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TIKHONOV REGULARIZATION METHODS FOR INVERSE MIXED QUASI-VARIATIONAL INEQUALITIES

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Abstract. The purpose of this paper is to study the Tikhonov regularization method for a new class of inverse mixed quasi-variational inequalities. The existence of solutions to the inverse mixed quasi-variational inequalities is proved via the Kakutani-Fan-Glicksberg fixed point theorem. We, based on a rather weak coercivity condition, obtain the Tikhonov regularization method for the inverse mixed quasi-variational inequalities and their regularized problems. Moreover, nonemptiness and boundedness of solution sets for regularized inverse mixed quasi-variational inequalities are also discussed.

1. Introduction

The theory of variational inequalities is a fascinating key component of nonlinear functional analysis, variational analysis, nonsmooth analysis and convex analysis. It's applications are ubiquitous in numerous scientific fields and real world applications, such as, game theory, differential equations, economics, network, transportation, control theory, and computer sciences; see, e.g., [1, 10, 15, 16, 18, 35, 36] and the references therein. The interplay between solutions of existence and solutions of iterative algorithms have kept researchers and engineers enthralled for decades; see, e.g., [11, 13, 14, 17, 31, 37] and the references therein.

In 1960's, Lescarret [29] and Browder [9] introduced an useful generalization of the variational inequality which is called the mixed variational inequality (for short, MVI). The mixed variational inequalities have a number of applications in diverse areas, such as general economic equilibrium problems, oligopolistic equilibrium problems, and electrical circuits; see, e.g., [19, 28]. As an important research field, quasi-variational inequalities were originally studied by Bensoussan, Goursat and Lion in [6–8] in connection with its numerous applications. Recently, quasi-variational inequalities have been an effective mathematical tool for describing a wide range of complex equilibrium situations appearing in different fields.

In this paper, we are concerned with a class of inverse variational inequalities, which was studied in [23, 24]. The inverse variational inequalities, which are closely related with the classical variational inequalities, find a number of real applications in bipartite market equilibrium problems and telecommunication networks. Recently, inverse variational inequalities (for short, IVIs), which characterize a particular kind of variational inequalities, have received increasing attention in the theory of optimization; see [2, 4, 5, 12, 22, 25, 26, 30] and the references therein.

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In this paper, we consider the following class of inverse mixed quasi-variational inequalities (in short, IMQVI(f,ϕ,S)): find an $x^* \in \mathbb{R}^n$ such that $\phi(x^*) \in S(x^*)$ and

$$\langle x - \phi(x^*), x^* \rangle + f(x) - f(\phi(x^*)) \geq 0, \quad \forall x \in S(x^*), \tag{1.1}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping, K is a nonempty subset of $\mathbb{R}^n$ and $S : \mathbb{R}^n \to 2^K$ is a set-valued mapping such that $S(x)$ is a nonempty closed convex subset of $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. Then $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous mapping. The solution set of (1.1) is denoted by SIMQVI(f,ϕ,S).

If $S(x) \equiv S$, a nonempty closed convex set, then IMQVI (1.1) is equivalent to the known inverse mixed variational inequality, denoted by IMVI, which consists of finding an $x^* \in \mathbb{R}^n$ such that $\phi(x^*) \in S$ and

$$\langle x - \phi(x^*), x^* \rangle + f(x) - f(\phi(x^*)) \geq 0, \quad \forall x \in S, \tag{1.2}$$

which was considered by Li, Li and Huang [30].

If

$$f(x) = \delta_S(x) = \begin{cases} 0, & x \in S(x), \\ +\infty, & x \notin S(x), \end{cases}$$

where δ_S is the indicator function of a closed convex set $S(x) \subset \mathbb{R}^n$, then IMQVI (1.1) is reduced to the inverse quasi-variational inequality, denoted by IQVI, which consists of finding an $x^* \in \mathbb{R}^n$ such that $\phi(x^*) \in S(x^*)$ and

$$\langle x - \phi(x^*), x^* \rangle \geq 0, \quad \forall x \in S(x^*). \tag{1.3}$$

This problem was studied by Aussel, Gupta and Mehra [4].

