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EXISTENCE OF SOLUTIONS IN THE SPACE OF HÖLDER FUNCTIONS TO CHANDRASEKHAR'S EQUATION

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Abstract. In this paper, we study sufficient conditions for the existence of solutions of a nonlinear quadratic integral equation which has as a particular case the well-known Chandrasekhar's equation which appears in the theory of radiative transfer, the kinetic theory of gases, the queuing theory, traffic theory, among others. Perhaps, the originality of the paper lies in that the solutions are placed in the space of Lipschitz functions. The main tools used in the proof of the results are a sufficient condition for the relative compactness in the Hölder spaces and the classical Schauder fixed point theorem.

1. Introduction

The following quadratic integral equation

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s}\, \varphi(s) x(s)\, ds, \quad t \in [0,1] \tag{1.1}$$

is the well-known Chandrasekhar's equation and it appears in the theory of radiative transfer, the queuing theory, the kinetic theory of gases, the theory of neutron transport, among others [2, 6].

Motivated by this equation, in this paper we study the theory of existence of solutions to the following nonlinear quadratic integral equation of Urysohn type

$$x(t) = h(t) + x(t) \int_0^1 K(t,s) g(s, x(s), (Hx)(s))\, ds, \quad t \in [0,1] \tag{1.2}$$

which is more general than the one appearing in (1.1), where H is an operator applying $C[0,1]$ into itself.

For the best of our knowledge, in the papers appearing in the literature which study the theory of existence for Eq. (1.1), their solutions are continuous on $[0,1]$, [5, 7, 8, 9]. In this paper, the solutions belong to the space of Hölder functions (see Section 2). The main tools used in the paper are a sufficient condition for the relative compactness in Hölder spaces, recently proved in [1], and the classical Schauder fixed point theorem.

2. Preliminaries

Our starting point in this section is to collect some basic facts about the functions satisfying a Hölder condition. This material can be found in [1].

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Let $[a, b]$ be a closed interval in $\mathbb{R}$. By $C[a, b]$ we denote the space of continuous functions with real values defined on $[a, b]$ and equipped with the classical norm of the supremum, that is, for $x \in C[a, b]$, $\|x\|_\infty = \sup\{|x(t)| : t \in [a, b]\}$.

For $0 < \alpha \leq 1$ fixed, $H_\alpha[a, b]$ will denote the space of the real functions x defined on $[a, b]$ and satisfying the Hölder condition, that is, those functions $x : [a, b] \to \mathbb{R}$ for which a constant H_x^α exists such that

$$|x(t) - x(s)| \leq H_x^\alpha |t - s|^\alpha, \tag{2.1}$$

for any $t, s \in [a, b]$. It is easily seen that $H_\alpha[a, b]$ is a linear subspace of $C[a, b]$. For $x \in H_\alpha[a, b]$, by H_x^α we denote the least constant satisfying (2.1), i.e.,

$$H_x^\alpha = \sup\left\{\frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s\right\}.$$

The space $H_\alpha[a, b]$ $(0 < \alpha \leq 1)$ can be normed by

$$\|x\|_\alpha = |x(a)| + H_x^\alpha = |x(a)| + \sup\left\{\frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s\right\}.$$

In [1], it is proved that $(H_\alpha[a, b], \|\cdot\|_\alpha)$ is a Banach space. When $\alpha = 1$, the space $(H_1[a, b], \|\cdot\|_1)$ is called Lipschitz space.

Lemma 2.1. *For $x \in H_\alpha[a, b]$, we have*

$$\|x\|_\infty \leq \max\{1, (b - a)^\alpha\} \|x\|_\alpha.$$

Lemma 2.2. *Suppose that $0 < \alpha < \gamma \leq 1$ then*

$$H_\gamma[a, b] \subset H_\alpha[a, b] \subset C[a, b].$$

Moreover, if $x \in H_\gamma[a, b]$ then the following inequality

$$\|x\|_\alpha \leq \max\left\{1, (b - a)^{\gamma - \alpha}\right\} \|x\|_\gamma.$$

holds.

The following result will be very important in our study.

Theorem 2.3. *Suppose that $0 < \alpha < \beta \leq 1$ and A is a bounded subset of $H_\beta[a, b]$ (this means that $\|x\|_\beta \leq M$ for any $x \in A$, where M is a positive constant). Then A is a relatively compact subset of $(H_\alpha[a, b], \|\cdot\|_\alpha)$.*

3. Main result

We begin this section studying the behaviour of the usual product of functions in Hölder spaces.

Lemma 3.1. *Suppose that $x, y \in H_\alpha[0, 1]$ $(0 < \alpha \leq 1)$.*
Then

(i) $xy \in H_\alpha[0, 1]$.
(ii) $\|xy\|_\alpha \leq 2\|x\|_\alpha \|y\|_\alpha$.

