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THE EDGE CALCULUS OF SINGULARITY ORDER ≥ 3

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Abstract. We study Mellin pseudo-differential algebras on singular straight cones and manifolds with singularity of order ≥ 3. Those are necessary to express parametrices of elliptic differential operators with a corresponding corner-degenerate behavior, and we obtain regularity in weighted spaces.

1. Introduction

Analysis of differential or pseudo-differential operators on singular manifolds, e.g., with conical singularities and edges, is a well-established topic in partial differential equations, motivated by numerous applications. Although concrete models of physics are usually connected with individual data, such as asymptotics of solutions which may be determined by opening angles of the geometric configurations in connection with piecewise smooth domains, it is also necessary to develop ideas for higher singularities. Those are natural in many-particle systems where "corners" of higher singularity orders belong to the geometry of the problems, cf., [4, 5]. It is common to pass to stretched coordinates, e.g., when we consider a cone

$$X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \tag{1.1}$$

for a smooth manifold X, analysis is formulated in variables $(r, x) \in X^{\wedge} := \mathbb{R}_+ \times X$, the associated open stretched cone, and then operators acquire a degenerate behavior in the axial variable $r \in \mathbb{R}_+$ which may measure the distance of a point to the respective singularity. An elementary example are polar coordinates in $\mathbb{R}^{n+1}_{\tilde{x}} \setminus \{0\}$. Then differentiations $D^{\alpha}_{\tilde{x}}$ expressed in polar coordinates can be written in terms of operators $r^{-|\alpha|}(-r\partial_r)^j D^{\beta_j}_x$ for $|\alpha| = j + |\beta_j|$, containing the Fuchs type differentiation $r\partial_r$ and differentiations in local coordinates x on the unit sphere in $\mathbb{R}^{n+1}$. Another example is the shape of the Laplace-Beltrami operator for a Riemannian metric on $\mathbb{R}_+ \times X$ of the form

$$dr^2 + r^2 g_X$$

where g_X is a Riemannian metric on X. The process of generating such operators can be iterated, and then we obtain operators with a degenerate behaviour in different axial variables $r_j \in \mathbb{R}_+, j = 1, \ldots, N$. On wedges (or manifolds with edges), locally described in stretched variables by $\mathbb{R}_+ \times X \times \mathbb{R}^q$, with a metric like

$$dr^2 + r^2 g_X + dy^2$$

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the corresponding Laplace-Beltrami operator takes the form

$$r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r,y)(-r\partial_r)^j (rD_y)^\alpha$$

for $\mu = 2$ and coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$. Here $\mathrm{Diff}^\nu(X)$ is the space of all differential operators of order $\nu \in \mathbb{N}$ with smooth coefficients over X. General orders are motivated by the task to understand corresponding operator algebras , and this aspect is essential in ellipticity and parametrix constructions where the corresponding operators are pseudo-differential and of opposite orders. Later on we outline more observations occurring in concrete models of applications. In any case, as noted before, interesting singularities are not only of conical or edge type, but also of higher order, such as cubes, etc., and singularities of any orders can be generated in Cartesian products

$$M_1^\Delta \times M_2^\Delta \times \cdots \times M_k^\Delta \tag{1.2}$$

for any smooth, say, compact manifolds $M_j, j = 1, \ldots, k$. It turns out that adequate approaches for solving elliptic equations on singular spaces like (1.2) for $k \geq 2$ are by no means straightforward, and it is still of help to study cases of lower singularity orders $k \geq 2$. The case $k = 2$ has been elaborated in several articles, see, e.g., in [1, 2], see also [27]. While [1, 2] is devoted to large singular orders k, in the present paper we intend to analyze the constructions for increasing k, starting from $k = 1, 2$ and to give more explicit material on the singular symbol structure for degenerate differential operators, see, [4, 9, 15, 26]. In particular, we consider the case of spaces which are locally modelled on $B^\Delta \times \mathbb{R}^{q_3}$ for a B of singular order 2, such that the cone B^Δ is of singular order 3. By iterating the arguments it becomes clear how to proceed for $k \geq 3$.

2. Operator-valued symbols

2.1. **The Mellin operator calculus.** We show in this paper the interplay between Mellin pseudo-differential operators on infinite (stretched) cones $B_2^\wedge$ for a base B_2 of singularity order 2 and on a singular manifold M with edge Y which is locally close to the edge in local coordinate $y \in \mathbb{R}^{q_3}$ modeled on $B_2^\Delta \times \mathbb{R}^{q_3}$.

Let us first outline some tools and notation on spaces of singularity order $k \in \mathbb{N} = \{0, 1, \ldots\}$. By $\mathfrak{M}_0$ we denote the category of smooth oriented manifolds where diffeomorphisms are the isomorphisms. A topological space M is said to belong the category $\mathfrak{M}_k$, $k \geq 1$ if

(i) M contains a subset $s_k(M) \in \mathfrak{M}_0$ such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$,
(ii) there is a neighborhood V_k of $s_k(M)$ in M which has the structure of a (locally trivial) B_{k-1}^Δ-bundle over $s_k(M)$ for some $B_{k-1} \in \mathfrak{M}_{k-1}$.

This is an iterative definition, and by applying those assumptions to $M \setminus s_k(M)$ we obtain another subspace $s_{k-1}(M) := s_{k-1}(M \setminus s_k(M)) \in \mathfrak{M}_0$ such that $M \setminus (s_{k-1}(M) \cup s_k(M))$ belongs to $\mathfrak{M}_{k-2}$, and $s_{k-1}(M)$ contains a neighborhood V_{k-1}

with the structure of a B^{Δ}_{k-2}-bundle over $s_{k-1}(M)$ for a $B_{k-2} \in \mathfrak{M}_{k-2}$. After finitely many steps we arrive at $s_0(M) \in \mathfrak{M}_0$ and we get a stratification

$$s(M) = (s_0(M), s_1(M), \ldots, s_k(M)) \tag{2.1}$$

such that M is the disjoint union of smooth manifolds $s_j(M), j = 0, \ldots, k$ of different dimensions

$$\dim M := \dim s_0(M) > \dim s_1(M) > \cdots > \dim s_k(M).$$

The set $s_k(M)$ is interpreted as the most singular stratum of M. Such situations also may be "fictitious", for instance, in $\mathbb{R}^m$ we may fix a single point and call it $s_1(\mathbb{R}^m)$; it plays the role of a conical singularity and then $\mathbb{R}^m$ turns to a cone with the corresponding vertex. Moreover, if Y is any hyperplane of dimension $0 < q < m$ in $\mathbb{R}^m$ we can set $s_1(\mathbb{R}^m) := Y$ and obtain in this case an edge. Examples of elements in $\mathfrak{M}_3$ are cubes embedded in $\mathbb{R}^3$ or Cartesian products $X^{\Delta}_1 \times X^{\Delta}_2 \times X^{\Delta}_3$ for some $X_i \in \mathfrak{M}_0$.