If $S(x) \equiv S \subset \mathbb{R}^n$ is a nonempty closed convex set, then problem IQVI (1.3) collapses to the classical inverse variational inequality (IVI): find an $x^* \in \mathbb{R}^n$ such that $\phi(x^*) \in S$ and

$$\langle x - \phi(x^*), x^* \rangle \geq 0, \quad \forall x \in S. \tag{1.4}$$

We note that the class IMQVI(f,ϕ,S) is a special case of a more general class of inverse mixed quasi-variational inequalities considered in [32] where some existence result and error bound are investigated.

The Tikhonov regularization method which introduced in [27,34] is one of the important methods for the ill-posed variational inequality problem. In 2014, Luo [33] proposed Tikhonov regularization methods for solving inverse variational inequality problem. In 2018, Chen et al. [12] established the Tikhonov regularization method for inverse mixed variational inequalities.

In this paper, inspired by the presented results, we develop a Tikhonov regularization method for the IMQVI, which improves some known results in the literature. We focus on the following regularized inverse mixed quasi-variational inequalities: find an $x^* \in \mathbb{R}^n$ such that $\phi(x^*) \in S(x^*)$ and

$$\langle x - \phi(x^*), x^* + \varepsilon\phi(x^*) \rangle + f(x) - f(\phi(x^*)) \geq 0, \quad \forall x \in S(x^*), \tag{1.5}$$

we denote the problem (1.5) by IMQVI (f,ϕ_ε,S), where $\phi_\varepsilon := I + \varepsilon\phi, \varepsilon > 0$. Hopefully, the solution of the IMQVI (f,ϕ_ε,S) converges to a solution of the IMQVI (f,ϕ,S) as $\varepsilon \to 0$.

One of the two concerns of this paper is to establish two existence theorems of solutions for IMQVI(f, ϕ, S) using the Kakutani-Fan-Glicksberg fixed point theorem and the Tikhonov regularization method under a coercivity condition. The other concern of this paper is to give the perturbation analysis for the solution set of IMQVI (f, ϕ_ε, S) under some suitable condition.

This paper is organized as follows. In Section 2, we present definitions and some lemmas which will be used in the proof of our main results. In Section 3, the solution set of IMQVI(f, ϕ, S) and its regularized problem are investigated under the coercivity condition. Finally, in Section 4, Tikhonov regularization methods for inverse mixed quasi-variational inequalities are provided.

2. Preliminaries

Throughout this paper, we denote by $\|\cdot\|$ the norm and by $\langle \cdot, \cdot \rangle$ the inner product in the Euclidean space $\mathbb{R}^n$. Let $\mathbb{R}^n_+$ be the nonnegative orthant of $\mathbb{R}^n$ and K be a nonempty subset of $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and $S : \mathbb{R}^n \to 2^K$ be a set-valued mapping such that $S(x)$ is a nonempty closed convex subset of $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous on $S(x)$, and for $t \in \mathbb{N}$, $S_t(x) = \{y \in S(x) : \|y\| \leq t\}$.

Definition 2.1 ([22])**.** Let T_1 and T_2 be two topological spaces. A set-valued mapping $S : T_1 \to 2^{T_2}$ is said to be

(i) upper semicontinuous (u.s.c.) at $x_0 \in T_1$ if, for any neighborhood $N(S(x_0))$ of $S(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that, for every $x \in N(x_0)$, $S(x) \subseteq N(S(x_0))$;

(ii) lower semicontinuous (l.s.c.) at $x_0 \in T_1$ if, for any $y \in S(x_0)$ and any neighborhood $N(y)$ of y, there exists a neighborhood $N(x_0)$ of x_0 such that, for every $x \in N(x_0)$, $S(x) \cap N(y) \neq \emptyset$.

We say that S is u.s.c. and l.s.c. on T_1 if it is u.s.c. and l.s.c. at each $x \in T_1$, respectively. S is said to be continuous on T_1 if it is both u.s.c. and l.s.c. on T_1 .

We need the following lemmas which are crucial for the proof of our convergence theorem.