Proof. (i) We take $t, s \in [0,1]$ with $t \neq s$. Taking into account Lemma 2.1, we get

$$\begin{aligned}\frac{|x(t)y(t)-x(s)y(s)|}{|t-s|^\alpha} &\le \frac{|x(t)y(t)-x(s)y(t)|}{|t-s|^\alpha} + \frac{|x(s)y(t)-x(s)y(s)|}{|t-s|^\alpha} \\ &\le |y(t)|\frac{|x(t)-x(s)|}{|t-s|^\alpha} + |x(s)|\frac{|y(t)-y(s)|}{|t-s|^\alpha} \\ &\le \|y\|_\infty H_x^\alpha + \|x\|_\infty H_y^\alpha \\ &\le \|y\|_\alpha H_x^\alpha + \|x\|_\alpha H_y^\alpha < \infty.\end{aligned}$$

This proves (i).

(ii) From the last estimate, it follows that

$$H_{xy}^\alpha \le \|y\|_\infty H_x^\alpha + \|x\|_\infty H_y^\alpha.$$

Now, taking into account the definition of $\|\cdot\|_\alpha$, we have

$$\begin{aligned}\|xy\|_\alpha &= |x(0)y(0)| + H_{xy}^\alpha \\ &\le |x(0)||y(0)| + \|y\|_\infty H_x^\alpha + \|x\|_\infty H_y^\alpha \\ &= |x(0)||y(0)| + |x(0)|\|y\|_\infty - |x(0)|\|y\|_\infty + \|y\|_\infty H_x^\alpha \\ &\quad + |y(0)|\|x\|_\infty - |y(0)|\|x\|_\infty + \|x\|_\infty H_y^\alpha \\ &= \|y\|_\infty (|x(0)| + H_x^\alpha) + \|x\|_\infty \left(|y(0)| + H_y^\alpha\right) \\ &\quad + |x(0)||y(0)| - |x(0)|\|y\|_\infty - |y(0)|\|x\|_\infty \\ &= \|y\|_\infty \|x\|_\alpha + \|x\|_\infty \|y\|_\alpha + |x(0)||y(0)| - |x(0)|\|y\|_\infty - |y(0)|\|x\|_\infty \\ &\le \|y\|_\alpha \|x\|_\alpha + \|x\|_\alpha \|y\|_\alpha + |x(0)||y(0)| - |x(0)||y(0)| - |y(0)|\|x\|_\infty \\ &= 2\|x\|_\alpha \|y\|_\alpha - |y(0)|\|x\|_\infty \\ &\le 2\|x\|_\alpha \|y\|_\alpha,\end{aligned}$$

where we have used Lemma 2.1.

This completes the proof. ☐

Remark 3.2. We do not know the sharp constant appearing in the inequality of (ii) of Lemma 3.1.

Lemma 3.3. *Let $M : H_\alpha[0,1] \times H_\alpha[0,1] \longrightarrow H_\alpha[0,1]$ be the usual product of functions, i.e., $M(x,y) = xy$ (which is well defined in virtue of Lemma 3.1). Then M is continuous respect to the norm $\|\cdot\|_\alpha$.*

Proof. Suppose that $(x_n), (y_n) \subset H_\alpha[0,1]$ are two sequences such that $x_n \xrightarrow{\|\cdot\|_\alpha} x$ and $y_n \xrightarrow{\|\cdot\|_\alpha} y$ where $x, y \in H_\alpha[0,1]$.

In order to prove the continuity of M it is necessary to prove that

$$M(x_n, y_n) = x_n y_n \xrightarrow{\|\cdot\|_\alpha} xy.$$

In fact, taking into account Lemma 3.1, we infer that

$$\begin{aligned}\|M(x_n, y_n) - M(x, y)\|_\alpha &= \|x_n y_n - xy\|_\alpha \\ &\leq \|x_n y_n - x y_n\|_\alpha + \|x y_n - xy\|_\alpha \\ &= \|(x_n - x) y_n\|_\alpha + \|x(y_n - y)\|_\alpha \\ &\leq 2\|x_n - x\|_\alpha \|y_n\|_\alpha + 2\|x\|_\alpha \|y_n - y\|_\alpha.\end{aligned}$$

Since $y_n \overset{\|\cdot\|_\alpha}{\longrightarrow} y$, the sequence $\|y_n\|_\alpha$ is bounded in $(H_\alpha[0,1], \|\cdot\|_\alpha)$ and, consequently, from the last inequality, it follows that $M(x_n, y_n) \longrightarrow M(x, y)$ when $n \to \infty$. This completes the proof. □