Below, for convenience, for $M \in \mathfrak{M}_k$ we also set

$$M_{\text{int}} := M \setminus s_k(M). \tag{2.2}$$

Moreover, in some constructions we need global (with respect to $s_k(M)$) cut-off functions

$$\omega''_{\text{glob}_k} \prec \omega_{\text{glob}_k} \prec \omega'_{\text{glob}_k}. \tag{2.3}$$

These are real-valued functions in $C(M)$ vanishing off V_k and smooth in $V_k \setminus s_k(M)$ which are $\equiv 1$ in a small neighborhood of $s_k(M)$. Smoothness of functions refers to correspondig stretched variables, iteratively defined like corner-degenerate differential operators of order zero, cf., the considerations below. Notation $\phi \prec \phi'$ means that $\phi' \equiv 1$ on $\operatorname{supp}\phi$. Let us now consider differential operators with some specific degenerate behavior. The motivation comes from differential operators in the Euclidean space $\mathbb{R}^{1+n+q}$ in variables $(\tilde{x}, y) \in \mathbb{R}^{1+n} \times \mathbb{R}^q$

$$\tilde{A} = \sum_{|\beta| \leq \mu} \tilde{a}_\beta(\tilde{x}, y) D^\beta_{\tilde{x},y} \tag{2.4}$$

for smooth coefficients $\tilde{a}_\beta(\tilde{x}, y)$ which turn to edge-degenerate operators after substituting polar coordinates in $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{0\}$ with variables $(r, x) \in \mathbb{R}_+ \times S^n$. We then obtain operators of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j (rD_y)^\alpha \tag{2.5}$$

for coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \text{Diff}^{\mu-(j+|\alpha|)}(S^n))$. Those are called edge-degenerate in this special case. The corresponding space of edge-degenerate differential operators A of order μ will be denoted by $\text{Diff}^\mu_{\text{deg}}(\mathbb{R}^{1+n+q})$.

If $M \in \mathfrak{M}_1$ is a manifold with edge $Y = s_1(M)$ then for $q := \dim Y > 0$, the space M is locally close to Y modelled on $X^{\Delta} \times \mathbb{R}^q$, for some compact X of dimension n and a differential operator $A \in \text{Diff}^\mu(s_0(M))$ is called edge-degenerate when it is locally close to Y of the form (2.5) with coefficiients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \text{Diff}^{\mu-(j+|\alpha|)}(X))$, i.e., referring to X rather than to S^n. We then obtain the space $\mathrm{Diff}^{\mu}_{\deg}(M)$ of edge-degenerate differential operators on M.

More generally, considering an $M \in \mathfrak{M}_k$ for $k \geq 1$ we have spaces of corner-degenerate differential operators $\mathrm{Diff}^{\mu}_{\deg}(M)$ which can be defined by iteration. An $A \in \mathrm{Diff}^{\mu}_{\deg}(M \setminus s_k(M))$ for $M \in \mathfrak{M}_k$ is said to be corner-degenerate on $M \in \mathfrak{M}_k$, i.e., belongs to the space $\mathrm{Diff}^{\mu}_{\deg}(M)$ if it has the form

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r, y_k)(-r\frac{\partial}{\partial r})^j (rD_{y_k})^{\alpha} \tag{2.6}$$

with coefficients $a_{j\alpha}(r, y_k) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q_k}, \mathrm{Diff}^{\mu-(j+|\alpha|)}(B_{k-1}))$ for $B_{k-1} \in \mathfrak{M}_{k-1}$. It is assumed here that M is locally close to $s_k(M)$ modelled on $B_{k-1} \times \mathbb{R}^{q_k}$. The variable y in formula (2.5) has the meaning of $y_1 \in \mathbb{R}^{q_1}$ but for simplicity we often drop the subscript when $k = 1$. Similarly the various half-axis variables should be indicated by r_k rather than r and edge covariables by $\eta_k \in \mathbb{R}^{q_k}$; thus notation in (2.6) has to be interpreted in this way. If necessary we will employ the more detailed notation, indicating variables and covariables together with k. In that sense, after finitely many steps we arrive at a principal symbol hierarchy

$$\sigma(A) := (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)) \tag{2.7}$$

associated with (2.1), where $\sigma_0(A)$ is the homogeneous principal symbol of A, with A being interpreted as an element of $\mathrm{Diff}^{\mu}(s_0(M))$ and

$$\begin{aligned} \sigma_h(A)(y_h, \eta_h) := r_h^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0, y_h)(-r_h\frac{\partial}{\partial r_h})^j (r_h\eta_h)^{\alpha} : \\ \mathcal{K}^{s,\beta^{[h]}}(B^{\wedge}_{h-1}) \to \mathcal{K}^{s-\mu,\beta^{[h]}-\mu,\gamma_h-\mu}(B^{\wedge}_{h-1}) \end{aligned} \tag{2.8}$$

for $\eta_h \neq 0$ and $h = 1, \ldots, k$. The Kegel spaces $\mathcal{K}^{s,\beta^{[h]}}(B^{\wedge}_{h-1})$ in (2.8) are of Sobolev smoothness $s \in \mathbb{R}$ and are equipped with weight data $\beta^{[h]} := (\beta_1, \beta_2, \ldots, \beta_h)$. Here $\beta^{[h-1]}$ belongs to the base B_{h-1} and β_h to the axial variable r_h. Corresponding spaces for $k = 2$ in the notation $\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})$ are explicitly defined in [27]; this is compatible with a more indirect definition in [1, 2]. In a forthcoming paper [3] we shall study those spaces once again from the viewpoint of methods from [27] for higher singularity orders k.

The idea of the singular analysis is to employ these symbols for constructing parametrices and solutions to equations $Au = f$ for $A \in \mathrm{Diff}^{\mu}_{\deg}(M)$ under an ellipticity condition defined in terms of symbols in (2.7).

