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STRONG CONVERGENCE OF THE SPLIT EQUALITY FIXED POINT PROBLEM FOR ASYMPTOTICALLY QUASI-PSEUDOCONTRACTIVE OPERATORS

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Dedicated to Professor Do Sang Kim on the occasion of his 65th birthday

Abstract. In this paper, we consider a split equality fixed point problem which includes split feasibility problem, split equality problem, split fixed point problem etc, as special cases. Furthermore we propose a new algorithm combining viscosity approximation methods for solving the split equality fixed point problem with asymptotically quasi-pseudocontractive operators, and establish a strong convergence theorem. The results obtained in this paper generalize and improve the recent ones announced by many others.

1. Introduction

Let H_1, H_2, H_3 be real Hilbert spaces, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two nonlinear mappings with nonempty fixed point sets $F(T)$ and $F(S)$, $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. In 2013, Moudafi and Al-Shemas [5] proposed a new split feasibility problem, which is called the split equality fixed point problem (SEFP) as follows:

$$\text{find } x^* \in F(T),\ y^* \in F(S) \text{ such that } Ax^* = By^*. \tag{1.1}$$

If $H_2 = H_3$ and $B = I$, then the SEFP reduces to the split common fixed-point problem (SCFP):

$$\text{find } x^* \in F(T) \text{ such that } Ay^* \in F(S).$$

Furthermore, if T and S are projection operators, i.e., $T = P_C$ and $S = P_Q$, then the SCFP reduces to the split feasibility problem(SFP):

$$\text{find } x^* \in C \text{ and } Ay^* \in Q.$$

These problems have been extensively studied in recent years; see [1], [2], [3], [5], [7], [8], [10], [12], [13], [14] for instance.

For solving the SEFP (1.1), Moudafi and Al-Shemas [5] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = T(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = S(y_k + \gamma_k B^*(Ax_k - By_k)) \end{cases}$$

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for firmly quasi-nonexpansive mappings T and S, where $\gamma_k \in (\epsilon, \frac{2}{\lambda_A+\lambda_B} - \epsilon)$, λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively.

Recently, Chang et al. [1] considered the following iterative algorithm for solving the SEFP (1.1):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi)I + \xi T((1-\eta)I + \eta T))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi)I + \xi S((1-\eta)I + \eta S))v_n, \end{cases} \tag{1.2}$$

where T and S are quasi-pseudocontractive operators. Furthermore, they established the weak and strong convergence of the scheme (1.2).

Note that the class of quasi-pseudocontractive operators which properly includes the classes of quasi-nonexpansive operators, directed operators and demicontractive operators, is more desirable for example in fixed point methods in image recovery where in many cases, and the class of asymptotically quasi-pseudocontractive operators is an important generalization of it.

The purpose of this paper is to extend the results in [1] from quasi-pseudocontractive operators to asymptotically quasi-pseudocontractive operators. We construct a new iterative algorithm combining viscosity approximation methods and establish its strong convergence result.

2. Preliminaries

Throughout this paper, let $\mathbb{R}$ be the set of real numbers. We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively, and $F(T)$ denotes the set of the fixed points of a mapping T. We use $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_n\}$.

Let C be a nonempty closed convex subset of a Hilbert space H. The metric (or nearest point) projection P_C from H onto C is defined as follows: Given $x \in H$, the unique point $P_C x \in C$ satisfies the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|.$$

It is well known [9] that P_C is a nonexpansive mapping and is characterized by the inequality

$$P_C x \in C,\ \langle x - P_C x, y - P_C x\rangle \le 0,\ \forall\, y \in C. \tag{2.1}$$

Definition 2.1. An operator $T : C \to C$ is said to be

(i) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x,\ y \in C,$$

especially, if $L \in (0, 1)$, T is said to be a contraction with constant L;

(ii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|$$

for all $x,\ y \in C$ and for all $n \ge 1$.

Definition 2.2. An operator $T : C \to C$ is said to be

(i) firmly quasi-nonexpansive if

$$\|Tx - q\|^2 \le \|x - q\|^2 - \|(I - T)x\|^2, \quad \forall\, x \in C,\ q \in F(T);$$

(ii) μ-strictly pseudocontractive if there exists $\mu \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \mu\|x - Tx - (y - Ty)\|^2, \quad \forall\, x,\ y \in C;$$

(iii) μ-demicontractive if $F(T) \neq \emptyset$ and the exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \le \|x - q\|^2 + \mu\|x - Tx\|^2, \quad \forall\, x \in H,\ q \in F(T);$$

(iv) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall\, x,\ y \in C.$$

Especially, T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|T^n x - q\| \le k_n\|x - q\|, \quad \forall\, x \in C, q \in F(T).$$

Definition 2.3. An operator $T : C \to C$ is said to be

(i) pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \ \forall\, x,\ y \in C.$$

It is well-known that T is pseudocontractive if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2, \quad \forall\, x,\ y \in C;$$

(ii) quasi-pseudocontractive if $F(T) \neq \emptyset$ and

$$\|Tx - q\|^2 \le \|x - q\|^2 + \|x - Tx\|^2, \quad \forall\, x \in C,\ q \in F(T).$$

It is obvious that the class of quasi-pseudocontractive operators includes the class of demicontractive operators as its special case;

