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FIXED POINT AND WEAK CONVERGENCE THEOREMS FOR NONCOMMUTATIVE TWO EXTENDED GENERALIZED HYBRID MAPPINGS IN BANACH SPACES

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Abstract. Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid [8] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. In this paper, we first obtain a common fixed point theorem for two extended generalized hybrid mappings in a Banach space. Then, we prove a weak convergence theorem of Mann's type iteration for such two mappings in a Banach space satisfying Opial's condition. Using this result, we get well-known and new weak convergence theorems in a Banach space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty, where $F(T)$ is the set of fixed points of T. Furthermore, from Baillon [3] we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space. Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

In 2010, Kocourek, Takahashi and Yao [14] defined a broad class of nonlinear mappings in a Hilbert space: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called (α, β)-*generalized hybrid*. We also know the following: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

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for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [16, 17] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [23] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [11]. We also know that λ-hybrid mappings are contained in the class of generalized hybrid mappings; see [9]. The nonlinear ergodic theorem by Baillon [3] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [14]. Kohsaka [15] proved the following theorem.

Theorem 1.1 ([15]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

Motivated by Kohsaka's theorem 1.1, Takahashi [24] proved a weak convergence theorem for finding a common fixed point of noncommutative two generalized hybrid mappings in a Hilbert space by using Mann's type iteration [18].

Theorem 1.2 ([24]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the following:

$$0 < a \leq \alpha_n \leq b < 1 \quad \textit{and} \quad 0 < c \leq \gamma_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$ and $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

On the other hand, Hojo and Takahashi [8] extended the concept of generalized hybrid mappings in a Hilbert space to that in a Banach space as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{1.3}$$

for all $x, y \in C$.

In this paper, we first obtain a common fixed point theorem for two extended generalized hybrid mappings in a Banach space. Then, we prove a weak convergence theorem of Mann's type iteration for such two mappings in a Banach space satisfying

Opial's condition. Using this result, we get well-known and new weak convergence theorems in a Banach space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [12]. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. For more details, see [21, 22]. The following result is also in [21].

Lemma 2.1 ([21])**.** *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j \rangle,$$

where $j \in Jy$.

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I+rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition. If C is a nonempty, closed and convex subset of a Banach space and T is a nonexpansive mapping of C into itself, then $A = I - T$ is an accretive operator and $C = D(A) \subset R(I + rA)$ for all $r > 0$; see [21, Theorem 4.6.4].

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [5] if

$$\|Tx - Ty\|^2 \le \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx-Ty)$; see also [4, 7]. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping of the closure of the domain into itself. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [21]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1,\ y_1 \in Az_1$ and $x_2 = z_2 + ry_2,\ y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j\rangle \ge 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \ge 0.$$

Furthermore, we have that

$$\begin{aligned} &\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \ge 0 \\ &\qquad \Longleftrightarrow \langle x_1 - z_1 - (x_2 - z_2), j\rangle \ge 0 \\ &\qquad \Longleftrightarrow \langle x_1 - x_2, j\rangle \ge \|z_1 - z_2\|^2. \end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping of C into itself; see also [5], [6] and [26]. Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid if it satisfies (1.3), that is, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \le 0$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. We note that an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping is nonexpansive for $\alpha = 1$, $\beta = \gamma = 0$ and $\delta = -1$, nonspreading for $\alpha = 2$, $\beta = \gamma = -1$ and $\delta = 0$, and hybrid for $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = -1$. Nonexpansive mappings, nonspreading mappings and hybrid mappings in a Banach space are deduced from firmly nonexpansive mappings as follows: Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\|Tx - Ty\|^2 \le \langle x - y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned} &\|Tx-Ty\|^2 \le \langle x - y, j\rangle \\ &\qquad \Longleftrightarrow 0 \le 2\langle x - Tx - (y - Ty), j\rangle \end{aligned} \tag{2.2}$$

$$\Longrightarrow 0 \leq \|x-y\|^2 - \|Tx-Ty\|^2$$
$$\Longleftrightarrow \|Tx-Ty\|^2 \leq \|x-y\|^2.$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx-Ty)$,