Lemma 2.2 ([3])**.** *Let $S : K \to 2^K$. Then, S is l.s.c. at $u_0 \in K$ if and only if, for any sequence $\{u_n\} \subseteq K$ with $u_n \to u_0$ and for any $x_0 \in S(u_0)$, there exists $x_n \in S(u_n)$ such that $x_n \to x_0$.*

Lemma 2.3 ([21])**.** *Let $S : K \to 2^K$. If $S(u_0)$ is compact, then S is u.s.c. at $u_0 \in K$ if and only if, for any sequence $\{u_n\} \subseteq K$ with $u_n \to u_0$ and for any $x_n \in S(u_n)$, there exists $x_0 \in S(u_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

Lemma 2.4 ([3])**.** *Let T_1, T_2, and T_3 be three topological spaces, and let $A : T_1 \to 2^{T_2}$ and $S : T_2 \to 2^{T_3}$ be two set-valued mappings. Defined a set-valued mapping $P : T_1 \to 2^{T_3}$ by $P(u) = S(A(u)) = \bigcup_{x \in A(u)} S(x), u \in T_1$ satisfying*

(i) *if A is u.s.c. on T_1 and S is u.s.c. on T_2, then P is u.s.c. on T_1;*

(ii) *if A is l.s.c. on T_1 and S is l.s.c. on T_2, then P is l.s.c. on T_1.*

Lemma 2.5 ([22]). *Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a mapping and let H be a nonempty subset of $f(\mathbb{R}^n)$. Assume that $f^{-1}(H)$ is compact and f is continuous on $\mathbb{R}^n$. Then, $f^{-1}(\cdot)$ is u.s.c. on H.*

Lemma 2.6 ([20]). *Let H be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X, and let $A:H\to 2^H$ be an u.s.c. set-valued mapping with nonempty compact convex values. Then, there exists $x_0\in H$ such that $x_0\in A(x_0)$.*

Definition 2.7. Let K be nonempty subset of $\mathbb{R}^n$, $F:\mathbb{R}^n\to\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$. The map F is said to be

(i) monotone on K if $\langle F(x)-F(y),x-y\rangle\ge 0$ for all $x,y\in K$;

(ii) f-pseudomonotone on K if, for $x,y\in K$,

$$\langle F(x),y-x\rangle+f(y)-f(x)\ge 0\ \Rightarrow\ \langle F(y),y-x\rangle+f(y)-f(x)\ge 0.$$

Remark 2.8. If F is monotone on K, then it is f-pseudomonotone. The inverse is not true.

Lemma 2.9. *Assume that K is convex, ϕ is f-pseudomonotone on $\phi^{-1}(K)$, $\phi^{-1}(\cdot)$ is l.s.c. on K, and $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is proper convex and lower semicontinuous. Then, the following two statements are equivalent:*

(i) *x^* is a solution to inverse mixed variational inequalities, that is,*

$$\phi(x^*)\in K,\quad \langle y-\phi(x^*),x^*\rangle+f(y)-f(\phi(x^*))\ge 0,\quad \forall y\in K.$$

(ii) *x^* is a solution to the following associated inverse mixed variational inequalities: find $x^*\in\mathbb{R}^n$ such that $\phi(x^*)\in K$, and*

$$\langle y-\phi(x^*),x\rangle+f(y)-f(\phi(x^*))\ge 0,\quad \forall y\in K,\forall x\in\phi^{-1}(y).$$

Proof. From [22, Lemma 7], one can conclude the desired conclusion immediately. □

3. Existence of solutions and coercivity conditions

In this section, we mainly discuss existence theorems for IMQVI(f,ϕ,S) and its regularized problem. We prove the existence of solutions for these problems by the Kakutani-Fan-Glicksberg fixed point theorem and the coercivity condition, respectively.