Let us now outline some tools and notation around pseudo-differential operators in common form, locally based on the Fourier transform and, what concerns axial variables $r \in \mathbb{R}_+$, on the Mellin transform. The space of pseudo-differential operators of order $\mu \in \mathbb{R}$ on a closed manifold X will be denoted by $L^{\mu}(X)$. Here we assume $n = \dim X$, and in local coordinates $x \in \mathbb{R}^n$ we express operators A in terms of amplitude functions

$$a(x, x', \xi) \in S^{\mu}(\mathbb{R}^{2n} \times \mathbb{R}^n), \tag{2.9}$$

in standard Hörmander's classes. The choice of notation here is similar to [24] or [25]. Classical elements, i.e., with homogeneous components, are indicated by "cl". If a consideration is valid both in the classical and general case we use subscript "(cl)". Because of coordinate in variance this notation makes sense both for (2.9) as well as on the level of operators. It will be important also to employ parameter-dependent variants with a parameter $\lambda \in \mathbb{R}^d$ which is treated like a covariable. In this case (2.9) is replaced by

$$a(x, x', \xi, \lambda) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{2n} \times \mathbb{R}^{n+d})$$

and the parameter-dependent classes of pseudo-differential operators are denoted by $L^{\mu}_{(\mathrm{cl})}(X; \mathbb{R}^d)$. Locally we also write

$$\mathrm{Op}_x(a)(\lambda)u(x) = \iint e^{i(x-x')\xi} a(x, \xi, \lambda)u(x')dx' đ\xi$$

for $đ\xi = (2\pi)^{-n}d\xi$. We also employ another variant of pseudo-differential operators, namely, on $\mathbb{R}_+$ in the variable r, formulated by means of the Mellin transform, in order to control distribution spaces and operators close to the singularities of underlying spaces, i.e., for $r \to 0$. The Mellin transform is defined by

$$Mu(v) = \int_0^\infty r^v u(r) \frac{dr}{r},$$

first for $u \in C_0^\infty(\mathbb{R}_+)$; then v, the Mellin covariable v runs over $\mathbb{C}$, and $Mu(v)$ is holomorphic, and we have

$$Mu(v)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta),$$

$\Gamma_\beta := \{v \in \mathbb{C} : \mathrm{Re}\, v = \beta\}$ for any real β, uniformly in compact β-intervals. We will also extend M in many ways to larger distribution spaces, e.g., $r^\gamma L^2(\mathbb{R}_+)$ for some weight $\gamma \in \mathbb{R}$, and then

$$M_\gamma u := Mu|_{\Gamma_{\frac{1}{2}-\gamma}}$$

induces an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$$

for every γ. The inverse is of the form

$$M_\gamma^{-1} g(r) = \int_{-\infty}^{\infty} r^{-(\frac{1}{2}-\gamma+i\varrho)} g(\varrho) đ\varrho$$

for any $g \in L^2(\Gamma_{\frac{1}{2}-\gamma})$. This gives rise to the weighted Mellin pseudo-differential operators

$$\mathrm{Op}_M^\gamma(f)u(r) := (M_\gamma^{-1} f(r, r', v) M_\gamma u)(r)$$

where M_γ transforms r' to v, often indicated by $M_{r,r' \to v}$ where r' from f is included in this action and the resulting function of r, v is transformed via $M^{-1}_{\gamma, v \to r}$ to a function of r. Concerning the Mellin symbols we assume classes

$$f(r, r', v) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}). \tag{2.10}$$

The crucial technique is now to admit operator-valued symbols and parameter-dependent Mellin symbols, namely,

$$f(r, r', v) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^{\mu}_{(\mathrm{cl})}(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^d)). \tag{2.11}$$

Then the corresponding Mellin operators induce λ-dependent families of continuous maps

$$\mathrm{Op}_M^{\gamma-n/2}(f)(\lambda):\mathcal{H}^{s,\gamma}(X^\wedge)\to\mathcal{H}^{s-\mu,\gamma-\mu}(X^\wedge)$$

for $X^\wedge:=\mathbb{R}_+\times X$, for any $s\in\mathbb{R}$. The weight shift $-n/2$ has a normalizing effect in terms of L^2 spaces, to be illustrated later on. The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ are locally in variables $x\in\mathbb{R}^n$ defined as completion of $C_0^\infty(\mathbb{R}_+\times\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}^n}\int_{\Gamma_{\frac{n+1}{2}-\gamma}}\langle v,\xi\rangle^{2s}\,|F_{x\to\xi}M_{r\to v}u(v,\xi)|^2 đv đ\xi\right\}^{1/2}$$

for $đv=(2\pi i)^{-1}dv$. Then $\mathcal{H}^{s,\gamma}(X^\wedge)$ is globally defined by the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)}=\left\{\sum\|(\varphi_j u)\circ\chi_j^{-1}\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}\right\}^{1/2}$$

for an open covering of X by coordinate neighborhoods $\{U_1,\dots,U_N\}$, a subordinate partition of unity $\{\varphi_1,\dots,\varphi_N\}$, and charts $\chi_j:U_j\to\mathbb{R}^n$. Another technique to be used here is the kernel cut-off which turns symbols like (2.10) or (2.11) to holomorphic symbols in the complex covariable.

2.2. **Symbols in the group action set-up.** In the edge operator analysis below we employ the concept of group actions in Hilbert spaces and operator-valued symbols with twisted symbolic estimates. This has been introduced in [23], after previous work jointly with Rempel, cf. [16] on concrete boundary value symbol on manifolds with edge and boundary. However, in order to make the present exposition self-contained concerning this point, we briefly recall some crucial notions.

Let H be a separable Hilbert space and $\kappa=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ be a group of isomorphisms $\kappa_\delta:H\to H$ such that $\kappa_\lambda\kappa_{\lambda'}=\kappa_{\lambda\lambda'}$ for all $\lambda,\lambda'\in\mathbb{R}_+$ and $\kappa_1=\mathrm{id}_H$, and that $h\to\kappa_\delta h$ determines a function in $C(\mathbb{R}_+,H)$ for every $h\in H$. Then we briefly say that H is a Hilbert space with group action.

Remark 2.1. *Let H be a Hilbert space with group action κ. Then there are constants $C,M>0$ such that the operator norm can be estimated by*

$$\|\kappa_\delta\|_{\mathcal{L}(H)}\le C(1+\max\{\delta,\delta^{-1}\})^M$$

for all $\delta\in\mathbb{R}_+$.