Proposition 3.4. *Suppose that* $x \in H_\alpha[0,1]$ $(0 < \alpha \leq 1)$ *and let* Tx *be the function defined by*

$$(Tx)(t) = \int_0^1 K(t,s) g(s, x(s), (Hx)(s))\, ds, \quad t \in [0,1].$$

Under the following assumptions:

(a) $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ *is a continuous function and there exists a constant* k_1 *such that*

$$|K(t,s) - K(\tau, s)| \leq k_1 |t - \tau|^\alpha,$$

for any $t, \tau, s \in [0,1]$,

(b) $g : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function satisfying*

$$|g(t, x, y) - g(t, x_1, y_1)| \leq \varphi(\max\{|x - x_1|, |y - y_1|\}),$$

for any $t \in [0,1]$ *and* $x, x_1, y, y_1 \in \mathbb{R}$*, where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function,*

(c) $H : C[0,1] \longrightarrow C[0,1]$ *(not necessarily linear) and it satisfies*
  (c1) $Hx \geq 0$ *for* $x \geq 0$,
  (c2) $\|Hx\|_\infty \leq \|x\|_\infty$,

we have that

(1) $Tx \in H_\alpha[0,1]$,
(2) $\|Tx\|_\alpha \leq (\varphi(\|x\|_\alpha) + M)(k_1 + N)$*, where* $M = \sup\{|g(t,0,0)| : t \in [0,1]\}$ *and* $N = \sup\{|K(0,s)| : s \in [0,1]\}$ *(the existence of the constants* M *and* N *are guaranteed in virtue of assumptions (a) and (b)).*

Proof. We take $t, \tau \in [0,1]$ with $t \neq \tau$.

By our assumptions and Lemma 2.1, it follows

$$\begin{aligned}
\frac{|(Tx)(t)-(Tx)(\tau)|}{|t-\tau|^\alpha} &= \frac{1}{|t-\tau|^\alpha}\left|\int_0^1 (K(t,s)-K(\tau,s))g(s,x(s),(Hx)(s))\,ds\right| \\
&\leq \frac{1}{|t-\tau|^\alpha}\int_0^1 |K(t,s)-K(\tau,s)||g(s,x(s),(Hx)(s)|\,ds \\
&\leq \frac{k_1|t-\tau|^\alpha}{|t-\tau|^\alpha}\int_0^1 [|g(s,x(s),(Hx)(s))-g(s,0,0)|+|g(s,0,0)|]\,ds \\
&\leq k_1\int_0^1 (\varphi(\max\{|x(s)|,|(Hx)(s)|\})+M)\,ds \\
&\leq k_1\int_0^1 (\varphi(\max\{\|x\|_\infty,\|Hx\|_\alpha\})+M)\,ds \\
&\leq k_1\int_0^1 (\varphi(\|x\|_\infty)+M)\,ds \\
&\leq k_1(\varphi(\|x\|_\alpha)+M)<\infty,
\end{aligned}$$

where in the last inequality we have used Lemma 2.1.

This proves that $Tx \in H_\alpha[0,1]$.

Now, taking into account the definition of $\|\cdot\|_\alpha$, we have

$$\begin{aligned}
\|Tx\|_\alpha &= |(Tx)(0)| + \sup\left\{\frac{|(Tx)(t)-(Tx)(\tau)|}{|t-\tau|^\alpha} : t,\tau\in[a,b], t\neq\tau\right\} \\
&\leq \left|\int_0^1 K(0,s)g(s,x(s),(Hx)(s))\,ds\right| + k_1(\varphi(\|x\|_\alpha)+M) \\
&\leq \int_0^1 |K(0,s)|[|g(s,x(s),(Hx)(s))-g(s,0,0)| \\
&\quad + |g(s,0,0)|]\,ds + k_1(\varphi(\|x\|_\alpha)+M) \\
&\leq N(\varphi(\|x\|_\alpha)+M) + k_1(\varphi(\|x\|_\alpha)+M) \\
&= (\varphi(\|x\|_\alpha)+M)(N+k_1).
\end{aligned}$$

This completes the proof. □

Proposition 3.5. *Let F be the operator defined on $H_\alpha[0,1]$ as*

$$(Fx)(t) = h(t) + x(t)\int_0^1 K(t,s)g(s,x(s),(Hx)(s))\,ds, \quad t\in[0,1]$$

where $x \in H_\alpha[0,1]$ and $h \in H_\alpha[0,1]$.