Theorem 3.1. *Let $S:\mathbb{R}^n\to 2^K$ be a set-valued mapping such that S is continuous on $\mathbb{R}^n$ and let $S(x)$ be a nonempty closed convex set for each $x\in\mathbb{R}^n$. Let ϕ be a linear continuous mapping on $\mathbb{R}^n$, f-pseudomonotone on $\phi^{-1}(K)$ and ϕ^{-1} be l.s.c. on K. Set $\phi^{-1}(S(x))=\{\bar{x}\in\mathbb{R}^n:\phi(\bar{x})\in S(x)\}$. Assume that*

(i) *$\phi^{-1}(K)$ is bounded convex and $K\subseteq\phi(\mathbb{R}^n)$ is compact;*

(ii) *$f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is a proper convex and lower semicontinuous mapping.*

Then IMQVI(f,ϕ,S) has a solution.

Proof. Letting $\Omega=\phi^{-1}(K)$, for each $z\in\Omega$, we define

$$G(z)=\{x\in\Omega:\phi(x)\in S(z),\langle y-\phi(x),x\rangle+f(y)-f(\phi(x))\ge 0,\forall y\in S(z)\}.$$

The following proof is split into 3 steps.

Step 1. Show that $G(z)$ is closed.

IMQVI (f, ϕ, S) has a solution is equivalent to the face that the set-valued mapping $G : \Omega \to 2^{\Omega}$ has a fixed point. By Theorem 3.1 in [12], we know that $G(z)$ is nonempty. Since f is l.s.c., ϕ is continuous and $S(z)$ is closed, one sees that $G(z)$ is a closed and bounded set.

Step 2. Show that $G(z)$ is convex.

Now, we define a mapping A as follows

$$\begin{aligned} A(z) = \{x \in \Omega : \phi(x) \in S(z), &\langle y - \phi(x), \alpha\rangle \\ &+ f(y) - f(\phi(x)) \geq 0, \forall y \in S(z), \forall \alpha \in \phi^{-1}(y)\}. \end{aligned}$$

Then, it follows from Lemma 2.9 that $G(z) = A(z)$. Hence, we can prove that $A(z)$ is convex. For any $x_1, x_2 \in A(z)$, and for any $t \in [0, 1]$. Since ϕ is linear, one has

$$\phi(tx_1 + (1 - t)x_2) = t\phi(x_1) + (1 - t)\phi(x_2).$$

Moreover, since $S(z)$ is convex and $\phi(x_i) \in S(z)$, $i = 1, 2$, it holds that

$$\phi(tx_1 + (1 - t)x_2) \in S(z).$$

Thus,

$$tx_1 + (1 - t)x_2 \in \phi^{-1}(S(z)) \subseteq \Omega.$$

Since $x_1, x_2 \in A(z)$ for each $y \in S(z)$ and for each $\alpha \in \phi^{-1}(y)$, we obtain

$$\langle y - \phi(x_i), \alpha\rangle + f(y) - f(\phi(x_i)) \geq 0, \quad i = 1, 2. \tag{3.1}$$

Since $\phi(tx_1 + (1 - t)x_2) = t\phi(x_1) + (1 - t)\phi(x_2)$, it follows from the convexity of f that

$$\begin{aligned} tf(\phi(x_1)) + (1 - t)f(\phi(x_2)) &\geq f(\phi(tx_1 + (1 - t)x_2)) \\ &= f(t\phi(x_1) + (1 - t)\phi(x_2)). \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\begin{aligned} &\langle y - \phi(tx_1 + (1 - t)x_2), \alpha\rangle + f(y) - f(\phi(tx_1 + (1 - t)x_2)) \\ &= \langle y - \phi(tx_1 + (1 - t)x_2), \alpha\rangle + f(y) - f(t\phi(x_1) + (1 - t)\phi(x_2)) \\ &\geq \langle y - t\phi(x_1) + (1 - t)\phi(x_2), \alpha\rangle + f(y) - tf(\phi(x_1)) - (1 - t)f(\phi(x_2)) \\ &\geq 0, \end{aligned}$$

so $tx_1 + (1 - t)x_2 \in A(z)$. Thus $G(z)$ is convex.

Step 3. Show that G is u.s.c. on Ω.

