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COINCIDENCE RESULTS FOR COMPOSITIONS OF MULTIVALUED MAPS BASED ON COUNTABLE COMPACTNESS PRINCIPLES

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Abstract. We present a general Mönch coincidence type result for set–valued maps on Hausdorff topological spaces.

1. Introduction

Coincidence results (based on compositions of set–valued maps) for compact maps is discussed extensively in the literature [1, 5, 6, 7] and a typical example is the following result due to Granas and Liu [6]: Let X, Y be convex subsets of topological vector spaces and A, $B : X \to Y$ be set valued maps satisfying:

(i). A is upper semicontinuous and has nonempty compact acyclic values,
(ii). B has nonempty convex values and open fibres.

If A is compact then there exists $x_0 \in X$ with $A(x_0) \cap B(x_0) \neq \emptyset$. A special case of this result is when A compact is replaced by X compact.

There are also results in the literature in the noncompact case [3, 10]. A typical example is the following result due to Furi, Martelli and Vignoli [3]: Let E and F be Banach spaces, $f : E \to F$ (single valued) a continuous strong surjection (i.e. $f(x) = \phi(x)$ has a solution for any continuous map $\phi : E \to F$ with $\overline{\phi(E)}$ compact), $X \subseteq E$ closed, $h : X \to F$ (single valued) is continuous and assume

(i). $f^{-1}\left(\overline{co}\, h(X)\right) \subseteq X$,
(ii). $h(X)$ is bounded,
(iii). $\alpha(f(A)) = \alpha(h(A))$ implies $\overline{A}$ is compact;

here α denotes the Kuratowski measure of noncompactness. Then the equation $f(x) = h(x)$ has a solution $x \in X$.

In this paper we obtain coincidence results of Mönch type for set–valued maps defined on Hausdorff topological spaces satisfying compactness conditions on countable sets. Our theory is based on coincidence results for maps defined on compact spaces (in fact these coincidence results are based on fixed point theory for compositions of set–valued maps; see Remark 2.5 (c)). As an application we will consider fixed points of maps with weakly sequentially closed graphs.

We now present two results which will be needed in Section 2.

(a). Let E be a Hausdorff topological vector space and M a separable subset of E. Then $span\, M$ is separable.

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To see this for convenience we let $span_K\, M$ (i.e here K is the standard countable set dense in the scalar field) denote the space spanned with rational coefficients. Let $A\,(\subseteq M)$ be countable and dense. It is easy to see that $span_K\, A$ is dense in $span\, A$, so

$$span_K\, A \subseteq span\, A \subseteq \overline{span_K\, A}.$$

Note $span_K\, A$ is countable and is a dense subset of $span\, M$; to see this note $\overline{span_K\, A} = \overline{span\, A}$ so $\overline{span_K\, A}$ is a closed linear subspace of E containing M (since $M \subseteq \overline{A} \subseteq \overline{span_K\, A}$) and thus $span\, M \subseteq \overline{span_K\, A}$.

(b). Let X be a Hausdorff locally convex topological vector space. Now X is separable if and only if X is weakly separable.

Suppose X is separable and let A be countable and dense in X so $\overline{A} = X$. Then immediately we have $X = \overline{A} \subseteq \overline{A^w}$, so X is weakly separable.

Conversely suppose X is weakly separable and let B be countable and weakly dense in X. Let $W = \overline{span\, B}$. Then $W = X$ since $X = \overline{B^w} \subseteq \overline{span\, B^w} = \overline{span\, B}$. Now the linear combinations of elements of B with rational coefficients are dense in X so X is separable.

2. Coincidence theory

By a space we mean a Hausdorff topological space. Let X and Y be spaces. For a multivalued map $G : X \to 2^Y$ we will consider the lower inverse G^l defined by $G^l(A) = \{x \in X : G(x) \cap A \neq \emptyset\}$ (here $A \subseteq Y$) and we will denote it by G^{-1}.

In this paper we could consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Let X and E be spaces.

Definition 2.1. We say $F \in M(X,E)$ if $F : X \to 2^E$ and $F \in \mathbf{A}(X,E)$; here 2^E denotes the family of nonempty subsets of E.

Definition 2.2. We say $G \in MB(X,E)$ (respectively $MC(X,E)$) if $G : X \to 2^E$ and $G \in \mathbf{B}(X,E)$ (respectively $G \in \mathbf{C}(X,E)$).

Remark 2.3. (i). Examples of the classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ can be found for example in [1, 6, 10, 11].
(ii). In fact it is enough to consider one class of maps here since we will consider the composition (of course the individual maps might belong to different classes).