(iii) asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \le k_n\|x - y\|^2, \quad \forall\, x,\ y \in C.$$

It is easy to see that T is asymptotically pseudocontractive if and only if

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x - y\|^2 + \|x - T^n x - (y - T^n y)\|^2, \quad \forall\, x,\ y \in C;$$

(iv) asymptotically quasi-pseudocontractive if $F(T) \neq \emptyset$ and if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - q, x - q\rangle \le k_n\|x - q\|^2, \quad \forall\, x \in C,\ q \in F(T). \tag{2.2}$$

It is clear that T is asymptotically quasi-pseudocontractive if and only if

$$\|T^n x - q\|^2 \le (2k_n - 1)\|x - q\|^2 + \|x - T^n x\|^2, \quad \forall\, x \in C,\ q \in F(T).$$

It is worth noting that the class of asymptotically quasi-pseudocontractive operators is more general than the class of asymptotically quasi-nonexpansive operators.

Definition 2.4. An operator $T : H \to H$ is said to be demiclosed at 0 if, for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

Definition 2.5. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $F : C \to H$ is said to be

(i) monotone if $\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall\, x, y \in C$;
(ii) strictly monotone if $\langle Fx - Fy, x - y\rangle > 0, \quad \forall\, x, y \in C, x \neq y$;
(iii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall\, x, y \in C.$$

Definition 2.6. ([15]) A mapping $T : C \to C$ is called asymptotically regular if for any bounded $\widetilde{C}$ of C, there holds the following equality:

$$\lim_{n\to\infty} \sup_{x\in\widetilde{C}} \|T^{n+1}x - T^n x\| = 0.$$

In any Hilbert space, the following conclusion holds:

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2. \tag{2.3}$$

Lemma 2.7. *Let C be a nonempty closed convex subset of H and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically quasi-pseudocontractive operator. Then $F(T)$ is a closed convex subset of C.*

Proof. Since T is L-Lipschitzian, i.e., T is continuous, $F(T)$ is closed. Next we only need to prove that $F(T)$ is convex. To this aim, let $p_1, p_2 \in F(T)$ and write $p = tp_1 + (1 - t)p_2$ for any $t \in (0, 1)$. We plan to show $p = Tp$, i.e., $p \in F(T)$. Take $\alpha \in (0, \frac{1}{1+L})$ and define $y_{\alpha,n} = (1 - \alpha)p + \alpha T^n p$. Since T is L-Lipschitzian, we have

$$\begin{aligned}
& \langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle \\
= \;& \langle p - y_{\alpha,n}, (p - y_{\alpha,n}) - (T^n p - T^n y_{\alpha,n})\rangle \\
\leq \;& \|p - y_{\alpha,n}\|^2 + \|p - y_{\alpha,n}\|\|T^n p - T^n y_{\alpha,n}\| \\
\leq \;& (1 + L)\|p - y_{\alpha,n}\|^2 \\
= \;& (1 + L)\alpha^2\|p - T^n p\|^2.
\end{aligned} \tag{2.4}$$

Hence, for any $q \in F(T)$, from (2.2) and (2.4) we obtain

$$\begin{aligned}
\|p - T^n p\|^2 &= \langle p - T^n p, p - T^n p\rangle \\
&= \frac{1}{\alpha}\langle p - y_{\alpha,n}, p - T^n p\rangle \\
&= \frac{1}{\alpha}\langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle \\
&\quad + \frac{1}{\alpha}\langle p - y_{\alpha,n}, y_{\alpha,n} - T^n y_{\alpha,n}\rangle \\
&= \frac{1}{\alpha}\langle p - y_{\alpha,n}, (p - T^n p) - (y_{\alpha,n} - T^n y_{\alpha,n})\rangle \\
&\quad + \frac{1}{\alpha}\langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle \\
&\quad + \frac{1}{\alpha}\langle q - y_{\alpha,n}, y_{\alpha,n} - q\rangle + \frac{1}{\alpha}\langle q - y_{\alpha,n}, q - T^n y_{\alpha,n}\rangle \\
&\leq \alpha(1 + L)\|p - T^n p\|^2 + \frac{1}{\alpha}\langle p - q, y_{\alpha,n} - T^n y_{\alpha,n}\rangle
\end{aligned}$$

$$+\frac{1}{\alpha}(k_n-1)\|y_{\alpha,n}-q\|^2,$$

which implies that

$$\begin{aligned} &\alpha(1-\alpha(1+L))\|p-T^np\|^2 \\ \leq\ &\langle p-q, y_{\alpha,n}-T^ny_{\alpha,n}\rangle+(k_n-1)\|y_{\alpha,n}-q\|^2. \end{aligned} \tag{2.5}$$

Since T is L-Lipschitzian, we get

$$\begin{aligned} \|y_{\alpha,n}-q\|^2 &= \|(1-\alpha)(p-q)+\alpha(T^np-q)\|^2 \\ &\leq (1-\alpha)\|p-q\|^2+\alpha\|T^np-q\|^2 \\ &\leq (1-\alpha+\alpha L^2)\|p-q\|^2 = M\|p-q\|^2. \end{aligned} \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} &\alpha(1-\alpha(1+L))\|p-T^np\|^2 \\ \leq\ &\langle p-q, y_{\alpha,n}-T^ny_{\alpha,n}\rangle+(k_n-1)M\|p-q\|^2. \end{aligned} \tag{2.7}$$

Taking $q=p_1$ and $q=p_2$ in (2.7), multiplying t and $1-t$ on both sides of (2.7), respectively, and adding up yields

$$\alpha(1-\alpha(1+L))\|p-T^np\|^2 \leq (k_n-1)M(\|p-p_1\|^2+\|p-p_2\|^2)\to 0,$$

which implies that $T^np\to p$, and the continuity of T gives

$$Tp=\lim_{n\to\infty}T^{n+1}p=p,$$

completing the proof. □

Remark 2.8. Comparing with Lemma 1.3 in [6], we do not require that the subset C of H is bounded in Lemma 2.7.