Under assumptions of Proposition 3.4, we have

$$\|Fx\|_\alpha \leq \|h\|_\alpha + (\varphi(\|x\|_\alpha)+M)(N+2k_1)\|x\|_\alpha.$$

Proof. In order to estimate $\|Fx\|_\alpha$, we take $t,\tau\in[0,1]$ with $t\neq\tau$. Taking into account our assumptions, we get

$$\frac{|(Fx)(t)-(Fx)(\tau)|}{|t-\tau|^\alpha} \leq \frac{|h(t)-h(\tau)|}{|t-\tau|^\alpha}$$

$$
\begin{aligned}
&+ \frac{1}{|t-\tau|^\alpha}\left|x(t)\int_0^1 K(t,s)g(s,x(s),(Hx)(s))\,ds\right. \\
&\left.-x(\tau)\int_0^1 K(\tau,s)g(s,x(s),(Hx)(s))\,ds\right| \\
&\le H_h^\alpha + \frac{1}{|t-\tau|^\alpha}\left|x(t)\int_0^1 K(t,s)g(s,x(s),(Hx)(s))\,ds\right. \\
&\quad\left.-x(\tau)\int_0^1 K(t,s)g(s,x(s),(Hx)(s))\,ds\right| \\
&\quad+ \frac{1}{|t-\tau|^\alpha}\left|x(\tau)\int_0^1 K(t,s)g(s,x(s),(Hx)(s))\,ds\right. \\
&\quad\left.-x(\tau)\int_0^1 K(\tau,s)g(s,x(s),(Hx)(s))\,ds\right| \\
&\le H_h^\alpha + \frac{|x(t)-x(\tau)|}{|t-\tau|^\alpha}\int_0^1 |K(t,s)|[|g(s,x(s),(Hx)(s)) \\
&\quad - g(s,0,0)| + |g(s,0,0)|]\,ds \\
&\quad+ \frac{|x(\tau)|}{|t-\tau|^\alpha}\int_0^1 |K(t,s)-K(\tau,s)|[|g(s,x(s),(Hx)(s)) \\
&\quad - g(s,0,0)| + |g(s,0,0)|]\,ds \\
&\le H_h^\alpha + H_x^\alpha \int_0^1 [K(t,s)-K(0,s)|+|K(0,s)|](\varphi(\|x\|_\alpha)+M)\,ds \\
&\quad+ \frac{\|x\|_\infty k_1|t-\tau|^\alpha}{|t-\tau|^\alpha}\int_0^1 (\varphi(\|x\|_\alpha)+M)\,ds \\
&\le H_h^\alpha + H_x^\alpha(k_1|t|+N)(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&\le H_h^\alpha + H_x^\alpha(k_1+N)(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M).
\end{aligned}
$$

Now, by the last estimate and, taking into account the definition of $\|\cdot\|_\alpha$, we deduce

$$
\begin{aligned}
\|Fx\|_\alpha &= |(Fx)(0)| + \sup\left\{\frac{|(Fx)(t)-(Fx)(\tau)|}{|t-\tau|^\alpha} : t,\tau\in[0,1], t\neq\tau\right\} \\
&= \left|h(0)+x(0)\int_0^1 K(0,s)g(s,x(s),(Hx)(s))\,ds\right| \\
&\quad + H_h^\alpha + H_x^\alpha(k_1+N)(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&\le |h(0)| + H_h^\alpha + |x(0)|N(\varphi(\|x\|_\alpha)+M) \\
&\quad + H_x^\alpha(k_1+N)(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&\le \|h\|_\alpha + (|x(0)|+H_x^\alpha)\,N(\varphi(\|x\|_\alpha)+M) \\
&\quad + H_x^\alpha k_1(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&\le \|h\|_\alpha + \|x\|_\alpha N(\varphi(\|x\|_\alpha)+M)+\|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&\qquad + \|x\|_\alpha k_1(\varphi(\|x\|_\alpha)+M) \\
&= \|h\|_\alpha + (\varphi(\|x\|_\alpha)+M)\|x\|_\alpha(N+2k_1).
\end{aligned}
$$

This proves the desired result. □

Remark 3.6. In virtue of Proposition 3.5, if there exists $r_0 > 0$ such that

$$\|h\|_\alpha + (\varphi(r_0) + M)(N + 2k_1)r_0 \leq r_0$$

then the operator F applies $B_\alpha^{r_0}$ into itself, where $B_\alpha^{r_0}$ is the ball centered at zero and radius r_0 in the space $(H_\alpha[0,1], \|\cdot\|_\alpha)$.