For example if we consider the strongly admissible maps of Gorniewicz [4](pp. 199) (a special case is (i). the Kakutani maps (ii). the acyclic maps) from X to E then $\mathbf{A}(X,E)$ is the class of strongly admissible maps from X to E, $\mathbf{B}(\Gamma, X)$ (here Γ is the space as defined in the definition of strongly admissible (see [4] (pp. 199)) is the class of Vietoris maps from Γ to X and $\mathbf{C}(\Gamma, E)$ is the class of continuous maps from Γ to E. In this example $E = X$, $Y = \Gamma$, Φ is from the class $\mathbf{B}(Y,X)$ and F is from the class $\mathbf{C}(Y,X)$.

As a result we will only consider the class M in this paper.

We will present two Mönch type coincidence results (see [8, 9, 10, 11]).

Theorem 2.4. *Let X be a Hausdorff topological vector space and Y a space. Suppose $\Phi : Y \to 2^X$, $F : Y \to 2^X$ and $x_0 \in \Phi(Y)$. Assume the following conditions hold:*

$$\Phi^{-1}\left(co\left(\{x_0\} \cup F(Y)\right)\right) \subseteq Y \tag{2.1}$$

$$\Phi^{-1} F \textit{ is a closed map (i.e. has closed graph)} \tag{2.2}$$

$$\begin{cases} A \subseteq Y,\ A = \Phi^{-1}\left(co\left(\{x_0\} \cup F(A)\right)\right) \quad \textit{with} \ \ \overline{A} = \overline{C} \\ \textit{and} \ \ C \subseteq A \ \ \textit{countable, implies} \ \ \overline{A} \ \ \textit{is compact} \end{cases} \tag{2.3}$$

$$\begin{cases} F \ \ \textit{maps separable sets in} \ \ Y \ \ \textit{to separable sets in} \ \ X, \ \ \textit{and} \\ \Phi^{-1} \ \ \textit{maps separable sets in} \ \ X \ \ \textit{to separable sets in} \ \ Y \end{cases} \tag{2.4}$$

and

(2.5)

$$\begin{cases} \textit{for any nonempty subset} \ \ W \ \ \textit{of} \ \ Y \ \ \textit{with} \ W = \Phi^{-1}\left(co\left(\{x_0\} \cup F(W)\right)\right), \\ \overline{W} = \overline{C} \ \textit{and} \ \ C \subseteq W \ \ \textit{countable (so} \ \ \overline{W} \ \ \textit{is compact), we have that} \\ \textit{the map} \ \ G \ \ \textit{given by} \ G(x) = \Phi^{-1} F(x) \cap \overline{W},\ x \in \overline{W} \ \textit{is in} \\ M(\overline{W}, \overline{W}) \ \ \textit{and there exists} \ \ x \in \overline{W} \ \ \textit{with} \ \ F(x) \cap \Phi(x) \neq \emptyset. \end{cases}$$

Then there exists $x \in Y$ *with* $F(x) \cap \Phi(x) \neq \emptyset$.

Remark 2.5. (a). Notice G is well defined i.e. $G(x) \neq \emptyset$ for $x \in \overline{W}$. If $x \in \overline{W}$ then $x_\alpha \to x$ for some net $\{x_\alpha\}$ in W. Take any $y_\alpha \in \Phi^{-1} F(x_\alpha)$. Since $W = \Phi^{-1}\left(co\left(\{x_0\} \cup F(W)\right)\right)$ we have $\Phi^{-1} F(W) \subseteq W$ so $y_\alpha \in W \subseteq \overline{W}$. The compactness of $\overline{W}$ guarantees that we may assume without loss of generality that $y_\alpha \to y$ for some $y \in \overline{W}$. Since $(x_\alpha, y_\alpha) \in graph\, \Phi^{-1} F$ and $graph\, \Phi^{-1} F$ is closed, we have $(x, y) \in graph\, \Phi^{-1} F$. Thus $y \in \Phi^{-1} F(x) \cap \overline{W}$ i.e. $y \in G(x)$ so $G(x) \neq \emptyset$.

(b). As an example of (2.4) note an upper semicontinuous map in metric spaces with separable values maps separable sets to separable sets (see [12] (pp. 345)).

(c). Note in (2.5) if we show $G \in M(\overline{W}, \overline{W})$ has a fixed point in $\overline{W}$ then automatically there exists a $x \in \overline{W}$ with $F(x) \cap \Phi(x) \neq \emptyset$.