Lemma 2.9 ([13])**.** *Let H be a real Hilbert space. Let $T:H\to H$ be a uniformly L-Lipschitzian asymptotically pseudocontractive operator with coefficient k_n. If $0<\zeta<\eta<\frac{1}{\sqrt{k_n^2+L^2}+k_n}$ for all $n\geq 1$, then*

$$\|(1-\zeta)x+\zeta T^n((1-\eta)I+\eta T^n)x-x^*\|^2 \leq [1+2(k_n-1)\zeta+2(k_n-1)(2k_n-1)\eta\zeta]\|x-x^*\|^2,$$

for all $x\in H$ and $x^*\in F(T)$.

Remark 2.10. From the proof of Proposition 3.2 in [13], we know that if the asymptotically pseudocontractive operator T is an asymptotically quasi-pseudocontractive operator with coefficient k_n, the conclusion of Lemma 2.9 still holds.

Lemma 2.11. *([4]). Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1}\leq (1-\lambda_n)s_n+\lambda_n\delta_n, \\ s_{n+1}\leq s_n-\gamma_n+\mu_n, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in (0,1), $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\mu_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\lambda_n=\infty$;
(ii) $\lim_{n\to\infty}\mu_n=0$;
(iii) $\lim_{l\to\infty}\gamma_{n_l}=0$ *implies* $\limsup_{l\to\infty}\delta_{n_l}\leq 0$ *for any subsequence* $\{n_l\}\subset\{n\}$.

Then $\lim_{n\to\infty}s_n=0$.

3. Main results

In this section, we always assume that H_1, H_2, H_3 are real Hilbert spaces. Let $T : H_1 \to H_1$ be a uniformly L_1-Lipschitzian asymptotically quasi-pseudocontractive operator with coefficient $k_n^{(1)}$ and $S : H_2 \to H_2$ be a uniformly L_2-Lipschitzian asymptotically quasi-pseudocontractive operator with coefficient $k_n^{(2)}$, $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Assume that T and S are asymptotically regular on H_1 and H_2, respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with their adjoints A^* and B^*, respectively. In the sequel, we use Γ to stand for the solution set of the SEFP (1.1), i.e.,

$$\Gamma = \{(x, y)|\ x \in F(T),\ y \in F(S) \text{ such that } Ax = By\}.$$

Put $H^* = H_1 \times H_2$. Define the inner product of H^* as follows:

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle,\ \forall\ (x_1, y_1), (x_2, y_2) \in H^*.$$

It is easy to see that H^* is also a real Hilbert space and

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}},\ \forall\ (x, y) \in H^*.$$

Algorithm 3.1. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants $\sigma_1,\ \sigma_2 \in (0, 1)$ and choose $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. Take arbitrary $x_1 \in H_1, y_1 \in H_2$. Assume that the nth iterate $x_n \in H_1,\ y_n \in H_2$ has been constructed, then we calculate $(n + 1)$th iterate $(x_{n+1},\ y_{n+1})$ via the formula

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ \widetilde{x}_n = \alpha_n x_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n T^n((1 - \eta_n)I + \eta_n T^n))u_n, \\ x_{n+1} = \beta_n f_1(\widetilde{x}_n) + (1 - \beta_n)\widetilde{x}_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ \widetilde{y}_n = \alpha_n y_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n S^n((1 - \eta_n)I + \eta_n S^n))v_n, \\ y_{n+1} = \beta_n f_2(\widetilde{y}_n) + (1 - \beta_n)\widetilde{y}_n. \end{cases} \tag{3.1}$$

Put $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$, and $L = \max\{L_1, L_2\}$. Based on the assumption on the operators T and S, we can easy to see that S and T are both uniformly L-Lipschitzian asymptotically quasi-pseudocontractive operators with coefficient k_n.

Theorem 3.2. *Let $H_1, H_2, H_3, A, B, S, T$ and Γ be the same as above and $\sigma_1,\ \sigma_2 \in (0, \frac{1}{\sqrt{2}})$. If $I - T$ and $I - S$ are demiclosed at 0 and the following conditions are satisfied:*

(a) $\gamma_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon),\quad \forall n \geq 1$;
(b) $0 < a^* < \xi_n < \eta_n < b^* < \frac{1}{\sqrt{k_n^2+L^2}+k_n}$;
(c) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(d) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(e) $\Sigma_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$,

where λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively and $\varepsilon > 0$ is small enough. Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$ which is the unique solution of the following variational inequality problem (VIP)

$$\langle ((I - f_1)x^*, (I - f_2)y^*), (x, y) - (x^*, y^*)\rangle \geq 0,\ \forall\ (x, y) \in \Gamma. \tag{3.2}$$

Proof. By Lemma 2.7 we have $F(T)$ and $F(S)$ are both closed convex sets, and since A and B are both bounded linear, it is easy to see that Γ is a closed convex subset in H^*. Be similar to the proof of Step 1 of Theorem 3.3 in [11], we have the VIP (3.2) has a unique solution $(x^*, y^*) \in \Gamma$. Now we divide the following proof into several steps.