Theorem 3.7. *If we add to assumptions of Proposition 3.5 the following ones:*

(i) *There exists $r_0 > 0$ such that*

$$\|h\|_\alpha + (\varphi(r_0) + M)(N + 2k_1)r_0 \leq r_0.$$

(ii) *The function φ appearing in assumption (b) of Proposition 3.4 is continuous at $t_0 = 0$.*

(iii) *The operator H appearing in assumption (c) of Proposition 3.4 satisfies that*

$$\|Hx - Hy\|_\infty \leq \|x - y\|_\infty.$$

then Eq. (1.2) *has at least one solution x^* located in $H_\alpha[0,1]$ with $\|x^*\|_\alpha \leq r_0$.*

Proof. Let F be the operator defined on $H_\alpha[0,1]$ by

$$(Fx)(t) = h(t) + x(t)\int_0^1 K(t,s)g(s, x(s), (Hx)(s))\,ds, \quad t \in [0,1].$$

By Proposition 3.5, F applies $H_\alpha[0,1]$ into itself.

Moreover, by assumption (i) and Remark 3.2, F applies the ball $B_\alpha^{r_0}$ into itself.

Since $B_\alpha^{r_0}$ is a bounded subset of $H_\alpha[0,1]$, by Theorem 2.3, $B_\alpha^{r_0}$ is a relatively compact subset of $(H_\beta[0,1], \|\cdot\|_\beta)$ for $0 < \beta < \alpha \leq 1$.

Using a similar argument to the one appearing in appendix of [3] it can be proved that $B_\alpha^{r_0}$ is closed in $(H_\beta[0,1], \|\cdot\|_\beta)$ and consequently, $B_\alpha^{r_0}$ is compact in $(H_\beta[0,1], \|\cdot\|_\beta)$.

In order to apply the Schauder fixed point theorem, we only need to prove that $F : B_\alpha^{r_0} \longrightarrow B_\alpha^{r_0}$ is continuous for the norm $\|\cdot\|_\beta$.

Since $h \in H_\alpha[0,1] \subset H_\beta[0,1]$ (Lemma 2.2) and the addition and product of functions in $H_\beta[0,1]$ is continuous for the norm $\|\cdot\|_\beta$ (Lemma 3.3), F will be continuous for the norm $\|\cdot\|_\beta$ if the operator defined on $B_\alpha^{r_0}$ by

$$(Gx)(t) = \int_0^1 K(t,s)g(s, x(s), (Hx)(s))\,ds, \quad t \in [0,1]$$

also is.

To prove this, we take a sequence $(x_n) \subset B_\alpha^{r_0}$ such that $x_n \overset{\|\cdot\|_\beta}{\longrightarrow} x$ with $x \in B_\alpha^{r_0}$ and we have that to prove that $Gx_n \overset{\|\cdot\|_\beta}{\longrightarrow} Gx$.

In fact, we take $t, \tau \in [0,1]$ with $t \neq \tau$, and, taking into account our assumptions, we have

$$\begin{aligned}
&\frac{1}{|t-\tau|^{\beta}} |[(Gx_n)(t)-(Gx)(t)]-[(Gx_n)(\tau)-(Gx)(\tau)]| \\
&= \frac{1}{|t-\tau|^{\beta}} \left| \int_0^1 K(t,s)[g(s,x_n(s),(Hx_n)(s))-g(s,x(s),(Hx)(s))]\,ds \right. \\
&\quad \left. - \int_0^1 K(\tau,s)[g(s,x_n(s),(Hx_n)(s))-g(s,x(s),(Hx)(s))]\,ds \right| \\
&= \frac{1}{|t-\tau|^{\beta}} \left| \int_0^1 [K(t,s)-K(\tau,s)][g(s,x_n(s),(Hx_n)(s))-g(s,x(s),(Hx)(s))]\,ds \right| \\
&\le \frac{1}{|t-\tau|^{\beta}} \int_0^1 |K(t,s)-K(\tau,s)||g(s,x_n(s),(Hx_n)(s))-g(s,x(s),(Hx)(s))|\,ds \\
&\le \frac{k_1|t-\tau|^{\alpha}}{|t-\tau|^{\beta}} \int_0^1 \varphi(\max\{|x_n(s)-x(s)|,|(Hx_n)(s)-(Hx)(s)|\})\,ds \\
&\le k_1|t-\tau|^{\alpha-\beta} \int_0^1 \varphi(\max\{\|x_n-x\|_\infty,\|Hx_n-Hx\|_\infty\})\,ds \\
&\le k_1 \int_0^1 \varphi(\|x_n-x\|_\infty)\,ds \\
&\le k_1\varphi(\|x_n-x\|_\beta),
\end{aligned}$$

where we have used (iii) and Lemma 2.1.

