| \\ &\leq \left(\frac{(1+\alpha_n\theta)}{\alpha}+1\right)\|x_n-y_n\| \end{aligned}$$

Thus $\lim_{n\to\infty}\|x_n-T_n(x_n)\|=0$.

Step 5. The sequence $\{x_n\}$ converges strongly to $z=P_{F(\mathfrak{T})}(x_1)$.
Since $\{T_n\}$ satisfies the NST*-condition with $\mathfrak{T}$,

$$\lim_{n\to\infty}\|x_n-T(x_n)\|=0 \tag{4.8}$$

for all $T\in\mathfrak{T}$. In view of demiclosedness principle and (4.8), it can be seen that $\omega_w(x_n)\subseteq F(\mathfrak{T})$. By (4.3) we get

$$\|x_n-x_1\|\leq\|z-x_1\|$$

for all $n\in\mathbb{N}$, where $z=P_{F(\mathfrak{T})}(x_1)\in\mathfrak{T}$. Using Lemma 2.9 we conclude that $\{x_n\}$ converges strongly to $z=P_{F(\mathfrak{T})}(x_1)$. □

Now we present another strong convergence theorem.

Theorem 4.2. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{T_n\}$ and $\mathfrak{T}$ be two families of nonexpansive self-mappings of $\mathcal{K}$ with $F(\mathfrak{T}) = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, assume that $\{T_n\}$ satisfies the NST^*-condition with $\mathfrak{T}$. Let $\theta \in [0,1]$ and $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$ for $n \in \mathbb{N}$. Let $u_0 \in \mathcal{H}$, $\mathcal{C}_1 = \mathcal{K}$, $x_1 = P_{\mathcal{C}_1}(u_0)$, then, by Lemma 3.1 we can generate a sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\{\theta T_n(x_n) + (1-\theta)T_n(y_n)\} \\ \mathcal{C}_{n+1} := \{z \in \mathcal{C}_n : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{\mathcal{C}_{n+1}}(u_0). \end{cases} \tag{4.9}$$

Then the sequence $\{x_n\}$ converges strongly to $z = P_{F(\mathfrak{T})}(u_0)$.

Proof. By induction, we show that $F(\mathfrak{T}) \subseteq \mathcal{C}_n$ for all $n \in \mathbb{N}$. For $n = 1$, $F(\mathfrak{T}) \subseteq \mathcal{C}_1 = \mathcal{K}$ is obvious. Assume that $F(\mathfrak{T}) \subseteq \mathcal{C}_m$ for some $m \in \mathbb{N}$. Let $p \in F(\mathfrak{T}) \subseteq \mathcal{C}_m$ and by (4.9), we have

$$\begin{aligned} \|y_m - p\| &= \|(1-\alpha_m)x_m + \alpha_m\{\theta T_m(x_m) + (1-\theta)T_m(y_m)\} - p\| \\ &\leq (1-\alpha_m)\|x_m - p\| + \alpha_m\theta\|T_m(x_m) - p\| + (1-\theta)\|T_m(y_m) - p\| \\ &\leq (1-\alpha_m)\|x_m - p\| + \alpha_m\theta\|x_m - p\| + (1-\theta)\|y_m - p\|. \end{aligned}$$

This implies that

$$\|y_m - p\| \leq \|x_m - p\|$$

for all $m \in \mathbb{N}$ and $p \in \mathcal{C}_{m+1}$. This proves that $F(\mathfrak{T}) \subseteq \mathcal{C}_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathcal{C}_1 = \mathcal{K}$ is closed and convex so in view of Lemma 2.8, $\mathcal{C}_n$ is closed and convex. Following largely the proof of [19, Theorem 3.3] we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Rest of the proof may be completed following Theorem 4.1. □

Theorem 4.3. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$ a one-parameter nonexpansive semigroup on $\mathcal{K}$ with $F(\mathfrak{S}) \neq \emptyset$. Let $\theta \in [0,1]$, $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$, $\mu_n \in (0,\infty)$ and $\lim_{n\to\infty} \mu_n = \infty$. For given $x_1 \in \mathcal{K}$ the sequence $\{x_n\}$ in $\mathcal{K}$ defined as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\left\{\theta \frac{1}{\mu_n}\int_0^{\mu_n} T(t)(x_n)dt + (1-\theta)\frac{1}{\mu_n}\int_0^{\mu_n} T(t)(y_n)dt\right\} \\ \mathcal{C}_n := \{z \in \mathcal{K} : \|y_n - z\| \leq \|x_n - z\|\} \\ \mathcal{Q}_n := \{z \in \mathcal{K} : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_1). \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $z = P_{F(\mathfrak{S})}(x_1)$.