Step I. We show that $\{(x_n, y_n)\}$ is bounded. Since $(x^*, y^*) \in \Gamma$, then $x^* \in F(T)$, $y^* \in F(S)$ such that $Ax^* = By^*$. By (3.1) and the definitions of λ_A and λ_B, we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|x_n - x^*\|^2 - 2\gamma_n\langle x_n - x^*, A^*(Ax_n - By_n)\rangle + \gamma_n^2\|A^*(Ax_n - By_n)\|^2\\ &= \|x_n - x^*\|^2 - 2\gamma_n\langle Ax_n - Ax^*, Ax_n - By_n\rangle\\ &\quad + \gamma_n^2\langle Ax_n - By_n, AA^*(Ax_n - By_n)\rangle\\ &\le \|x_n - x^*\|^2 - 2\gamma_n\langle Ax_n - Ax^*, Ax_n - By_n\rangle + \gamma_n^2\lambda_A\|Ax_n - By_n\|^2,\\ \|v_n - y^*\|^2 &= \|y_n - y^*\|^2 + 2\gamma_n\langle y_n - y^*, B^*(Ax_n - By_n)\rangle + \gamma_n^2\|B^*(Ax_n - By_n)\|^2\\ &= \|y_n - y^*\|^2 + 2\gamma_n\langle By_n - By^*, Ax_n - By_n\rangle\\ &\quad + \gamma_n^2\langle Ax_n - By_n, BB^*(Ax_n - By_n)\rangle\\ &\le \|y_n - y^*\|^2 + 2\gamma_n\langle By_n - By^*, Ax_n - By_n\rangle + \gamma_n^2\lambda_B\|Ax_n - By_n\|^2.\end{aligned}$$

By adding the above two inequalities and noticing $Ax^* = By^*$, we have

$$\begin{aligned}\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \le{}& \|x_n - x^*\|^2 + \|y_n - y^*\|^2\\ &- \gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2.\end{aligned} \tag{3.3}$$

Putting $\sigma = \max\{\sigma_1, \sigma_2\}$, then we have $\sigma \in (0, \frac{1}{\sqrt{2}})$. And put

$$\begin{aligned}K_n &:= (1 - \xi_n)I + \xi_n T^n((1 - \eta_n)I + \eta_n T^n),\\ G_n &:= (1 - \xi_n)I + \xi_n S^n((1 - \eta_n)I + \eta_n S^n).\end{aligned}$$

By Algorithm 3.1, (2.3) and Lemma 2.9 we obtain

$$\begin{aligned}\|\widetilde{x}_n - x^*\|^2 &= \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|K_n u_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|K_n u_n - x_n\|^2\\ &\le \alpha_n\|x_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|K_n u_n - x_n\|^2\\ &\quad + (1 - \alpha_n)[1 + 2(k_n - 1)\xi_n + 2(k_n - 1)(2k_n - 1)\eta_n\xi_n]\|u_n - x^*\|^2,\\ \|\widetilde{y}_n - y^*\|^2 &= \alpha_n\|y_n - y^*\|^2 + (1 - \alpha_n)\|G_n v_n - y^*\|^2 - \alpha_n(1 - \alpha_n)\|G_n v_n - y_n\|^2\\ &\le \alpha_n\|y_n - y^*\|^2 - \alpha_n(1 - \alpha_n)\|G_n v_n - y_n\|^2\\ &\quad + (1 - \alpha_n)[1 + 2(k_n - 1)\xi_n + 2(k_n - 1)(2k_n - 1)\eta_n\xi_n]\|v_n - y^*\|^2.\end{aligned}$$

Adding the above two inequalities, from (3.3) we have

$$\begin{aligned}&\|\widetilde{x}_n - x^*\|^2 + \|\widetilde{y}_n - y^*\|^2\\ &\le \alpha_n(\|x_n - x^*\|^2 + \|y_n - y^*\|^2)\\ &\quad - \alpha_n(1 - \alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) + (1 - \alpha_n)\\ &\quad \times [1 + 2(k_n - 1)\xi_n + 2(k_n - 1)(2k_n - 1)\eta_n\xi_n](\|u_n - x^*\|^2 + \|v_n - y^*\|^2)\\ &\le \alpha_n(\|x_n - x^*\|^2 + \|y_n - y^*\|^2)\end{aligned}$$