Now, by definition of $\|\cdot\|_\beta$, we have

$$\begin{aligned}
&\|Gx_n-Gx\|_\beta \\
&= |(Gx_n)(0)-(Gx)(0)| \\
&\quad + \sup\left\{ \frac{|[(Gx_n)(t)-(Gx)(t)]-[(Gx_n)(\tau)-(Gx)(\tau)]|}{|t-\tau|^{\beta}} : t,\tau \in [0,1], t\neq\tau \right\} \\
&\le \left| \int_0^1 K(0,s)[g(s,x_n(s),(Hx_n)(s))-g(s,x(s),(Hx)(s))]\,ds \right| + k_1\varphi(\|x_n-x\|_\beta) \\
&\le \int_0^1 |K(0,s)|\varphi(\|x_n-x\|_\infty)\,ds + k_1\varphi(\|x_n-x\|_\beta) \\
&\le N\varphi(\|x_n-x\|_\beta) + k_1\varphi(\|x_n-x\|_\beta) \\
&= (N+k_1)\varphi(\|x_n-x\|_\beta).
\end{aligned}$$

Since $x_n \overset{\|\cdot\|_\beta}{\longrightarrow} x$ and φ is continuous at $t_0 = 0$ (assumption (ii)) from the last inequality, it follows that $Gx_n \overset{\|\cdot\|_\beta}{\longrightarrow} Gx$.

Therefore, $F : B_\alpha^{r_0} \longrightarrow B_\alpha^{r_0}$, $B_\alpha^{r_0}$ is compact in $(H_\beta[0,1], \|\cdot\|_\beta)$ and F is continuous for the norm $\|\cdot\|_\beta$. Applying the Schauder fixed point theorem, we get the desired result, i.e., there exists $x^* \in B_\alpha^{r_0}$ such that

$$x^*(t) = h(t) + x^*(t)\int_0^1 K(t,s)g(s,x^*(s),(Hx^*)(s))\,ds, \quad t\in[0,1]$$

and $\|x^*\|_\alpha \le r_0$. □

Remark 3.8. When the function φ appearing in assumption (b) of Proposition 3.4 has the expression $\varphi(t) = \lambda t$ for $t \in [0,1]$ with $\lambda > 0$, the inequality in (i) of Theorem 3.7 has the expression

$$\|h\|_\alpha + \lambda(N + 2k_1)r_0^2 + M(N + 2k_1) \le r_0,$$

or, equivalently,

$$\lambda(N + 2k_1)r_0^2 - r_0 + (\|h\|_\alpha + M(N + 2k_1)) \le 0.$$

Under assumption

$$4\lambda(N + 2k_1)(\|h\|_\alpha + M(N + 2k_1)) \le 1$$

the above inequality is satisfied by

$$r_0 = \frac{1 - \sqrt{1 - 4\lambda(N + 2k_1)(\|h\|_\alpha + M(N + 2k_1))}}{2\lambda(N + 2k_1)} > 0.$$

4. Examples

In what follows, we present some examples of operators $H : C[0,1] \longrightarrow C[0,1]$ satisfying assumptions (i) of Proposition 3.4 and (iii) of Theorem 3.7.

Example 4.1. Suppose that $\varphi : [0,1] \longrightarrow [0,1]$ is a continuous function and consider the composition operator C_φ defined on $C[0,1]$ by $(C_\varphi x)(t) = x(\varphi(t))$.

It is easily seen that C_φ satisfies the required conditions.

Example 4.2. Suppose that $\varphi : [0,1] \longrightarrow [0,1]$ is a continuous function then the multiplication operator M_φ defined on $C[0,1]$ by $(M_\varphi x)(t) = \varphi(t)x(t)$ satisfies our conditions.

Example 4.3. Let I be integral operator defined on $C[0,1]$ by

$$(Ix)(t) = \int_0^t x(s)\,ds.$$

This operator satisfies our assumptions.

The operators in Examples 4.1, 4.2 and 4.3 are linear.

An example of nonlinear operator satisfying our hypotheses is the following.

Example 4.4. Consider the operator Q defined on $C[0,1]$ by

$$(Qx)(t) = \max\{|x(\tau)| : 0 \le \tau \le t\}.$$

This operator is nonlinear and satisfies our assumptions [4].

Remark 4.5. It can be easily proved that the composition of operators defined on $C[0,1]$ and satisfying assumptions (i) of Proposition 3.4 and (iii) of Theorem 3.7 also satisfies them.

Next, we present a numerical example illustrating our results.

Example 4.6. Consider the following integral equation

$$x(t) = \frac{1}{5}\arctan t$$
$$+\frac{x(t)}{10}\int_0^1 e^{-s}\ln(1+t)\sqrt{s(|x(s)|+\max\{|x(\tau)|:0\le\tau\le s\})}\,ds,\ t\in[0,1]. \tag{4.1}$$

Notice that Eq. (4.1) is a particular case of Eq. (1.2), where

$$h(t)=\frac{1}{5}\arctan t,\quad K(t,s)=e^{-s}\ln(1+t),$$

and

$$g(s,x,y)=\frac{1}{10}\sqrt{s(|x|+|y|)},\quad (Hx)(t)=\max\{|x(\tau)|:0\le\tau\le t\}.$$

Since, for $t,\tau\in[0,1]$

$$|\arctan t-\arctan\tau|\le\arctan|t-\tau|\le|t-\tau|$$

we have that $\|h\|_1=1/5$ and $\alpha=1$.