An explicit proof of this property is given in [13] and [14].

A similar notion can be formulated for a Fréchet space E written as a projective limit $E=\varprojlim_{j\in\mathbb{N}}E^j$ of Hilbert spaces E^j with continuous embeddings $E^j\subset E^0$ for all j. Then E is equipped with κ, when E^j is a Hilbert space with group action which is the restriction of κ on E^0 to E^j.

Techniques of studying pseudo-differential operators on singular manifolds can be formulated in terms of pseudo-differential operators with symbols in

$$S^\mu(\mathbb{R}^q\times\mathbb{R}^q;H,\widetilde{H}) \tag{2.12}$$

where $H, \widetilde{H}$ are Hilbert spaces with group action κ and $\tilde{\kappa}$, respectively. The space (2.12) for $\mu \in \mathbb{R}$ is defined as the set of all $a(y,\eta) \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that the following symbolic estimates hold:

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \le c\,\langle\eta\rangle^{\mu-|\beta|} \tag{2.13}$$

for all $(y,\eta) \in K \times \mathbb{R}^q, K \Subset \mathbb{R}^q$, multi-indices $\alpha, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha,\beta,K) > 0$.
The best constants in estimates (2.13) form a semi-norm system in (2.12) which turn it to a Fréchet space. Recall that standard properties, e.g., on asymptotic summation, etc. are valid in the operator-valued symbol spaces (2.12), too. Note that in the case $H = \widetilde{H} = \mathbb{C}$ and $\kappa_\delta = \mathrm{id}_\mathbb{C}, \tilde{\kappa}_\delta = \mathrm{id}_\mathbb{C}$ we recover Hörmander's symbol spaces. It is also useful to formulate and analogue

$$S^\mu_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; H, \widetilde{H}) \tag{2.14}$$

of classical symbols. The definition employs "twisted homogeneity". We say that an $f(y,\eta) \in C^\infty(\mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \widetilde{H}))$ belongs to

$$S^{(\nu)}(\mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}) \tag{2.15}$$

for some $\nu \in \mathbb{R}$ if

$$f(y,\delta\eta) = \delta^\nu \tilde{\kappa}_\delta f(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+, \eta \neq 0$. Then (2.14) is defines as the set of all $a(y,\eta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; H, \widetilde{H})$ such that there is a sequence $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(\mathbb{R}^q \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$ for $j \in \mathbb{N}$, and for every N and any excision function $\chi(\eta)$ in $\mathbb{R}^q$ we have

$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(\mathbb{R}^q \times \mathbb{R}^q; H, \widetilde{H}).$$

3. Operators on spaces of singularity order 3

3.1. **Iteratively constructed singular operators.** In the article [23] we studied the edge algebra which contains parametrices of elliptic edge-degenerate operators. More details may be found in the above-mentioned monographs [24, 25], as well as in [6, 7, 12–14, 17, 18, 20–22]. The investigations concern various aspects of the operator calculus on a manifold with edge in connection with corresponding model cones of the wedges and observations on asymptotics, see also [8, 11, 16]. The extension of methods to spaces of higher singularity order is a complex programs. The iterative approach suggests more refined structures, and careful functional analytic managements, since the involved data become more and more complex with increasing singularity order, cf. also the monograph [10].

Starting with a space $B_2 \in \mathfrak{M}_2$ we first look at the corresponding parameter-dependent algebra

$$L^\mu(B_2, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^d), \tag{3.1}$$

and we choose a third order edge Y_3 with local variables $y_3 \in \mathbb{R}^{q_3}$ such that we reach spaces $B_3 \in \mathfrak{M}_3$ by studying local wedges $B_2^{\Delta} \times Y_3$ in terms of singular cone algebras of the kind

$$L^\mu(B_2^\wedge, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d_\lambda \setminus \{0\})$$

which depend on (y_3, η_3) and also on $\lambda \in \mathbb{R}^d$ as the range of third-order operator-valued edge symbols of elements in $L^\mu(B_3, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d_\lambda)$. Here $\boldsymbol{g}_j = (\beta_j, \beta_j - \mu)$. This point of view is connected with weighted Kegel spaces on $B_2^\wedge$, denoted by

$$\mathcal{K}^{s,\beta^{[3]}}(B_2^\wedge) \tag{3.2}$$

for weights $\beta^{[3]} = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3, s \in \mathbb{R}$. As is illustrated in [1] those spaces are formulated in terms of operator structures on spaces in $\mathfrak{M}_2$ and also on $\mathfrak{M}_1$, see also the order reducing definition of such spaces from [17, 21]. In order to keep the procedure transparent we briefly outline the method. As noted before, alternative ideas are developed in [27]. Considering $\mathcal{K}^{s,\beta^{[2]}}(B_1^\wedge)$-spaces for a $B_1 \in \mathfrak{M}_1$ and weights $\beta^{[2]} = (\beta_1, \beta_2) \in \mathbb{R}^2$ we analyze holomorphic and parameter-dependent Mellin symbols. Doing this first for any order $\mu \in \mathbb{R}$, we start with local terms containing degenerate symbols. In the present case when $B_2 \in \mathfrak{M}_2$ is the base of the (stretched) cone $B_2^\wedge$ we mainly look at local models like $B_1^\Delta \times \mathbb{R}^{q_2}$ for some compact $B_1 \in \mathfrak{M}_1$. The non-smoothing holomorphic Mellin symbols in $v_2 \in \mathbb{C}$ may be understood in terms of parameter-dependent degenerate Laplacians together with some kernel cut-offs, and the global operators are linear combinations of localized contributions of this kind. Since far from the highest singularities we have degenerate Laplacians of less singularity the arising operators behave as desired because of the invariance of the corresponding homogeneous principal symbols. In order to illustrate the process we first look at the case

$$X^\Delta \times \mathbb{R}^{q_1}$$

for some $X \in \mathfrak{M}_0$. Then we start the construction in local coordinates

$$(r_1, x, y_1),\ r_1 \in \mathbb{R}_+, x \in \mathbb{R}^n, y_1 \in \mathbb{R}^{q_1},$$

and choose a degenerate parameter-dependent symbol

$$p_0(r_1, x, \varrho_1, \xi, \eta_1, \lambda) := \tilde{p}_0(x, r_1\rho_1, \xi, r_1\eta_1, r_1\lambda) \tag{3.3}$$