Suppose that there exists $z_0 \in \Omega$ such that G is not u.s.c at z_0. Hence, for any neighborhood $U(z_0)$ of z_0, there exists a neighborhood V_0 of $G(z_0)$ and exists $\bar{z} \in U(z_0)$ such that $G(\bar{z}) \nsubseteq V_0$. Thus, there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $G(z_n) \nsubseteq V_0$ for $n = 1, 2, \cdots$. This indicates that there exist

$$x_n \in G(z_n) \text{ such that } \quad x_n \notin V_0, \quad n = 1, 2, \cdots. \tag{3.3}$$

By Lemma 2.5, we can obtain that $\phi^{-1}(\cdot)$ is u.s.c. on K. According to the assumptions and Lemma 2.4, we can get that the mapping $\phi^{-1}(S(\cdot))$ is u.s.c. on Ω with compact values. Using $x_n \in \phi^{-1}(S(z_n))$ and Lemma 2.3, there exist $x_0 \in \phi^{-1}(S(z_0))$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we may assume that $x_n \to x_0$. If $x_0 \notin G(z_0)$, then, there exists $y_0 \in S(z_0)$ such that

$$\langle y_0 - \phi(x_0), x_0\rangle + f(y_0) - f(\phi(x_0)) < 0. \tag{3.4}$$

On the other hand, $S(\cdot)$ is l.s.c. at z_0 and $y_0 \in S(z_0)$. From Lemma 2.2, there exist $y_n \in S(z_n)$ such that $y_n \to y_0$. From (3.3), we have

$$\langle y_n - \phi(x_n), x_n\rangle + f(y_n) - f(\phi(x_n)) \geq 0.$$

Combining $y_n \to y_0$ and $x_n \to x_0$, we obtain

$$\langle y_0 - \phi(x_0), x_0\rangle + f(y_0) - f(\phi(x_0)) \geq 0,$$

which contradicts (3.4). Hence, $x_0 \in G(z_0)$. So $x_n \to x_0 \in V_0$, which contradicts (3.3). Therefore, for each $z \in \Omega$, $G(z)$ is nonempty convex closed, and G is u.s.c. on Ω. By Lemma 2.6, G has a fixed point. This completes the proof. □

Now, we give an example to illustrate Theorem 3.1.

Example 3.2. Let $f : \mathbb{R}^2_+ \to \mathbb{R}^2$ be defined as $f(x) = 2x_1 + \frac{3}{2}x_2$ and $\phi : \mathbb{R}^2_+ \to \mathbb{R}^2$ be defined as $\phi(x) = (x_1, 2x_2)$. Clearly, we can see that ϕ is linear, continuous and monotone on $\mathbb{R}^2_+$ with $\phi^{-1}(\cdot)$ is l.s.c. on $\phi(\mathbb{R}^2_+)$, and f is a proper convex and lower semicontinuous mapping. Let $K = [0, 3] \times [0, 3]$. Then we can prove $\phi^{-1}(K)$ is bounded and convex.

Let $S : \mathbb{R}^n \to 2^K$ be defined as follows:

$$S(x) = [0, 2 + \sin(x_1 - x_2)] \times [0, 2 + \cos(x_1 + x_2)].$$

Then, it is clear that S is continuous on $\mathbb{R}^2$. For each $x \in \mathbb{R}^2$, $S(x)$ is closed and convex, and $\phi^{-1}(S(x))$ is bounded and convex. Hence, all the conditions of Theorem 3.1 hold. Obviously, $x = (0, 0)$ is a solution of IMQVI(f, ϕ, S).

Theorem 3.3. *Assume that all conditions of Theorem 3.1 hold. Then IMQVI(f, ϕ_ε, S) has a solution.*

Proof. By using Theorem 3.1, we can obtain that the desired conclusion immediately. □

Before deriving the existence of solutions of IMQVI(f, ϕ, S) and its regularized problem when the set $S(\cdot)$ is unbounded, we need to compare some coercivity conditions for the inverse mixed quasi-variational inequality problem. The relations of these coercivity conditions are as following:

Lemma 3.4. *Let $S(x)$ be a nonempty, closed and convex set of $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. Consider the following coercivity conditions:*

(A) *there exists $t > 0$ such that, for every $\phi(x) \in S(x)\backslash S_t(x)$, there exists $y_0 \in S(x)$ with $\|y_0\| < \|\phi(x)\|$ satisfying*

$$\langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \geq 0.$$

(B) *there exists $t > 0$ such that, for every $\phi(x) \in S(x)\backslash S_t(x)$, there exists $y_0 \in S_t(x)$ satisfying*

$$\langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \geq 0.$$

(C) *there exists* $y_0 \in S(x)$ *such that the set*

$$Q(y_0) = \{\phi(x) \in S(x) : \langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) < 0\}.$$

is bounded provide that it is nonempty.