$$\begin{aligned} \|Tx-Ty\|^2 &\leq \langle x-y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x-Tx-(y-Ty), j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x-Tx, j\rangle + 2\langle Ty-y, j\rangle \\ &\Longrightarrow 0 \leq \|x-Ty\|^2 - \|Tx-Ty\|^2 + \|Tx-y\|^2 - \|Tx-Ty\|^2 \\ &\Longleftrightarrow 0 \leq \|x-Ty\|^2 + \|y-Tx\|^2 - 2\|Tx-Ty\|^2 \\ &\Longleftrightarrow 2\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2. \end{aligned} \tag{2.3}$$

Therefore, using (2.2) and (2.3), we have that

$$\begin{aligned} \|Tx-Ty\|^2 &\leq \langle x-y, j\rangle \\ &\Longrightarrow 3\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2 + \|x-y\|^2. \end{aligned}$$

Hojo and Takahashi [8] proved the following result.

Lemma 2.2 ([8])**.** *Let E be a Banach space, let C be a nonempty, closed and convex subset of E. Then an extended generalized hybrid mapping which has a fixed point is quasi-nonexpansive.*

The following result was proved by Xu [28].

Lemma 2.3 ([28])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\mu x + (1-\mu)y\|^2 \leq \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)g(\|x-y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [19] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [21] for the proof of the existence of a Banach limit and its other elementary properties.

3. Fixed point theorems

Using Takahashi and Jeong's result [25], Hsu, Takahashi and Yao [10] also proved the following lemma for nonlinear mappings in a Banach space; see also [2, 13].

Lemma 3.1 ([10]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a mapping of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. If*

$$\mu_n\|x_n - Ty\|^2 \le \mu_n\|x_n - y\|^2$$

for all $y \in C$, then T has a fixed point in C.

Using Lemma 3.1, we get the following fixed point theorem.

Theorem 3.2. *Let E be a uniformly convex Banach space and let C be a noempty, closed and convex subset of E. Let S and T be commutative extended generalized hybrid mappings from C into itself such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded for some $z \in C$. Then $F(S) \cap F(T) \neq \emptyset$.*

Proof. Suppose that there exists $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Since $S : C \to C$ is an extended generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$ and

$$\alpha\|Sx - Sy\|^2 + \beta\|x - Sy\|^2 + \gamma\|Sx - y\|^2 + \delta\|x - y\|^2 \le 0$$

for all $x, y \in C$. Putting $x = S^nz$, we obtain that

$$\alpha\|S^{n+1}z - Sy\|^2 + \beta\|S^nz - Sy\|^2 + \gamma\|S^{n+1}z - y\|^2 + \delta\|S^nz - y\|^2 \le 0$$

for any $n \in \mathbb{N} \cup \{0\}$ and $y \in C$. Applying a Banach limit μ to both sides of this inequality, we obtain that

$$\mu_n\big(\alpha\|S^{n+1}z - Sy\|^2 + \beta\|S^nz - Sy\|^2 + \gamma\|S^{n+1}z - y\|^2 + \delta\|S^nz - y\|^2\big) \le 0$$

and hence

$$(\alpha + \beta)\mu_n\|S^nz - Sy\|^2 + (\gamma + \delta)\mu_n\|S^nz - y\|^2 \le 0.$$

By $\alpha + \beta + \gamma + \delta \ge 0$ and $\alpha + \beta > 0$ we obtain that

$$\begin{aligned}\mu_n\|S^nz - Sy\|^2 &\le \frac{-(\gamma + \delta)}{\alpha + \beta}\mu_n\|S^nz - y\|^2 \\ &\le \mu_n\|S^nz - y\|^2\end{aligned}$$

for all $y \in C$. Since $\{S^nz\} \subset C$ is bounded, by Lemma 3.1 we obtain a fixed point $p \in C$. Furthermore, S is quasi-nonexpansive. Then $F(S)$ is nonempty, closed and convex. Since S and T is commutative, we have that, for any $u \in F(S)$,

$$Tu = TSu = STu.$$

This implies that $Tu \in F(S)$. Then $TF(S) \subset F(S)$. Since $T : F(S) \to F(S)$ is an extended generalized hybrid mapping, as in the above proof, we have that $v \in F(S)$ such that $Tv = v$. This implies that $v \in F(S) \cap F(T)$. This completes the proof. □

We also have the following theorem.