with $\tilde{p}_0(x, \tilde{\rho}_1, \xi, \tilde{\eta}_1, \tilde{\lambda})$ being an ordinary classical scalar symbol in covariables $(\tilde{\rho}_1, \xi, \tilde{\eta}_1, \tilde{\lambda})$ of order $\mu \in \mathbb{R}$. For instance, we could take

$$p_0(c, r_1, \varrho_1, \xi, \eta_1, \lambda) := \tilde{p}_0(c, r_1\rho_1, \xi, r_1\eta_1, r_1\lambda) \tag{3.4}$$

for

$$\tilde{p}_0(c, \tilde{\rho}_1, \xi, \tilde{\eta}_1, \tilde{\lambda}) := (c + |\tilde{\rho}_1|^2 + |\xi|^2 + |\tilde{\eta}_1|^2 + |\tilde{\lambda}|^2)^{\mu/2} \tag{3.5}$$

for some $c > 0$. The parameter λ is of sufficiently large dimension d. From $\tilde{p}_0$ we pass to a holomorphic function $\tilde{f}_0(v_1, \xi, \tilde{\eta}_1, \tilde{\lambda})$ by applying the Mellin quantization, where $\tilde{\rho}_1$ turns to the complex Mellin-covariable v_1, and then we form an

$$\tilde{h}_1(v_1, \tilde{\eta}_1, \tilde{\lambda}) \in M^\mu_{\mathcal{O}_{v_1}}(X; \mathbb{R}^{q_1+d}_{\tilde{\eta}_1, \tilde{\lambda}}) \tag{3.6}$$

by

$$\tilde{h}_1(v_1,\tilde{\eta}_1,\tilde{\lambda}) := \sum_{\varphi_0 \prec \varphi_0'} \varphi_0 \mathrm{Op}_x(\tilde{f}_0)(v_1,\tilde{\eta}_1,\tilde{\lambda})\varphi_0'. \tag{3.7}$$

Notation $\sum_{\varphi_0 \prec \varphi_0'}$ means that the operator family (3.7) living over X is determined by localizations via charts

$$\chi_j : U_j \to \mathbb{R}^n$$

for a finite open covering $\{U_1,\dots,U_N\}$ of X where φ_0 indicates a subordinate partition of unity (which is basically a sequence $\{\varphi_{0,1},\dots,\varphi_{0,N}\}$ of functions $\varphi_{0,j} \in C_0^\infty(U_j)$), and φ_0' also stands for a sequence $\varphi_{0,j}' \in C_0^\infty(U_j)$, $\varphi_{0,j}' \succ \varphi_{0,j}$. Under the integrals $\mathrm{Op}_x(\tilde{f}_0)$ is to be interpreted as the operator push forward $(\chi_j^{-1})_* \mathrm{Op}_x(\tilde{f}_0)$ with respect to χ_j^{-1}. We hope our abbreviations do not cause confusions. Similar notation is used below in formulas (3.10) and (3.19).

The next steps of formulating higher singular operators are determined by the sequence of singular spaces

$$X^\wedge \Rightarrow B_1 \Rightarrow B_1^\wedge \Rightarrow B_2 \Rightarrow B_2^\wedge \Rightarrow B_3 \Rightarrow \dots \tag{3.8}$$

where B_1 is locally close to $s_1(B_1) = Y_1$ in variables $y_1 \in \mathbb{R}^{q_1}$ modelled on $X^\Delta \times \mathbb{R}^{q_1}$. Moreover, $B_1^\wedge = \mathbb{R}_{+,r_2} \times B_1$, which leads to $B_2 \in \mathfrak{M}_2$ locally close to $s_2(B_2) = Y_2$ in variables $y_2 \in \mathbb{R}^{q_2}$ modelled on $B_1^\Delta \times \mathbb{R}^{q_2}$. Then we pass to $B_2^\wedge = \mathbb{R}_{+,r_3} \times B_2$, which gives rise to $B_3 \in \mathfrak{M}_3$, locally close to $s_3(B_3) = Y_3$ in variables $y_3 \in \mathbb{R}^{q_3}$ modelled on $B_2^\Delta \times \mathbb{R}^{q_3}$, etc..

Now let

$$h_1^\wedge(r_1,v_1,\eta_1,\lambda) := \tilde{h}_1(v_1,r_1\eta_1,r_1\lambda)$$

and form the operator family

$$b_0^\wedge(\eta_1,\lambda) := r_1^{-\mu}\mathrm{Op}_{M_{r_1}}^{\gamma_1-n/2}(h_1^\wedge)(\eta_1,\lambda) + (m_1^\wedge + g_1^\wedge)(\eta_1,\lambda) \in L^\mu(X^\wedge,\boldsymbol{g}_1;\mathbb{R}^{q_1+d}_{\eta_1,\lambda} \setminus \{0\}). \tag{3.9}$$

Moreover, using set (2.3) for $B_1 \in \mathfrak{M}_1$ rather than M_1 we define $L^\mu(B_1,\boldsymbol{g}_1;\mathbb{R}^d_\lambda)$ for $\boldsymbol{g}_1 := (\beta_1,\beta_1-\mu)$

$$b_1(\lambda) := H_1(\lambda) + (M_1+G_1)(\lambda) + (1-\omega_{\mathrm{glob}_1})A_{1,\mathrm{int}}(\lambda)(1-\omega''_{\mathrm{glob}_1}) + C_1(\lambda) \tag{3.10}$$

for

$$A_{1,\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{cl}}(\mathrm{int}\,B_1), \quad \mathrm{int}\,B_1 = B_1 \setminus s_1B_1 \tag{3.11}$$

$$H_1(\lambda) := \omega_{\mathrm{glob}_1} \sum_{\varphi_1 \prec \varphi_1'} \varphi_1 \mathrm{Op}_{y_1}\{\omega_1 r_1^{-\mu}\mathrm{Op}_{M_{r_1}}^{\beta_1-n/2}(h_1)(\eta_1,\lambda)\omega_1'\}\varphi_1'\omega'_{\mathrm{glob}_1}, \tag{3.12}$$

where

$$h_1(r_1,v_1,\eta_1,\lambda) := \tilde{h}_1(r_1,v_1,r_1\eta_1,r_1\lambda),$$

for

$$\tilde{h}_1(r_1,v_1,\tilde{\eta}_1,\tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_{v_1}}(X;\mathbb{R}^{q_1+d}_{\tilde{\eta}_1,\tilde{\lambda}})).$$