Then $(C) \Rightarrow (B) \Rightarrow (A)$.

Proof. (C) $\Rightarrow$ (B). If $Q(y_0) = \emptyset$, then, for any $\phi(x) \in S(x)$,

$$\langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \geq 0.$$

Hence, (B) holds.

If $Q(y_0) \neq \emptyset$, then by (C) there exists $t > 0$ such that $Q(y_0) \cup \{y_0\} \subset S_t(x)$. Therefore, for every $\phi(x) \in S(x)\backslash S_t(x)$, we have

$$\langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \geq 0.$$

Since $y_0 \in S_t(x)$, we find that (B) is still true.

(B) $\Rightarrow$ (A). If (B) holds, then there exist $t > r > 0$ such that $S_{t-r}(x) \subset S_t(x)$. For each $\phi(x) \in S(x)\backslash S_t(x)$, we find from the condition (B) that there is $y_0 \in S_{t-r}(x) \subset S(x)$ such that

$$\langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \geq 0$$

with $\|y_0\| < t - r < t < \|\phi(x)\|$. Thus (A) is verified. □

Remark 3.5. From Lemma 3.4, we can see that condition (A) is a rather weak coercivity condition.

The following theorem shows that the IMQVI(f, ϕ_ε, S) has a solution by the relatively weak coercivity condition (A).

Theorem 3.6. *Let* $S : \mathbb{R}^n \to 2^K$ *be a set-valued mapping such that* S *is continuous and let* $S(x)$ *be a nonempty closed convex set for each* $x \in \mathbb{R}^n$. *Assume that*

(i) ϕ *is a linear continuous mapping on* $\mathbb{R}^n$;
(ii) $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex and lower semicontinuous mapping;*
(iii) $\phi^{-1}(S(x))$ *is bounded.*

If coercivity condition (A) holds, then IMQVI(f, ϕ_ε, S) *has a solution.*

Proof. Let $t > 0$ be as in condition (A) and let $m > t$. From Theorem 3.3 and the fact that $S_m(x)$ is a bounded closed convex set, we find that there exists $x_m \in \mathbb{R}^n$ such that $\phi(x_m) \in S_m(x_m)$ and

$$\langle y - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y) - f(\phi(x_m)) \geq 0, \quad \forall y \in S_m(x_m). \tag{3.5}$$

Case 1. If $\|\phi(x_m)\| = m$, then $\|\phi(x_m)\| > t$. From assumption (A), one sees that there exists $y_0 \in S(x_m)$ with $\|y_0\| < \|\phi(x_m)\|$ such that

$$\langle \phi(x_m) - y_0, x_m + \varepsilon\phi(x_m)\rangle - f(y_0) + f(\phi(x_m)) \geq 0, \quad y_0 \in S_m(x_m). \tag{3.6}$$

For any $y \in S(x_m)$, since $\|y_0\| < \|\phi(x_m)\| = m$, there exists $r \in (0, 1)$ such that

$$y_r = y_0 + r(y - y_0) \in S_m(x_m).$$

Using (3.5), (3.6) and the convexity of f, we arrive at

$$\begin{aligned}0 &\le \langle y_r - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y_r) - f(\phi(x_m)) \\ &= \langle r(y - \phi(x_m)) + (1-r)(y_0 - \phi(x_m)), x_m + \varepsilon\phi(x_m)\rangle + f(y_r) - f(\phi(x_m)) \\ &\le r[\langle y - \phi(x_m)), x_m + \varepsilon\phi(x_m)\rangle + f(y) - f(\phi(x_m))] \\ &+ (1-r)[\langle y_0 - \phi(x_m)), x_m + \varepsilon\phi(x_m)\rangle + f(y_0) - f(\phi(x_m))] \\ &\le r[\langle y - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y) - f(\phi(x_m))].\end{aligned}$$

Hence,

$$\langle y - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y) - f(\phi(x_m)) \ge 0.$$

Since $y \in S(x_m)$ is arbitrary, one concludes that x_m is a solution of IMQVI(f, ϕ_ε, S).