Theorem 3.3. *Let H be a uniformly convex Banach space and let C be a nonempty, closed and convex subset of E. Let $S, T : C \to H$ be extended generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$ and let $\{\gamma_n\}$ be a sequence of real numbers. Assume that there exist $c, d \in \mathbb{R}$ such that $0 < c \leq \gamma_n \leq d < 1$ for all $n \in \mathbb{N}$. If $T_n = \gamma_n S + (1 - \gamma_n)T$ for all $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$.*

Proof. Since S and T are extended generalized hybrid mappings and $F(S) \cap F(T) \neq \emptyset$, S and T are quasi-nonexpansive mappings. For $z_0 \in F(S) \cap F(T)$ and $z \in \cap_{n=1}^{\infty} F(T_n)$, let $r = \|z - z_0\|$. Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\mu x + (1 - \mu)y\|^2 \leq \mu\|x\|^2 + (1 - \mu)\|y\|^2 - \mu(1 - \mu)g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. Using this, we have that for $z_0 \in F(S) \cap F(T)$, $z \in \cap_{n=1}^{\infty} F(T_n)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|z - z_0\|^2 &= \|T_n z - z_0\|^2 \\
&= \|(\gamma_n S + (1 - \gamma_n)T)z - z_0\|^2 \\
&= \|\gamma_n(Sz - z_0) + (1 - \gamma_n)(Tz - z_0)\|^2 \\
&\leq \gamma_n\|Sz - z_0\|^2 + (1 - \gamma_n)\|Tz - z_0\|^2 - \gamma_n(1 - \gamma_n)g(\|Sz - Tz\|) \\
&\leq \gamma_n\|z - z_0\|^2 + (1 - \gamma_n)\|z - z_0\|^2 - \gamma_n(1 - \gamma_n)g(\|Sz - Tz\|) \\
&= \|z - z_0\|^2 - \gamma_n(1 - \gamma_n)g(\|Sz - Tz\|).
\end{aligned}$$

This means that $\gamma_n(1 - \gamma_n)g(\|Sz - Tz\|) \leq 0$. Since $0 < c \leq \gamma_n \leq d < 1$ for all $n \in \mathbb{N}$, we have $Sz = Tz$. Since

$$\begin{aligned}
\|Sz - z\| &= \|\gamma_n Sz + (1 - \gamma_n)Sz - z\| \\
&= \|\gamma_n Sz + (1 - \gamma_n)Tz - z\| \\
&= \|(\gamma_n S + (1 - \gamma_n)T)z - z\| \\
&= \|z - z\| \\
&= 0,
\end{aligned}$$

we have that $Sz = z$. Similarly, we have that $Tz = z$. This implies that $\cap_{n=1}^{\infty} F(T_n) \subset F(S) \cap F(T)$. It is obvious that $F(S) \cap F(T) \subset \cap_{n=1}^{\infty} F(T_n)$. Thus $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$. This completes the proof. □

4. Weak convergence theorems

In this section, we first prove a weak convergence theorem of Mann's type iteration [18] for extended generalized hybrid mappings in a Banach space satisfying Opial's condition. Before proving the theorem, we need the following result from Hojo and Takahashi [8]. Let E be a Banach space. Let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [20] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to E$ is said to be demiclosed if $\hat{F}(T) = F(T)$.

Lemma 4.1 ([8])**.** *Let E be a Banach space satisfying Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into E which satisfies $\beta \leq 0$ and $\gamma \leq 0$. Then $\hat{F}(T) = F(T)$, i.e., T is demiclosed.*

Theorem 4.2. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha', \beta', \gamma', \delta' \in \mathbb{R}$. Let S and T be $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma,' \delta')$-extended generalized hybrid mappings of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$ and $\beta' \leq 0$ and $\gamma' \leq 0$, respectively. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \alpha_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \gamma_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in F(S) \cap F(T)$.