$$
\begin{aligned}
&- \alpha_n(1-\alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) + (1-\alpha_n) \\
&\times [1 + 2(k_n-1)\xi_n + 2(k_n-1)(2k_n-1)\eta_n\xi_n](\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
&- (1-\alpha_n)[1 + 2(k_n-1)\xi_n \\
&+ 2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2 \\
= {}& \{1 + (1-\alpha_n)(k_n-1)[2\xi_n + 2(2k_n-1)\eta_n\xi_n]\}s_n \\
&- \alpha_n(1-\alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) \\
&- (1-\alpha_n)[1 + 2(k_n-1)\xi_n \\
&+ 2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2 \\
\le {}& (1 + (k_n-1)M)s_n - \alpha_n(1-\alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) \\
&- (1-\alpha_n)[1 + 2(k_n-1)\xi_n \\
&+ 2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2 \quad (3.4) \\
\le {}& (1 + (k_n-1)M)s_n, \quad (3.5)
\end{aligned}
$$

where $s_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, $M = \sup_{n\ge1}\{2\xi_n + 2(2k_n-1)\eta_n\xi_n\}$. By Algorithm 3.1 we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n\|f_1(\widetilde{x}_n) - x^*\|^2 + (1-\beta_n)\|\widetilde{x}_n - x^*\|^2 \\
&\le \beta_n(2\sigma_1^2\|\widetilde{x}_n - x^*\|^2 + 2\|f_1(x^*) - x^*\|^2) + (1-\beta_n)\|\widetilde{x}_n - x^*\|^2 \\
&\le [1 - \beta_n(1-2\sigma^2)]\|\widetilde{x}_n - x^*\|^2 + 2\beta_n\|f_1(x^*) - x^*\|^2, \quad (3.6) \\
\|y_{n+1} - y^*\|^2 &\le \beta_n\|f_2(\widetilde{y}_n) - y^*\|^2 + (1-\beta_n)\|\widetilde{y}_n - y^*\|^2 \\
&\le \beta_n(2\sigma_2^2\|\widetilde{y}_n) - y^*\|^2 + 2\|f_2(y^*) - y^*\|^2) + (1-\beta_n)\|\widetilde{y}_n - y^*\|^2 \\
&\le [1 - \beta_n(1-2\sigma^2)]\|\widetilde{y}_n - y^*\|^2 + 2\beta_n\|f_2(y^*) - y^*\|^2.
\end{aligned}
$$

Adding the above two inequalities, from (3.5) we get

$$
\begin{aligned}
s_{n+1} &\le [1 - \beta_n(1-2\sigma^2)](\|\widetilde{x}_n - x^*\|^2 + \|\widetilde{y}_n - y^*\|^2) \\
&\quad + 2\beta_n(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \\
&\le [1 - \beta_n(1-2\sigma^2)]s_n + 2\beta_n(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \\
&\quad + (k_n-1)Ms_n.
\end{aligned}
$$

Then by induction, we have

$$
\begin{aligned}
s_{n+1} &\le [1 - \beta_n(1-2\sigma^2)]s_n + \beta_n(1-2\sigma^2)\frac{2(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)}{1-2\sigma^2} \\
&\quad + (k_n-1)Ms_n \\
&\le [1 + (k_n-1)M]\max\{s_n, \frac{2(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)}{1-2\sigma^2}\} \\
&\vdots \\
&\le \Pi_{i=1}^n[1 + (k_i-1)M]\max\{s_1, \frac{2(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)}{1-2\sigma^2}\}.
\end{aligned}
$$

Therefore, by the condition (e) we have $\{\Pi_{i=1}^{n}[1+(k_i-1)M]\}$ is convergent. So $\{s_n\}$ is bounded, furthermore, $\{x_n\}$ and $\{y_n\}$ are both bounded. So are $\{u_n\}$, $\{v_n\}$,$\{\widetilde{x}_n\}$, $\{\widetilde{y}_n\}$,$\{f_1(\widetilde{x}_n)\}$ and $\{f_2(\widetilde{y}_n)\}$.

Step II. We show that $\lim_{l\to\infty}\|u_{n_l}-Tu_{n_l}\|=0$ and $\lim_{l\to\infty}\|v_{n_l}-Sv_{n_l}\|=0$, where $\{n_l\}$ is any subsequence of $\{n\}$ such that $\lim_{l\to\infty}\nu_{n_l}=0$.

Indeed, by Algorithm 3.1 we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \beta_n^2\|f_1(\widetilde{x}_n)-x^*\|^2+2\beta_n(1-\beta_n)\langle f_1(\widetilde{x}_n)-x^*,\widetilde{x}_n-x^*\rangle\\ &\quad+(1-\beta_n)^2\|\widetilde{x}_n-x^*\|^2\\ &\le \beta_n^2\|f_1(\widetilde{x}_n)-x^*\|^2+\beta_n(1-\beta_n)(\|f_1(\widetilde{x}_n)-f_1(x^*)\|^2+\|\widetilde{x}_n-x^*\|^2)\\ &\quad+2\beta_n(1-\beta_n)\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle+(1-\beta_n)^2\|\widetilde{x}_n-x^*\|^2\\ &\le \beta_n^2\|f_1(\widetilde{x}_n)-x^*\|^2+\beta_n(1-\beta_n)(\sigma^2\|\widetilde{x}_n-x^*\|^2+\|\widetilde{x}_n-x^*\|^2)\\ &\quad+2\beta_n(1-\beta_n)\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle+(1-\beta_n)^2\|\widetilde{x}_n-x^*\|^2\\ &= \{1-\beta_n[1-(1-\beta_n)\sigma^2]\}\|\widetilde{x}_n-x^*\|^2\\ &\quad+\beta_n(\beta_n\|f_1(\widetilde{x}_n)-x^*\|^2+2(1-\beta_n)\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle).\end{aligned}\tag{3.7}$$