Anlogously we set

$$(M_1+G_1)(\lambda) := \omega_{\mathrm{glob}_1} \sum_{\varphi_1 \prec \varphi_1'} \varphi_1 \mathrm{Op}_{y_1}\{\chi(\eta_1,\lambda)(m_1+g_1)(\eta_1,\lambda)\}\varphi_1'\omega_{\mathrm{glob}_1}' \tag{3.13}$$

for some excision function $\chi(\eta_1,\lambda)$ in $\mathbb{R}^{q_1+d}$. The operator functions $(m_1+g_1)(\eta_1,\lambda)$ are of analogous structure as those in [19, formula (1.19) and Definition 1.5]. The difference here is that we do not use capitals and the parameters λ are replaced by (η_1,λ). In particular, the smoothing Mellin amplitude functions are of the form

$$m(\eta_1,\lambda) := r_1^{-\mu}\omega_{\eta_1,\lambda}\sum_{l=0}^{N} r_1^l \sum_{|\alpha|\le l} \mathrm{Op}_M^{\gamma_{l\alpha}-n/2}(f_{l\alpha})(\eta_1,\lambda)^\alpha \omega_{\eta_1,\lambda}' \tag{3.14}$$

for $\omega_{\eta_1,\lambda}(r_1) := \omega(r_1\eta_1, r_1\lambda)$, and the same for cut-offs with prime. Finally,

$$C_1(\lambda) \in L^{-\infty}(B_1, \boldsymbol{g}_1; \mathbb{R}^d_\lambda)$$

is a smoothing family, following a straighforward definition, using the spaces $H^{s,\beta_1}(B_1)$.

Note that in our notation we suppressed y_1-variables, in order not to overload the expressions. Clearly those variables occur as well, and below we speak about spaces

$$C^\infty(\mathbb{R}^{q_1}_{y_1}, L^\mu(X^\wedge, \boldsymbol{g}_1; \mathbb{R}^{q_1+d}_{\eta_1,\lambda}\setminus\{0\})).$$

The definition of the full operator space on the right-hand side of (3.9) is contained in [19, formula (1.2)] while $L^\mu(B_1,\boldsymbol{g}_1;\mathbb{R}^d_\lambda)$ denotes the edge algebra, given in [19, Definition 1.12]. In particular, the meaning of $(M_1+G_1)(\lambda)$ is given there, where we employ a similar operator convention as that in H_1, based on smoothing Mellin- and Green amplitude functions in [19, formula (1.38)], and by $(m_1^\wedge + g_1^\wedge)(\eta_1,\lambda)$ we understand the respective twisted homogeneous principal edge symbols of order μ, i.e.,

$$(m_1^\wedge + g_1^\wedge)(\eta_1,\lambda) = \sigma_1((M_1+G_1)(\lambda)). \tag{3.15}$$

For successively constructing operators on the spaces (3.8) we replace the parameter by

$$\lambda \implies (\tilde\rho_2, \tilde\eta_2, \tilde\lambda) \tag{3.16}$$

and form a higher Mellin symbol

$$h_2^\wedge(r_2, v_2, \eta_2, \lambda) := \tilde h_2(v_2, r_2\eta_2, r_2\lambda), \tag{3.17}$$

where

$$\tilde h_2(v_2,\tilde\eta_2,\tilde\lambda) \in M^\mu_{\mathcal{O}_{v_2}}(B_1,\boldsymbol{g}_1;\mathbb{R}^{q_2+d}_{\tilde\eta_2,\tilde\lambda})$$

is obtained by applying the Mellin quantization to $b_1(\tilde\rho_2,\tilde\eta_2,\tilde\lambda)$ which turns $\tilde\rho_2$ to the complex variable v_2. Next let

$$\begin{aligned} b_2^\wedge(\eta_2,\lambda) &:= r_2^{-\mu}\mathrm{Op}_{M_{r_2}}^{\beta_2-\dim B_1/2}(h_2^\wedge)(\eta_2,\lambda) + (m_2+g_2)(\eta_2,\lambda) \\ &\in L^\mu(B_1^\wedge, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^{q_2+d}_{\eta_2,\lambda}\setminus\{0\}). \end{aligned} \tag{3.18}$$

Moreover, using (2.3) for $B_2 \in \mathfrak{M}_2$ rather than M_2 we define $L^\mu(B_2, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^d_\lambda)$ for $\boldsymbol{g}_i := (\beta_i, \beta_i - \mu),\ i = 1, 2,$

$$b_2(\lambda) := H_2(\lambda) + (M_2 + G_2)(\lambda) + (1 - \omega_{\mathrm{glob}_2}) A_{2,\mathrm{int}}(\lambda)(1 - \omega''_{\mathrm{glob}_2}) + C_2(\lambda) \tag{3.19}$$

for

$$A_{2,\mathrm{int}}(\lambda) \in L^\mu(\mathrm{int}\, B_2, \boldsymbol{g}_1; \mathbb{R}^d_\lambda), \quad \mathrm{int}\, B_2 = B_2 \setminus s_2 B_2 \tag{3.20}$$

$$H_2(\lambda) := \omega_{\mathrm{glob}_2} \sum_{\varphi_2 \prec \varphi'_2} \varphi_2 \mathrm{Op}_{y_2}\{\omega_2 r_2^{-\mu} \mathrm{Op}_{M_{r_2}}^{\beta_2 - \dim B_1/2}(h_2)(\eta_2, \lambda)\omega'_2\}\varphi'_2 \omega'_{\mathrm{glob}_2}, \tag{3.21}$$

where

$$h_2(r_2, v_2, \eta_2, \lambda) := \tilde{h}_2(r_2, v_2, r_2\eta_2, r_2\lambda),$$

for

$$\tilde{h}_2(r_2, v_2, \tilde{\eta}_2, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_{v_2}}(B_1, \boldsymbol{g}_1; \mathbb{R}^{q_2+d}_{\tilde{\eta}_2, \tilde{\lambda}})).$$