Moreover, for $t,\tau\in[0,1]$, we have

$$|K(t,s)-K(\tau,s)|=e^{-s}|\ln(1+t)-\ln(1+\tau)|\le e^{-s}|t-\tau|\le|t-\tau|$$

and, therefore, $k_1=1$.

Moreover,

$$N=\sup\{|K(0,s)|:s\in[0,1]\}=0$$

and

$$M=\sup\{|g(t,0,0)|:t\in[0,1]\}=0.$$

On the other hand, for $t\in[0,1]$ and $x,x_1,y,y_1\in\mathbb{R}$, we have

$$\begin{aligned}|g(t,x,y)-g(t,x_1,y_1)| &= \frac{1}{10}\left|\sqrt{t(|x|+|y|)}-\sqrt{t(|x_1|+|y_1|)}\right|\\ &\le \frac{1}{10}\sqrt{|t[(|x|-|x_1|)+(|y|-|y_1|)]|}\\ &\le \frac{1}{10}\sqrt{||x|-|x_1||+||y|-|y_1||}\\ &\le \frac{1}{10}\left(\sqrt{||x|-|x_1||}+\sqrt{||y|-|y_1||}\right)\\ &\le \frac{1}{10}\left(\sqrt{|x-x_1|}+\sqrt{|y-y_1|}\right)\\ &\le \frac{2}{10}\sqrt{\max\{|x-x_1|,|y-y_1|\}}\\ &= \frac{1}{5}\sqrt{\max\{|x-x_1|,|y-y_1|\}}.\end{aligned}$$

Therefore, assumption (b) of Proposition 3.4 is satisfied with $Q(t)=\frac{1}{5}\sqrt{t}$.

In this case, the inequality appearing in assumption (i) of Theorem 3.7 has the expression

$$\frac{1}{5}+2\left(\frac{1}{5}\sqrt{r_0}\right)r_0\le r_0.$$

It is easily checked that $r_0=1$ satisfies the last inequality. Applying Theorem 3.7, Eq. (4.1) has at least one solution $x^*\in H_1[0,1]$ with $\|x^*\|_1\le 1$.

Remark 4.7. Notice that if Eq. (4.1) we replace the operator $(Hx)(t) = \max\{|x(\tau)| : 0 \leq \tau \leq t\}$ by some of the operator presented in Examples 4.1, 4.2 and 4.3, then the same argument says us that the integral equation obtained has at least one solution in $H_1[0,1]$. This is interesting from a practical point of view.

5. Chandrasekhar's equation

Chandrasekhar's integral equation is given by

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s}\, \varphi(s) x(s)\, ds, \quad t \in [0,1] \tag{5.1}$$

and it appears in the theory of radiative transfer [6].

Notice that Eq. (5.1) is a particular case of Eq, (1.2) with

$$h(t) = 1, \quad K(t,s) = \frac{t}{t+s}\, \varphi(s), \quad g(s,x,y) = x.$$

Next, we will prove that under certain assumptions, Eq. (5.1) satisfies the conditions of Theorem 3.7.

Proposition 5.1. *Suppose that $\varphi : [0,1] \longrightarrow \mathbb{R}$ is a continuous function satifying $\varphi(0) = 0$. Let $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function defined by*

$$K(t,s) = \begin{cases} 0, & s = 0 \text{ and } t \geq 0, \\ \dfrac{t}{t+s}\, \varphi(s), & s \neq 0 \text{ and } t \geq 0. \end{cases}$$

Then K is a continuous function on $[0,1] \times [0,1]$.

Proof. It is clear that it is sufficient to prove the continuity of K at $(0,0)$.

In fact, for $\varepsilon > 0$ given, by the continuity of φ, we can find $\delta > 0$ such that $|\varphi(s)| < \varepsilon$ whenever $0 < s < \delta$.

Then , for $t, s \in [0,1]$ such that $0 < \sqrt{t^2 + s^2} < \delta$, we have

$$|K(t,s)| = \left| \frac{t}{t+s}\, \varphi(s) \right| \leq |\varphi(s)| < \varepsilon.$$

This proves the continuity of K at $(0,0)$.