Similarly, we have

$$\begin{aligned}\|y_{n+1}-y^*\|^2 &\le \{1-\beta_n[1-(1-\beta_n)\sigma^2]\}\|\widetilde{y}_n-y^*\|^2\\ &\quad+\beta_n(\beta_n\|f_2(\widetilde{y}_n)-y^*\|^2+2(1-\beta_n)\langle f_2(y^*)-y^*,\widetilde{y}_n-y^*\rangle).\end{aligned}\tag{3.8}$$

It follows from (3.5), (3.7) and (3.8) that

$$\begin{aligned}s_{n+1} &\le \{1-\beta_n[1-(1-\beta_n)\sigma^2]\}(\|\widetilde{x}_n-x^*\|^2+\|\widetilde{y}_n-y^*\|^2)\\ &\quad+\beta_n[\beta_n(\|f_1(\widetilde{x}_n)-x^*\|^2+\|f_2(\widetilde{y}_n)-y^*\|^2)\\ &\quad+2(1-\beta_n)(\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle+\langle f_2(y^*)-y^*,\widetilde{y}_n-y^*\rangle)]\\ &\le \{1-\beta_n[1-(1-\beta_n)\sigma^2]\}s_n+\beta_n[\beta_n(\|f_1(\widetilde{x}_n)-x^*\|^2+\|f_2(\widetilde{y}_n)-y^*\|^2)\\ &\quad+2(1-\beta_n)(\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle\\ &\quad+\langle f_2(y^*)-y^*,\widetilde{y}_n-y^*\rangle)]+(k_n-1)Ms_n\\ &= (1-\lambda_n)s_n+\lambda_n\delta_n,\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}\lambda_n &= \beta_n[1-(1-\beta_n)\sigma^2],\\ \delta_n &= \frac{\beta_n(\|f_1(\widetilde{x}_n)-x^*\|^2+\|f_2(\widetilde{y}_n)-y^*\|^2)}{1-(1-\beta_n)\sigma^2}+\frac{(k_n-1)Ms_n}{\beta_n[1-(1-\beta_n)\sigma^2]}\\ &\quad+\frac{2(1-\beta_n)(\langle f_1(x^*)-x^*,\widetilde{x}_n-x^*\rangle+\langle f_2(y^*)-y^*,\widetilde{y}_n-y^*\rangle)}{1-(1-\beta_n)\sigma^2}.\end{aligned}$$

On the other hand, by Algorithm 3.1,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|f_1(\widetilde{x}_n)-x^*\|^2+(1-\beta_n)\|\widetilde{x}_n-x^*\|^2,\\ \|y_{n+1}-y^*\|^2 &\le \beta_n\|f_2(\widetilde{y}_n)-y^*\|^2+(1-\beta_n)\|\widetilde{y}_n-y^*\|^2.\end{aligned}$$

Adding the above two inequalities, by (3.4) we have

$$\begin{aligned} s_{n+1} \leq\ & s_n + (k_n - 1)Ms_n + \beta_n(\|f_1(\widetilde{x}_n) - x^*\|^2 + \|f_2(\widetilde{y}_n) - y^*\|^2) \\ & - \alpha_n(1-\alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) \\ & - (1-\alpha_n)[1 + 2(k_n - 1)\xi_n \\ & + 2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2. \end{aligned} \tag{3.10}$$

Now, by setting

$$\begin{aligned} \mu_n &= \beta_n(\|f_1(\widetilde{x}_n) - y^*\|^2 + \|f_2(\widetilde{y}_n) - y^*\|^2) + (k_n - 1)Ms_n, \\ \nu_n &= \alpha_n(1-\alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2) \\ &\quad + (1-\alpha_n)[1 + 2(k_n-1)\xi_n \\ &\quad + 2(k_n-1)(2k_n-1)\eta_n\xi_n]\gamma_n[2 - (\lambda_A + \lambda_B)\gamma_n]\|Ax_n - By_n\|^2, \end{aligned}$$

(3.10) can be rewritten as the following form,

$$s_{n+1} \leq s_n - \nu_n + \mu_n, n \geq 1.$$

By the conditions $(d), (e)$ and the boundedness of $\{\widetilde{x}_n\}$ and $\{\widetilde{y}_n\}$, we get $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \mu_n = 0$. To use Lemma 2.11, it suffices to verify that, for all subsequence $\{n_l\} \subset \{n\}$, $\lim_{l\to\infty} \nu_{n_l} = 0$ implies that

$$\limsup_{l\to\infty} \delta_{n_l} \leq 0. \tag{3.11}$$

It follows from $\lim_{l\to\infty} \nu_{n_l} = 0$, $\lim_{n\to\infty} k_n = 1$, and the conditions $(a) - (c)$ that

$$\lim_{l\to\infty} \|K_{n_l} u_{n_l} - x_{n_l}\| = \lim_{l\to\infty} \|G_{n_l} v_{n_l} - y_{n_l}\| = \lim_{l\to\infty} \|Ax_{n_l} - By_{n_l}\| = 0. \tag{3.12}$$