(d). In some applications we are interested in maps $\Theta : Y \to 2^X$ and $\Psi : Y \to 2^X$ where F (maybe Θ itself) is a selection of Θ and Φ (maybe Ψ itself) is a selection of Ψ; note F is a selection of Θ if $F(x) \subseteq \Theta(x)$ for $x \in Y$. Now assuming the conditions in Theorem 2.4 we know there exists a $x \in Y$ with $F(x) \cap \Phi(x) \neq \emptyset$, so as a result $\Theta(x) \cap \Psi(x) \neq \emptyset$.

Proof. Let

$$D_0 = \Phi^{-1}(\{x_0\}), \quad D_n = \Phi^{-1}\left(co\left(\{x_0\} \cup F(D_{n-1})\right)\right) \quad \text{for} \ \ n \in N = \{1, 2, \dots\}$$

and $D = \cup_{n=0}^{\infty} D_n$. Note $D_0 \subseteq D_1$ so $F(D_0) \subseteq F(D_1)$ and as a result

$$D_1 = \Phi^{-1}\left(co\left(\{x_0\} \cup F(D_0)\right)\right) \subseteq \Phi^{-1}\left(co\left(\{x_0\} \cup F(D_1)\right)\right) = D_2.$$

By induction we have

$$D_0 \subseteq D_1 \subseteq \dots, \subseteq D_{n-1} \subseteq D_n \cdots \subseteq Y.$$

We now claim

$$D = \Phi^{-1}\left(co\left(\{x_0\} \cup F(D)\right)\right). \tag{2.6}$$

First note $D_0 \subseteq \Phi^{-1}\left(co\left(\{x_0\} \cup F(D)\right)\right)$. Also for <u>each</u> $n \in \{1, 2, \dots\}$ note $D_{n-1} \subseteq D$, $F(D_{n-1}) \subseteq F(D)$ so

$$D_n = \Phi^{-1}\left(co\left(\{x_0\} \cup F(D_{n-1})\right)\right) \subseteq \Phi^{-1}\left(co\left(\{x_0\} \cup F(D)\right)\right),$$

and as a result

$$D = \cup_{n=0}^{\infty} D_n \subseteq \Phi^{-1}\left(co\left(\{x_0\} \cup F(D)\right)\right).$$

To show the other side of (2.6) first note $F(D) = F(\cup_{n=0}^{\infty} D_n) = \cup_{n=0}^{\infty} F(D_n)$ and

$$co\left(\{x_0\} \cup F(D)\right) = co\ \left(\{x_0\} \cup\ (\cup_{n=0}^{\infty} F(D_n))\right) \subseteq \cup_{n=0}^{\infty} co\left(\{x_0\} \cup F(D_n)\right)$$

since $F(D_0) \subseteq F(D_1) \subseteq \cdots \subseteq F(D_{n-1}) \subseteq F(D_n) \subseteq \cdots$ (note if $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n \subseteq \cdots$ then it is easy to see that $co\ (\cup_{n=0}^{\infty} A_n) \subseteq \cup_{n=0}^{\infty} co(A_n)$). Now

$$\begin{aligned} \Phi^{-1}\left(co\left(\{x_0\} \cup F(D)\right)\right) &\subseteq \Phi^{-1}\left(\cup_{n=0}^{\infty} co\left(\{x_0\} \cup F(D_n)\right)\right) \\ &= \cup_{n=0}^{\infty} \left(\Phi^{-1}\left(co\left(\{x_0\} \cup F(D_n)\right)\right)\right) = \cup_{n=0}^{\infty} D_{n+1} = D \end{aligned}$$

(recall $\Phi^{-1}\left(\cup_{i\in I} E_i\right) = \Phi^{l}\left(\cup_{i\in I} E_i\right) = \cup_{i\in I} \Phi^{l}\left(E_i\right)$). Thus (2.6) holds.

Next we claim

$$D_n \ \text{ is separable (for each } \ n \in \{0,1,2,\dots\}). \tag{2.7}$$

Clearly D_0 is separable. Now assume D_k is separable for some $k \in \{0,1,\dots\}$. Then from (2.4) we have that $F(D_k)$ is separable so as a result $co\left(\{x_0\} \cup F(D_k)\right)$ is separable. Now from (2.4) we have that $\Phi^{-1}\left(co\left(\{x_0\} \cup F(D_k)\right)\right)$ $(= D_{k+1})$ is separable. By induction (2.7) is true. Thus for each $n \in \{0,1,\dots\}$, (2.7) guarantees that there exists $C_n \subseteq D_n$ with C_n countable and $\overline{C_n} = \overline{D_n}$. Let $C = \cup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n} \subseteq \overline{\bigcup_{n=0}^{\infty} D_n}$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} D_n} = \overline{D} \quad \text{and} \quad \overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C}.$$

Thus $\overline{C} = \overline{D}$ so from (2.3) (see (2.6)) we have that $\overline{D}$ is compact.