Anlogously we set

$$(M_2 + G_2)(\lambda) := \omega_{\mathrm{glob}_2} \sum_{\varphi_2 \prec \varphi'_2} \varphi_2 \mathrm{Op}_{y_2}\{\chi(\eta_2, \lambda)(m_2 + g_2)(\eta_2, \lambda)\}\varphi'_2 \omega'_{\mathrm{glob}_2} \tag{3.22}$$

for some excision function $\chi(\eta_2, \lambda)$ in $\mathbb{R}^{q_2+d}$. The operator functions $(m_2+g_2)(\eta_2, \lambda)$ are of analogous structure as those in [19, formula (1.19) and Definition 1.5]. The difference here is that we do not use capitals and the parameters λ are replaced by (η_2, λ). In particular, the smoothing Mellin amplitude functions are of the form

$$m(\eta_2, \lambda) := r_2^{-\mu} \omega_{\eta_2,\lambda} \sum_{l=0}^{N} r_2^l \sum_{|\alpha| \leq l} \mathrm{Op}_M^{\beta_{l\alpha} - \dim B_1/2}(f_{l\alpha})(\eta_2, \lambda)^\alpha \omega'_{\eta_2,\lambda} \tag{3.23}$$

for $\omega_{\eta_2,\lambda}(r_2) := \omega(r_2\eta_2, r_2\lambda)$, and the same for cut-offs with prime. Finally,

$$C_2(\lambda) \in L^{-\infty}(B_2, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^d_\lambda)$$

is a smoothing family, following a straighforward definition, using the spaces $H^{s,\beta_1,\beta_2}(B_2)$. Then, for the next higher singularity we repeat the game of replacing

the parameter

$$\lambda \quad \text{by} \quad (\tilde{\rho}_3, \tilde{\eta}_3, \tilde{\lambda}) \tag{3.24}$$

and form a Mellin symbol

$$h_3^\wedge(r_3, v_3, \eta_3, \lambda) := \tilde{h}_3(v_3, r_3\eta_3, r_3\lambda), \tag{3.25}$$

where

$$\tilde{h}_3(v_3, \tilde{\eta}_3, \tilde{\lambda}) \in M^\mu_{\mathcal{O}_{v_3}}(B_2, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^{q_3+d}_{\tilde{\eta}_3, \tilde{\lambda}})$$

is obtained by applying the Mellin quantization to $b_2(\tilde{\rho}_3, \tilde{\eta}_3, \tilde{\lambda})$ which turns $\tilde{\rho}_3$ to the complex variable v_3. It follows then

$$\begin{aligned} b_2^\wedge(\eta_3, \lambda) &:= r_3^{-\mu} \mathrm{Op}_{M_{r_3}}^{\gamma_3 - b_2/2}(h_3^\wedge)(\eta_3, \lambda) + (m_3 + g_3)(\eta_3, \lambda) \\ &\in L^\mu(B_2^\wedge, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^{q_3+d}_{\eta_3,\lambda} \setminus \{0\}). \end{aligned} \tag{3.26}$$

Moreover, using (2.3) for $B_3 \in \mathfrak{M}_3$ rather than M_2 we can define $L^\mu(B_3, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d_\lambda)$ for $\boldsymbol{g}_i := (\beta_i, \beta_i - \mu),\ i = 1, 2, 3$, by

$$b_3(\lambda) := H_3(\lambda) + (M_3 + G_3)(\lambda) + (1 - \omega_{\mathrm{glob}_3}) A_{3,\mathrm{int}}(\lambda)(1 - \omega''_{\mathrm{glob}_3}) + C_3(\lambda) \tag{3.27}$$

for

$$A_{3,\mathrm{int}}(\lambda) \in L^\mu(\mathrm{int}\, B_3, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^d_\lambda), \quad \mathrm{int}\, B_3 = B_\setminus s_3 B_3 \tag{3.28}$$

$$H_3(\lambda) := \omega_{\mathrm{glob}_3} \sum_{\varphi_3 \prec \varphi'_3} \varphi_3 \mathrm{Op}_{y_3}\{\omega_3 r_3^{-\mu} \mathrm{Op}_{M_{r_3}}^{\beta_3 - \mathrm{dim} B_2/2}(h_3)(\eta_3, \lambda)\omega'_3\}\varphi'_3 \omega'_{\mathrm{glob}_3}, \tag{3.29}$$

where

$$h_3(r_3, v_3, \eta_3, \lambda) := \tilde{h}_3(r_3, v_3, r_3\eta_3, r_3\lambda),$$

for

$$\tilde{h}_3(r_3, v_3, \tilde{\eta}_3, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_{v_3}}(B_2, \boldsymbol{g}_1, , \boldsymbol{g}_2; \mathbb{R}^{q_3+d}_{\tilde{\eta}_3, \tilde{\lambda}})).$$

Anlogously we set

$$(M_3 + G_3)(\lambda) := \omega_{\mathrm{glob}_3} \sum_{\varphi_3 \prec \varphi'_3} \varphi_3 \mathrm{Op}_{y_3}\{\chi(\eta_3, \lambda)(m_3 + g_3)(\eta_3, \lambda)\}\varphi'_3 \omega'_{\mathrm{glob}_3} \tag{3.30}$$

for some excision function $\chi(\eta_3, \lambda)$ in $\mathbb{R}^{q_3+d}$. The operator functions $(m_3 + g_3)(\eta_3, \lambda)$ are of analogous structure as those before. The difference here is that we do not use capitals and the parameters λ are replaced by (η_3, λ). In particular, the smoothing Mellin amplitude functions are of the form

$$m(\eta_3, \lambda) := r_3^{-\mu} \omega_{\eta_3, \lambda} \sum_{l=0}^{N} r_3^l \sum_{|\alpha| \le l} \mathrm{Op}_M^{\beta_{l\alpha} - \mathrm{dim} B_2/2}(f_{l\alpha})(\eta_3, \lambda)^\alpha \omega'_{\eta_3, \lambda} \tag{3.31}$$

for $\omega_{\eta_3,\lambda}(r_3) := \omega(r_3\eta_3, r_3\lambda)$, and the same for cut-offs with prime. Finally,

$$C_3(\lambda) \in L^{-\infty}(B_3, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d_\lambda)$$

is a smoothing family, following a straighforward definition, using the spaces $H^{s,\beta_1,\beta_2,\beta_3}(B_3)$.

It is now evident how we may iteratively continue the construction for singularities of order ≥ 3.