Then

$$\lim_{l\to\infty} \|u_{n_l} - x_{n_l}\| = \lim_{l\to\infty} \gamma_{n_l}\|A^*(Ax_{n_l} - By_{n_l})\| = 0, \tag{3.13}$$

$$\lim_{l\to\infty} \|v_{n_l} - y_{n_l}\| = \lim_{l\to\infty} \gamma_{n_l}\|B^*(Ax_{n_l} - By_{n_l})\| = 0. \tag{3.14}$$

It follows from (3.12) that

$$\begin{aligned} \lim_{l\to\infty} \|\widetilde{x}_{n_l} - x_{n_l}\| &= \lim_{l\to\infty} (1-\alpha_{n_l})\|K_{n_l} u_{n_l} - x_{n_l}\| = 0, \\ \lim_{l\to\infty} \|\widetilde{y}_{n_l} - y_{n_l}\| &= \lim_{l\to\infty} (1-\alpha_{n_l})\|G_{n_l} v_{n_l} - y_{n_l}\| = 0. \end{aligned}$$

From the condition (b) we have

$$0 < a^* < \xi_n < \eta_n < b^* < \frac{1}{\sqrt{k_n^2 + L^2} + k_n} < \frac{1}{L}. \tag{3.15}$$

Since T is uniformly L-Lipschitzian, we can derive

$$\begin{aligned} \|u_{n_l} - T^{n_l}u_{n_l}\| \leq\ & \|u_{n_l} - T^{n_l}((1-\eta_{n_l})I + \eta_{n_l}T^{n_l})u_{n_l}\| \\ & + \|T^{n_l}((1-\eta_{n_l})I + \eta_{n_l}T^{n_l})u_{n_l} - T^{n_l}u_{n_l}\| \\ \leq\ & \frac{1}{\xi_{n_l}}\|u_{n_l} - (1-\xi_{n_l})u_{n_l} - \xi_{n_l}T^{n_l}((1-\eta_{n_l})I + \eta_{n_l}T^{n_l})u_{n_l}\| \\ & + L\|(1-\eta_{n_l})u_{n_l} + \eta_{n_l}T^{n_l}u_{n_l} - u_{n_l}\| \end{aligned}$$

$$= \frac{1}{\xi_{n_l}}\|u_{n_l} - K_{n_l}u_{n_l}\| + L\eta_{n_l}\|u_{n_l} - T^{n_l}u_{n_l}\|,$$

which together with (3.12), (3.13) and (3.15) implies that

$$\begin{aligned} & \|u_{n_l} - T^{n_l}u_{n_l}\| \\ \leq & \frac{1}{\xi_{n_l}(1-L\eta_{n_l})}\|u_{n_l} - K_{n_l}u_{n_l}\| \\ \leq & \frac{1}{\xi_{n_l}(1-L\eta_{n_l})}\|u_{n_l} - x_{n_l}\| + \frac{1}{\xi_{n_l}(1-L\eta_{n_l})}\|x_{n_l} - K_{n_l}u_{n_l}\| \to 0. \end{aligned} \tag{3.16}$$

Since $\{u_n\}$ is bounded, put $K = \sup_{n\geq 1}\{u_n\}$, then K is bounded. Since T is uniformly L-Lipschitzian and asymptotically regular on H_1, from (3.16) we can obtain

$$\begin{aligned} \|u_{n_l} - Tu_{n_l}\| &\leq \|u_{n_l} - T^{n_l}u_{n_l}\| + \|T^{n_l}u_{n_l} - T^{n_l+1}u_{n_l}\| + \|T^{n_l+1}u_{n_l} - Tu_{n_l}\| \\ &\leq \|u_{n_l} - T^{n_l}u_{n_l}\| + \sup_{z\in K}\|T^{n_l}z - T^{n_l+1}z\| + L\|T^{n_l}u_{n_l} - u_{n_l}\| \to 0, \end{aligned}$$

i.e.,

$$\lim_{l\to\infty}\|u_{n_l} - Tu_{n_l}\| = 0. \tag{3.17}$$

Similarly, we can get

$$\lim_{l\to\infty}\|v_{n_l} - Sv_{n_l}\| = 0. \tag{3.18}$$

Step III. We show that $\omega_w(x_{n_l}, y_{n_l}) \subset \Gamma$. Indeed, for any $(\widetilde{x}, \widetilde{y}) \in \omega_w(x_{n_l}, y_{n_l})$, it follows from (3.13) and (3.14) that $(\widetilde{x}, \widetilde{y}) \in \omega_w(u_{n_l}, v_{n_l})$. Then $\widetilde{x} \in \omega_w(u_{n_l})$ and $\widetilde{y} \in \omega_w(v_{n_l})$. Since $I - T$ and $I - S$ are demiclosed at 0, it follows from (3.17) and (3.18) that $\widetilde{x} \in F(T)$ and $\widetilde{y} \in F(S)$. On the other hand, $A\widetilde{x} - B\widetilde{y} \in \omega_w(Ax_{n_l} - By_{n_l})$, which together with weakly lower semicontinuity of the norm and (3.12) implies that $\|A\widetilde{x} - B\widetilde{y}\| \leq \liminf_{l\to\infty}\|Ax_{n_l} - By_{n_l}\| = 0$. Therefore, $(\widetilde{x}, \widetilde{y}) \in \Gamma$. So $\omega_w(x_{n_l}, y_{n_l}) \subset \Gamma$.