Consider the map G given by $G(x) = \Phi^{-1} F(x) \cap \overline{D}$, $x \in \overline{D}$ (note Remark 2.5 guarantees that G is well defined). Now apply (2.5). □

Theorem 2.6. *Let X be a Hausdorff topological vector space and Y a space. Suppose $\Phi : Y \to 2^X$, $F : Y \to 2^X$, $x_0 \in \Phi(Y)$ and assume* (2.1) *and* (2.4) *hold. Also suppose the following conditions hold:*

$$F\,\Phi^{-1} \ \textit{is a closed map (i.e. has closed graph)} \tag{2.8}$$

$$\left\{ \begin{array}{l} A \subseteq Y,\ A = \Phi^{-1}\left(co\left(\{x_0\} \cup F(A)\right)\right) \ \ \textit{with} \ \ \overline{A} = \overline{C} \\ \textit{and} \ \ C \subseteq A \ \ \textit{countable, implies} \ \ \overline{co}\left(F(A)\right) \ \ \textit{is compact} \end{array} \right. \tag{2.9}$$

and

(2.10)

$$\left\{ \begin{array}{l} \textit{for any nonempty subset } W \textit{ of } Y \textit{ with } W = \Phi^{-1}\left(co\left(\{x_0\} \cup F(W)\right)\right), \\ \overline{W} = \overline{C} \textit{ and } C \subseteq W \textit{ countable (so } \overline{co}\left(F(W)\right) \textit{ is compact), we have that} \\ \textit{the map } G \textit{ given by } G(x) = F\,\Phi^{-1}(x) \cap \overline{co}\left(F(W)\right),\ x \in \overline{co}\left(F(W)\right) \textit{ is in} \\ M(\overline{co}\left(F(W)\right), \overline{co}\left(F(W)\right)) \textit{ and there exists } x \in \Phi^{-1}\left(\overline{co}\left(F(W)\right)\right) \\ \textit{with } F(x) \cap \Phi(x) \neq \emptyset. \end{array} \right.$$

Then there exists $x \in Y$ with $F(x) \cap \Phi(x) \neq \emptyset$.

Remark 2.7. Note (2.8) is only needed to guarantee that G in (2.10) is well defined i.e. $G(x) \neq \emptyset$ for $x \in \overline{co}\,(F(A))$. If $x \in \overline{co}\,(F(A))$ then $x_\alpha \to x$ for some net $\{x_\alpha\}$ in $co\,(F(A))$. Take any $y_\alpha \in F\,\Phi^{-1}\,(x_\alpha)$. Note $A = \Phi^{-1}\,(co\,(\{x_0\} \cup F(A)))$ so $F\,\Phi^{-1}\,(co\,F(A)) \subseteq F(A)$ and so $y_\alpha \in \overline{co}\,(F(A))$. The compactness of $\overline{co}\,(F(A))$ guarantees that we may assume without loss of generality that $y_\alpha \to y$ for some $y \in \overline{co}\,(F(A))$. Since $(x_\alpha, y_\alpha) \in graph\,F\,\Phi^{-1}$ and $graph\,F\,\Phi^{-1}$ is closed, we have $(x,y) \in graph\,F\Phi^{-1}$. Thus $y \in F\,\Phi^{-1}\,(x) \cap \overline{co}\,(F(A))$ i.e. $y \in G(x)$ so $G(x) \neq \emptyset$.

Proof. Let D_n, D, C_n and C (countable) be as in Theorem 2.4. Then as in Theorem 2.4 we have $D = \Phi^{-1}\,(co\,(\{x_0\} \cup F(D)))$ and $\overline{C} = \overline{D}$. Now (2.9) guarantees that $\overline{co}\,(F(D))$ is compact. Now apply (2.10). □

As an application we will consider fixed points of maps with weakly sequentially closed graphs. First we recall the following well known result in the literature [9].