Case 2. If $\|\phi(x_m)\| < m$, then, for any $y \in S(x_m)$, there exists $r \in (0,1)$ such that

$$y_r = \phi(x_m) + r(y - \phi(x_m)) \in S_m(x_m).$$

It follows that

$$\begin{aligned}0 &\le \langle y_r - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y_r) - f(\phi(x_m)) \\ &= r\langle y - \phi(x_m)), x_m + \varepsilon\phi(x_m)\rangle + f(y_r) - f(\phi(x_m)) \\ &\le r\langle y - \phi(x_m)), x_m + \varepsilon\phi(x_m)\rangle + rf(y) + (1-r)f(\phi(x_m)) - f(\phi(x_m)) \\ &= r\langle y - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + rf(y) - rf(\phi(x_m)).\end{aligned}$$

Therefore,

$$\langle y - \phi(x_m), x_m + \varepsilon\phi(x_m)\rangle + f(y) - f(\phi(x_m)) \ge 0, \quad \forall y \in S(x_m).$$

Thus x_m solves IMQVI(f, ϕ_ε, S). □

4. The Tikhonov regularization

In this section, we establish the Tikhonov regularization method for the inverse mixed quasi-variational inequality problem.

Theorem 4.1. *Let S be a set-valued mapping such that S is continuous and $S(x)$ is a nonempty closed convex set for each $x \in \mathbb{R}^n$. Let f be a convex mapping. Let ϕ be a linear continuous mapping and $\phi^{-1}(S(x))$ be bounded. If coercivity condition (A) holds, then, for any $\varepsilon > 0$,*

- (i) *IMQVI(f, ϕ_ε, S) has a solution;*
- (ii) *if $\phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded mapping, then set $\{SIMQVI(f, \phi_\tau, S) : \tau \in (0, \varepsilon]\}$ is bounded.*

Proof. Let $t > 0$ be as in the condition (A). (i) We claim that, for every $\phi(x) \in S(x)\backslash S_t(x)$, there exists $y_0 \in S(x)$ with $\|y_0\| < \|\phi(x)\|$ satisfying

$$\langle \phi(x) - y_0, x + \varepsilon\phi(x)\rangle - f(y_0) + f(\phi(x)) \ge 0$$

Then, we deduce from the assumption (A) that

$$\begin{aligned}
&\langle \phi(x) - y_0, x + \varepsilon\phi(x)\rangle - f(y_0) + f(\phi(x)) \\
&= \langle \phi(x) - y_0, x\rangle + \varepsilon\|\phi(x)\|^2 - \varepsilon\langle y_0, \phi(x)\rangle - f(y_0) + f(\phi(x)) \\
&\geq \langle \phi(x) - y_0, x\rangle + \varepsilon\|\phi(x)\|(\|\phi(x)\| - \|y_0\|) - f(y_0) + f(\phi(x)) \\
&> \langle \phi(x) - y_0, x\rangle - f(y_0) + f(\phi(x)) \\
&\geq 0.
\end{aligned}$$

It follows from Theorem 3.6 that IMQVI(f, ϕ_ε, S) has a solution.