Step IV. We prove that the sequence $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*). In fact, since

$$\lim_{l\to\infty}\beta_{n_l}(\|f_1(\widetilde{x}_{n_l}) - x^*\|^2 + \|f_2(\widetilde{y}_{n_l}) - y^*\|^2) = 0,\ \lim_{l\to\infty}(1 - (1-\beta_{n_l})\sigma^2) = 1 - \sigma^2$$

and the condition (e), to prove (3.11), so finally we only need to prove

$$\limsup_{l\to\infty}(\langle f_1(x^*) - x^*, \widetilde{x}_{n_l} - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{y}_{n_l} - y^*\rangle) \leq 0.$$

By (3.12), we obtain

$$\begin{aligned} & \limsup_{l\to\infty}(\langle f_1(x^*) - x^*, \widetilde{x}_{n_l} - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{y}_{n_l} - y^*\rangle) \\ = & \limsup_{l\to\infty}(\langle f_1(x^*) - x^*, \alpha_{n_l}x_{n_l} + (1-\alpha_{n_l})K_{n_l}u_{n_l} - x^*\rangle \\ & +\langle f_2(y^*) - y^*, \alpha_{n_l}y_{n_l} + (1-\alpha_{n_l})G_{n_l}y_{n_l} - y^*\rangle) \\ = & \limsup_{l\to\infty}[\langle f_1(x^*) - x^*, \alpha_{n_l}x_{n_l} + (1-\alpha_{n_l})x_{n_l} - x^*\rangle \\ & +\langle f_1(x^*) - x^*, (1-\alpha_{n_l})(K_{n_l}u_{n_l} - x_{n_l})\rangle \end{aligned}$$

$$
\begin{aligned}
& +\langle f_2(y^*)-y^*, \alpha_{n_l} y_{n_l}+(1-\alpha_{n_l}) y_{n_l}-y^*\rangle \\
& +\langle f_2(y^*)-y^*, \alpha_{n_l} y_{n_l}+(1-\alpha_{n_l})(G_{n_l} v_{n_l}-y_{n_l})\rangle] \\
\leq\ & \limsup_{l \to \infty}(\langle f_1(x^*)-x^*, x_{n_l}-x^*\rangle+\langle f_2(y^*)-y^*, y_{n_l}-y^*\rangle). \qquad (3.19)
\end{aligned}
$$

By the boundedness of $\{(x_{n_l}, y_{n_l})\}$ in H^*, there exists a point $(p,q) \in H^*$ and a subsequence $\{(x_{n_l'}, y_{n_l'})\}$ of $\{(x_{n_l}, y_{n_l})\}$ in H^* such that $(x_{n_l'}, y_{n_l'}) \rightharpoonup (p,q)$ and

$$
\begin{aligned}
& \limsup_{l \to \infty}(\langle f_1(x^*)-x^*, x_{n_l}-x^*\rangle+\langle f_2(y^*)-y^*, y_{n_l}-y^*\rangle) \\
=\ & \lim_{l \to \infty}(\langle f_1(x^*)-x^*, x_{n_l'}-x^*\rangle+\langle f_2(y^*)-y^*, y_{n_l'}-y^*\rangle). \qquad (3.20)
\end{aligned}
$$

Then $(p,q) \in \omega_w(x_{n_l}, y_{n_l})$. By Step III, we have $(p,q) \in \Gamma$. Thus by (3.2), (3.19) and (3.20) we obtain

$$
\begin{aligned}
& \limsup_{l \to \infty}(\langle f_1(x^*)-x^*, \widetilde{x}_{n_l}-x^*\rangle+\langle f_2(y^*)-y^*, \widetilde{y}_{n_l}-y^*\rangle) \\
\leq\ & \lim_{l \to \infty}(\langle f_1(x^*)-x^*, x_{n_l'}-x^*\rangle+\langle f_2(y^*)-y^*, y_{n_l'}-y^*\rangle) \\
=\ & \langle f_1(x^*)-x^*, p-x^*\rangle+\langle f_2(y^*)-y^*, q-y^*\rangle \\
=\ & -\langle((I-f_1)x^*, (I-f_2)y^*), (p,q)-(x^*,y^*)\rangle \leq 0,
\end{aligned}
$$

i.e., $\limsup_{l\to\infty} \delta_{n_l} \leq 0$. Therefore it follows from Lemma 2.11 that $\lim_{n\to\infty} s_n = 0$, that is

$$
\lim_{n \to \infty}(\|x_n-x^*\|^2+\|y_n-y^*\|^2)=0,
$$

which implies that $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$ which is the unique solution of the VIP (3.2), completing the proof. □

Remark 3.3. Theorem 3.2 extends and improves Theorem 3.2 in [1] from quasi-pseudocontractive operators to asymptotically quasi-pseudocontractive operators, and modifies the conditions on $\{\gamma_n\}$ and $\{\alpha_n\}$. Meanwhile, we remove the condition of semi-compactness on operators and obtain the strong convergence result by using viscosity approximation methods. Furthermore, our proof is different from that of Theorem 3.2 in [1].

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