Therefore, this completes the proof. □

Proposition 5.2. *If to assumptions Proposition 5.1 we add the following one: $\varphi(s) = O(s)$, i.e., $|\varphi(s)| \leq Ls$ for any $s \in [0,1]$, where L is a nonnegative constant, then the function K mentioned in Proposition 5.1 satisfies assumption (a) of Proposition 3.4, with $\alpha = 1$.*

Proof. We have that to prove that $K(\cdot, s) : [0,1] \longrightarrow \mathbb{R}$ belongs to $H_1[0,1]$ with the same constant $H^1_{K(\cdot,s)}$ for any $s \in [0,1]$.

In order to prove this, we fix $s \in [0,1]$.

We distinguish two cases.

Case 1. $s \neq 0$.
We take $t, \tau \in [0,1]$ and we deduce that

$$\begin{aligned}|K(t,s) - K(\tau,s)| &= \left|\frac{t}{t+s}\,\varphi(s) - \frac{\tau}{\tau+s}\,\varphi(s)\right| = |\varphi(s)|\left|\frac{t}{t+s} - \frac{\tau}{\tau+s}\right| \\ &= |\varphi(s)|\frac{|s(t-\tau)|}{(t+s)(\tau+s)} \leq \frac{|\varphi(s)|s}{s^2}|t-\tau| \\ &= \frac{|\varphi(s)|}{s}|t-\tau| \leq L|t-\tau|.\end{aligned}$$

Case 2. $s = 0$.
In this case, we have, for $t, \tau \in [0,1]$,

$$|K(t,s) - K(\tau,s)| = 0 \leq L|t-\tau|.$$

This finishes the proof. □

Next, we present the following result about the existence of solutions for Eq. (5.1) in the Lipschitz space.

Theorem 5.3. *Under the following assumptions:*

(1) $\varphi : [0,1] \longrightarrow \mathbb{R}$ *is a continuous function with* $\varphi(0) = 0$*,*
(2) $\varphi(s) = O(s)$*, for* $s \in [0,1]$*,*
(3) $\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\} < 1/8$*,*

then Eq. (5.1) *has at least one solution* $x^* \in H_1[0,1]$ *with*

$$\|x^*\|_1 \leq \frac{1 - \sqrt{1 - 8\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}}{4\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}.$$

Proof. As we mention above, Eq. (5.1) is a particular case of Eq. (1.2) with $h(t) = 1$ and, consequently, $\|h\|_1 = 1$, and

$$K(t,s) = \frac{t}{t+s}\,\varphi(s).$$

In virtue of our assumptions and by Proposition 5.1 and 5.2, $K(t,s)$ is continuous on $[0,1] \times [0,1]$ and satisfies

$$|K(t,s) - K(\tau,s)| \leq \sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}|t-\tau|,$$

for any $t, \tau, s \in [0,1]$. Therefore,

$$k_1 = \sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}.$$

Moreover,

$$N = \sup\{|K(0,s)| : s \in [0,1]\} = 0.$$

In this case, $g(t,x,y) = x$ and this function satisfies assumption (b) of Proposition 3.4 with $\varphi(t) = t$ for $t \in [0,1]$ and

$$M = \sup\{|g(t,0,0)| : t \in [0,1]\} = 0.$$

On the other hand, the inequality appearing in assumption (i) of Theorem 3.7 has the form

$$1 + 2\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\} r_0^2 \leq r_0,$$

and this inequality has a positive solution in

$$r_0 = \frac{1 - \sqrt{1 - 8\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}}{4\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}},$$

by assumption (3).

This proves that the conditions of Theorem 3.7 are satisfied and, by this theorem, Eq. (5.1) has at least one solution $x^* \in H_1[0,1]$ with

$$\|x^*\|_1 \leq r_0 = \frac{1 - \sqrt{1 - 8\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}}{4\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}.$$

□

In [Section 38, Corollary 1 of [6]], it is proved that a necessary condition for that a solution of Eq. (5.1) is real

$$\int_0^1 \varphi(s)\, ds \leq \frac{1}{2}.$$

We have proved in Theorem 5.3 that, under certain assumptions there exist a real solutions of Eq. (5.1) and, moreover, these solutions are located in $H_1[0,1]$.

On the other hand, by assumption (3) of Theorem 5.3, we have

$$\int_0^1 \varphi(s)\, ds \leq \int_0^1 |\varphi(s)|\, ds \leq \int_0^1 s\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\} ds$$
$$\leq \frac{1}{2}\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\} < \frac{1}{16},$$

and our result is consistent with the one obtained in [6], as we comment above. Notice that, under our assumptions, the solution x^* of Eq. (5.1) given by Theorem 5.3, satisfies

$$\|x^*\|_1 \leq \frac{1 - \sqrt{1 - 8\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}}{4\sup\left\{\frac{|\varphi(s)|}{s} : s \in [0,1]\right\}}$$

and to the best of our knowledgment this result does not appear in the literature.

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