Proof. For $n \in \mathbb{N}$, define

$$T_n(x) = \frac{1}{\mu_n}\int_0^{\mu_n} T(t)(x)dt$$

for all $x \in \mathcal{K}$. By Lemma 2.10 and Remark 2.4, $\{T_n\}$ satisfies the NST*-condition with $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$. Therefore, desired result follows by Theorem 4.1. □

Theorem 4.4. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $A : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone operator having resolvent J_r for $r > 0$ such that $A^{-1}(0) \neq \emptyset$. Let $\theta \in [0,1]$, $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$, $r_n \in (0,\infty)$ and $\lim_{n\to\infty} r_n = \infty$. For given $x_1 \in \mathcal{K}$ the sequence $\{x_n\}$ in $\mathcal{K}$ defined as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\{\theta J_{r_n}(x_n) + (1-\theta)J_{r_n}(y_n)\} \\ \mathcal{C}_n := \{z \in \mathcal{K} : \|y_n - z\| \leq \|x_n - z\|\} \\ \mathcal{Q}_n := \{z \in \mathcal{K} : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n}(x_1). \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $z = P_{A^{-1}(0)}(x_1)$.

Proof. Let $T_n(x) = J_{r_n}(x)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$. By Lemma 2.10 and Remark 2.4, $\{T_n\}$ satisfies the NST*-condition with $J_1 = (I + A)^{-1}$. Thus the rest of proof is followed by Theorem 4.1. □

Using Theorem 4.2 one may easily obtain the following two results:

Theorem 4.5. *Let $\mathcal{K}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $\mathfrak{S} := \{T(t) : 0 \leq t < \infty\}$ a one-parameter nonexpansive semigroup on $\mathcal{K}$ with $F(\mathfrak{S}) \neq \emptyset$. Let $\theta \in [0,1]$, $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$, $\mu_n \in (0,\infty)$ and $\lim_{n\to\infty} \mu_n = \infty$. Let $u_0 \in \mathcal{H}$, $\mathcal{C}_1 = \mathcal{K}$, $x_1 = P_{\mathcal{C}_1}(u_0)$, and the sequence $\{x_n\}$ in $\mathcal{K}$ defined as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\left\{\theta \frac{1}{\mu_n}\int_0^{\mu_n} T(t)(x_n)dt + (1-\theta)\frac{1}{\mu_n}\int_0^{\mu_n} T(t)(y_n)dt\right\} \\ \mathcal{C}_{n+1} := \{z \in \mathcal{C}_n : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{\mathcal{C}_{n+1}}(u_0) \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $z = P_{F(\mathfrak{S})}(u_0)$.

Theorem 4.6. *Let $\mathcal{H}$ be a Hilbert space and $A : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone operator having resolvent J_r for $r > 0$ such that $A^{-1}(0) \neq \emptyset$. Let $\theta \in [0,1]$, $\alpha_n \in [\alpha, 1]$ for some $\alpha \in (0,1)$ such that $(1-\theta)\alpha_n < 1$, for $n \in \mathbb{N}$, $r_n \in (0,\infty)$ and $\lim_{n\to\infty} r_n = \infty$. Let $u_0 \in \mathcal{H}$, $\mathcal{C}_1 = \mathcal{H}$, $x_1 = u_0$, and the sequence $\{x_n\}$ in $\mathcal{H}$ defined as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n\{\theta J_{r_n}(x_n) + (1-\theta)J_{r_n}(y_n)\} \\ \mathcal{C}_{n+1} := \{z \in \mathcal{C}_n : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{\mathcal{C}_{n+1}}(u_0). \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $z = P_{A^{-1}(0)}(u_0)$.

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