(ii) Let $\tau \in (0, \varepsilon]$ and $x(\tau) \in SIMQVI(f, \phi_\tau, S)$. We claim

$$\phi(x(\tau)) \in S_t(x(\tau)).$$

If not, by the assumption (A), there exists $y_0(\tau) \in S(x(\tau))$ with $\|y_0(\tau)\| < \|\phi(x(\tau))\|$ such that

$$\langle \phi(x(\tau)) - y_0(\tau), x(\tau)\rangle - f(y_0(\tau)) + f(\phi(x(\tau))) \geq 0.$$

Since $x(\tau) \in SIMQVI(f, \phi_\tau, S)$ and $y_0(\tau) \in S(x(\tau))$, we obtain $\phi(x(\tau)) \in S(x(\tau))$ and

$$\langle y_0(\tau) - \phi(x(\tau)), x(\tau) + \tau\phi(x(\tau))\rangle + f(y_0(\tau)) - f(\phi(x(\tau))) \geq 0.$$

Hence,

$$\begin{aligned}
&\tau[\langle y_0(\tau), \phi(x(\tau))\rangle - \|\phi(x(\tau))\|^2] \\
&\geq \langle \phi(x(\tau)) - y_0(\tau), x(\tau)\rangle - f(y_0(\tau)) + f(\phi(x(\tau))) \\
&\geq 0.
\end{aligned}$$

Using the Cauchy-Schwartz inequality, we can obtain $\|y_0(\tau)\| \geq \|\phi(x(\tau))\|$, which is a contradiction. Thus, $\phi(x(\tau)) \in S_t(x(\tau))$. Since $\phi^{-1}(x(\tau))$ is a bounded mapping, one concludes that $\{SIMQVI(f, \phi_\tau, S)\}$ is bounded. ☐

For every $\varepsilon > 0$, let $\{A_\varepsilon\}$ be a sequence of sets in $\mathbb{R}^n$ and define

$$\limsup_{\varepsilon\to 0_+} A_\varepsilon := \{x \in \mathbb{R}^n : \exists \varepsilon_n \to 0_+ \quad \text{and} \quad x_n \in A_{\varepsilon_n} \text{ such that } x_n \to x\}.$$

The following result gives the convergence analysis for SIMQVI(f, ϕ_ε, S).

Theorem 4.2. *Let $S : \mathbb{R}^n \to 2^K$ be a continuous mapping on $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$, $S(x)$ be a nonempty closed convex set. Let f be a proper convex and lower semicontinuous mapping. Let ϕ be a linear continuous mapping and $\phi^{-1}(S(x))$ be bounded. If coercivity condition (A) holds, then*

$$\emptyset \neq \limsup_{\varepsilon\to 0^+} SIMQVI(f, \phi_\varepsilon, S) \subset SIMQVI(f, \phi, S).$$

Proof. From Theorem 4.1, one sees that $\limsup_{\varepsilon\to 0^+}$ SIMQVI(f, ϕ_ε, S) is nonempty. Next, we let $x \in \limsup_{\varepsilon\to 0^+}$ SIMQVI(f, ϕ_ε, S). Then there exist a sequence $\varepsilon_n \to 0_+$ and $x_n \in$ SIMQVI$(f, \phi_{\varepsilon_n}, S)$ such that $x_n \to x$. This implies that $\phi(x_n) \in S(x_n)$ and

$$\langle y - \phi(x_n), x_n + \varepsilon_n\phi(x_n)\rangle + f(y) - f(\phi(x_n)) \geq 0, \quad \forall y \in S(x_n). \tag{4.1}$$

Since ϕ is continuous, one has $\phi(x_n) \to \phi(x)$. Since $S(x_n)$ is closed, one obtains $\phi(x) \in S(x_n)$. Thus, for every $y \in S(x_n), \phi(x) \in S(x_n)$ and

$$\begin{aligned}&\langle y-\phi(x_n), x_n+\varepsilon_n\phi(x_n)\rangle + f(y) - f(\phi(x_n))\\&= \langle y-\phi(x_n), x_n\rangle + \varepsilon_n\langle y, \phi(x_n)\rangle - \varepsilon_n\|\phi(x_n)\|^2 + f(y) - f(\phi(x_n))\\&\to \langle y-\phi(x), x\rangle + f(y) - f(\phi(x)), \text{as} \quad n\to\infty.\end{aligned}$$

Using (4.1) and the upper semicontinuous of S, we conclude $\phi(x) \in S(x)$ and

$$\langle y-\phi(x), x\rangle + f(y) - f(\phi(x)) \geq 0, \forall y \in S(x).$$

Therefore, $x \in \text{IMQVI}(f, \phi, S)$. □

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