3.2. **The sequence of higher operator algebras.** After the constructions of the preceding Subsection 3.1 the sequence of singular spaces (3.8) induces a sequence of operator spaces

$$\begin{aligned} &C^\infty(\mathbb{R}^{q_1}_{y_1}, L^\mu(X^\wedge, \boldsymbol{g}_1; \mathbb{R}^{q_1+d}_{\eta_1,\lambda} \setminus \{0\})) \Rightarrow L^\mu(B_1, \boldsymbol{g}_1; \mathbb{R}^d) \\ &\Rightarrow C^\infty(\mathbb{R}^{q_2}_{y_2}, L^\mu(B_1^\wedge, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^{q_2+d}_{\eta_2,\lambda} \setminus \{0\})) \Rightarrow L^\mu(B_2, \boldsymbol{g}_1, \boldsymbol{g}_2; \mathbb{R}^d) \\ &\Rightarrow C^\infty(\mathbb{R}^{q_3}_{y_3}, L^\mu(B_2^\wedge, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^{q_3+d}_{\eta_3,\lambda} \setminus \{0\})) \Rightarrow L^\mu(B_3, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d) \\ &\Rightarrow \dots \end{aligned} \tag{3.32}$$

where

$$L^\mu(B_j, \boldsymbol{g}_1, \cdots, \boldsymbol{g}_j; \mathbb{R}^d) \to C^\infty(\mathbb{R}^{q_j}_{y_j}, L^\mu(B_{j-1}^\wedge, \boldsymbol{g}_1, \cdots, \boldsymbol{g}_j; \mathbb{R}^{q_j+d}_{\eta_j,\lambda} \setminus \{0\})) \tag{3.33}$$

are the respective higher parameter-dependent principal edge symbol maps, indicated for differential operators in Subsection 1.1. By definition the non-smoothing holomorphic Mellin symbols

$$h_j(r_j, y_j, v_j, \eta_j, \lambda) = \tilde{h}_j(r_j, y_j, v_j, r_j\eta_j, r_j\lambda)$$

may also depend on y_j. On the other hand, when

$$\tilde{h}_j(y_j, v_j, \tilde{\eta}_j, \tilde{\lambda})$$

does not explicitly depend on r_j, and we form

$$h_j^{\wedge}(r_j, y_j, v_j, \eta_j, \lambda) = \tilde{h}_j(y_j, v_j, r_j\eta_j, r_j\lambda),$$

then the corresponding operators

$$k_j^{\wedge}(y_j, \eta_j, \lambda) := r_j^{-\mu}\mathrm{Op}_{M_{r_j}}^{\gamma_j - b_{j-1}/2}(h_j^{\wedge})(y_j, \eta_j, \lambda)$$

satisfy the homogeneity relation

$$k_j^{\wedge}(y_j, \delta\eta_j, \delta\lambda) = \delta^{\mu}\kappa_\delta k_j^{\wedge}(y_j, \eta_j, \lambda)\kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$ and $(\eta_j, \lambda) \neq 0$, where

$$\kappa_\delta u(r_j, \cdot) = \delta^{(1+\dim B_j)/2} u(\delta r_j, \cdot).$$

The other ingredients $r_j^l \sum_{|\alpha|\le l} \mathrm{Op}_M^{\beta_{l\alpha} - \dim B_j/2}(f_{l\alpha})(\eta_j, \lambda)^{\alpha}(m_j + g_j)(y_j, \eta_j, \lambda)$ involved in

$$C^\infty(\mathbb{R}^{q_j}_{y_j}, L^\mu(B_{j-1}^\wedge, \boldsymbol{g}_1, \cdots, \boldsymbol{g}_j; \mathbb{R}^{q_j+d}_{\eta_j,\lambda} \setminus \{0\}))$$

are classical symbols and belong to

$$S^\mu_{\mathrm{cl}}(\mathbb{R}^{q_j} \times \mathbb{R}^{q_j+d}; \mathcal{K}^{s,\beta^{[j]}}(B_{j-1}^\wedge), \mathcal{K}^{s-\mu,\beta^{[j]}-\mu}(B_{j-1}^\wedge)),$$

$\beta^{[j]} := (\beta_1, \ldots, \beta_j)$, $\beta^{[j]} - \mu := (\beta_1 - \mu, \ldots, \beta_j - \mu)$, but in our definition not necessarily homogeneous. The corresponding principal symbolic map (3.33) maps to the homogeneous principal part of order μ, and we now denote the image under (3.33), by

$$C^\infty(\mathbb{R}^{q_j}_{y_j}, L^{(\mu)}(B_{j-1}^\wedge, \boldsymbol{g}_1, \cdots, \boldsymbol{g}_j; \mathbb{R}^{q_j+d}_{\eta_j,\lambda} \setminus \{0\})).$$

Remark 3.1. *The parameter-dependent homogeneous principal symbols σ_j for $j = 1, 2, 3$ together with the scalar (degenerate) interior symbol σ_0 determine parameter-dependent ellipticity of the respective operator family $b_3(\lambda)$ by requiring bijectivity of all components, for $j > 0$ as operator families*

$$\mathcal{K}^{s,\beta^{[j]}}(B_{j-1}^\wedge) \to \mathcal{K}^{s-\mu,\beta^{[j]}-\mu}(B_{j-1}^\wedge).$$

Theorem 3.2. *Let $B_3 \in \mathfrak{M}_3$ be compact, and $b_3(\lambda) \in L^\mu(B_3, \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3; \mathbb{R}^d)$ be parameter-dependent elliptic. Then*

$$b_3(\lambda) : H^{s,\beta^{[3]}}(B_3) \to H^{s-\mu,\beta^{[3]}-\mu}(B_3) \tag{3.34}$$

is a family of Fredholm operators for all $s \in \mathbb{R}$ which become isomorphisms for every sufficiently large $|\lambda|$.

Proof. The ideas are close to the methods from [2]. We construct a parameter-dependent parametrix

$$b_3^{(-1)}(\lambda) \in L^{-\mu}(B_3, \boldsymbol{g}_1^{(-1)}, \boldsymbol{g}_2^{(-1)}, \boldsymbol{g}_3^{(-1)}; \mathbb{R}^d)$$

for $\boldsymbol{g}_j^{(-1)} := (\beta_j - \mu, \beta_j)$ which is an iterative process, using corresponding parametrices for singularity orders $0, 1, 2$. The Fredholm property then follows from the fact, that remainders are compact in the respective spaces. Since the two-sided remainders are Schwartz in the parameter with values in smoothing operators along B_3 the invertibility is a consequence of a Neumann series argument. □

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