Theorem 2.8. *Let Q be a nonempty, convex, weakly compact subset of a metrizable locally convex linear topological space E. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph; here $K(Q)$ denotes the family of nonempty, convex, weakly compact subsets of Q. Then F has a fixed point in Q.*

Theorem 2.9. *Let Q be a nonempty, closed, convex subset of a metrizable locally convex linear topological space E and let $x_0 \in Q$. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph and assume the following conditions hold:*

$$(2.11)\qquad \begin{cases} A \subseteq Q,\ A = co\,(\{x_0\} \cup F(A)) \ \textit{with}\ \overline{C^w} = \overline{A^w}\,(= \overline{A}) \\ \textit{and}\ C \subseteq A\ \textit{countable, implies}\ \overline{A^w}\,(= \overline{A})\ \textit{is weakly compact} \end{cases}$$

and

$$(2.12)\qquad F\ \textit{maps separable sets in}\ Q\ \textit{to separable sets in}\ Q.$$

Then F has a fixed point in Q.

Proof. Let

$$D_0 = \{x_0\},\quad D_n = co\,(\{x_0\} \cup F(D_{n-1}))\quad \text{for}\ n = 1, 2, \dots\ \text{and}\ D = \cup_{n=0}^{\infty} D_n.$$

Notice $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_{n-1} \subseteq D_n \subseteq \cdots \subseteq Q$. Also note D is convex and since (D_n) is increasing we have (a slight adjustment of the argument in Theorem 2.4)

$$(2.13)\qquad D = \bigcup_{n=1}^{\infty} co\ (\{x_0\} \cup F(D_{n-1})) = co\,(\{x_0\} \cup F(D)).$$

We claim

$$(2.14)\qquad D_n \text{ is weakly separable (for each } n \in \{0, 1, 2, \dots\}).$$

Assume D_k is weakly separable (so separable) for some $k \in \{0, 1, \dots\}$. Then from (2.12) we have that $F(D_k)$ is separable so as a result $co\,(\{x_0\} \cup F(D_k))$ $(= D_{k+1})$ is separable (so weakly separable). By induction (2.14) is true. Thus for each $n \in \{0, 1, \dots\}$, (2.14) guarantees that there exists $C_n \subseteq D_n$ with C_n countable and $\overline{C_n}^w = \overline{D_n}^w$. Let $C = \cup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n}^w \subseteq \overline{\bigcup_{n=0}^{\infty} D_n}^w$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} D_n}^w = \overline{D^w} \quad \text{and} \quad \overline{\bigcup_{n=0}^{\infty} \overline{D_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}^w}^w = \overline{\bigcup_{n=0}^{\infty} C_n}^w = \overline{C^w}.$$

Thus $\overline{C^w} = \overline{D^w}$ so from (2.11) (see (2.13)) we have that $\overline{D^w}\,(= \overline{D})$ is weakly compact.

Consider the map $F^\star : \overline{D^w} \to K(\overline{D^w})$ given by

$$F^\star(x) = F(x) \cap \overline{D^w};$$

this is clear once we show the map $F^\star$ is well defined i.e. once we show $F^\star(x) \neq \emptyset$ for each $x \in \overline{D^w}$. Note from (2.13) that $F(D) \subseteq D \subseteq \overline{D^w}$ so $D \subseteq F^{-1}(\overline{D^w})$. Now let $x \in \overline{D^w}$. Now since $\overline{D^w}$ is weakly compact the Eberlein–Šmulian theorem [2] (pg. 549) guarantees that there is a sequence (x_n) in D with $x_n \rightharpoonup x$ (here $\rightharpoonup$ denotes weak convergence). Take any $y_n \in F(x_n)$. Now since $F(D) \subseteq D$ we have $y_n \in D$. Also since $\overline{D^w}$ is weakly compact the Eberlein–Šmulian theorem [2] (pg. 549) guarantees that we may assume without loss of generality that $y_n \rightharpoonup y$ for some $y \in \overline{D^w}$. Note $y_n \in F(x_n)$, $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ implies $y \in F(x)$ since F has weakly sequentially closed graph. Thus $y \in F(x) \cap \overline{D^w}$ so $x \in F^{-1}(\overline{D^w})$. As a result $\overline{D^w} \subseteq F^{-1}(\overline{D^w})$ i.e. $F^\star(x) \neq \emptyset$ for each $x \in \overline{D^w}$.

Note $F^\star : \overline{D^w} \to K(\overline{D^w})$ has weakly sequentially closed graph. Now Theorem 2.8 guarantees a $x \in \overline{D^w}$ with $x \in F^\star(x) \subseteq F(x)$. □

Remark 2.10. In fact if you look at the proof of Theorem 2.8 (see [9]) and the proof of Theorem 2.9 one can see that E metrizable can be replaced by E a Šmulian space (i.e. E is such that if the weak closure of a subset Ω of E is weakly compact then (i). Ω is weakly sequentially compact, and (ii). if $x \in \overline{\Omega^w}$ then there exists a sequence (x_n) in Ω with $x_n \rightharpoonup x$).

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