Proof. Since S and T are quasi-nonexpansive, we have that $F(S) \cap F(T)$ is closed and convex. Put

$$T_n = \gamma_n S + (1 - \gamma_n) T$$

for all $n \in \mathbb{N}$ and let w be a point of $F(S) \cap F(T)$. We have that

$$\begin{aligned} \|T_n x_n - w\| &= \|(\gamma_n S + (1 - \gamma_n) T) x_n - w\| \\ &\leq \gamma_n \|S x_n - w\| + (1 - \gamma_n) \|T x_n - w\| \\ &\leq \gamma_n \|x_n - w\| + (1 - \gamma_n) \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{4.1}$$

Using (4.1), we have that

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n x_n + (1 - \alpha_n) T_n x_n - w\| \\ &\leq \alpha_n \|x_n - w\| + (1 - \alpha_n) \|T_n x_n - w\| \\ &\leq \alpha_n \|x_n - w\| + (1 - \alpha_n) \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{4.2}$$

Then $\lim_{n \to \infty} \|x_n - w\|$ exists. Thus we have that the sequence $\{x_n\}$ is bounded. This implies that $\{T_n x_n\}$ is bounded. Let

$$r = \max\{\sup_{n \in \mathbb{N}} \|x_n - w\|, \sup_{n \in \mathbb{N}} \|T_n x_n - w\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\mu x + (1 - \mu) y\|^2 \leq \mu \|x\|^2 + (1 - \mu) \|y\|^2 - \mu (1 - \mu) g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. Then we have that for $w \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \|\alpha_n x_n + (1 - \alpha_n) T_n x_n - w\|^2 \\ &= \|\alpha_n (x_n - w) + (1 - \alpha_n)(T_n x_n - w)\|^2 \end{aligned}$$

$$\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_n x_n - w\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|)$$
$$\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|)$$
$$= \|x_n - w\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|)$$

and hence

$$\alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|) \leq \|x_n - w\|^2 - \|x_{n+1} - w\|^2.$$

Since $\lim_{n\to\infty}\|x_n - w\|^2$ exists, we have from $0 < a \leq \alpha_n \leq b < 1$ that

$$\lim_{n\to\infty} g(\|x_n - T_n x_n\|) = 0.$$

From the properties of g, we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{4.3}$$

Using this, we have from Lemma 2.1 that, for $w \in F(S)\cap F(T)$,

$$\begin{aligned}
\|x_n - w\|^2 &= \|x_n - T_n x_n + T_n x_n - w\|^2 \\
&\leq \|T_n x_n - w\|^2 + 2\langle x_n - T_n x_n, j(x_n - w)\rangle \\
&= \|\gamma_n S x_n + (1-\gamma_n) T x_n - w\|^2 + 2\langle x_n - T_n x_n, J(x_n - w)\rangle \\
&\leq \gamma_n\|S x_n - w\|^2 + (1-\gamma_n)\|T x_n - w\|^2 \\
&\quad - \gamma_n(1-\gamma_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, J(x_n - w)\rangle \\
&\leq \gamma_n\|x_n - w\|^2 + (1-\gamma_n)\|x_n - w\|^2 \\
&\quad - \gamma_n(1-\gamma_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, j(x_n - w)\rangle \\
&= \|x_n - w\|^2 - \gamma_n(1-\gamma_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, j(x_n - w)\rangle
\end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)g(\|S x_n - T x_n\|) \leq 2\langle x_n - T_n x_n, j(x_n - w)\rangle.$$

Since $x_n - T_n x_n \to 0$ and $\{x_n\}$ is bounded, we have from $0 < c \leq \gamma_n \leq d < 1$ that $S x_n - T x_n \to 0$. Then we have that

$$\begin{aligned}
\|x_n - S x_n\| &= \|x_n - T_n x_n + T_n x_n - S x_n\| \\
&\leq \|x_n - T_n x_n\| + \|T_n x_n - S x_n\| \\
&= \|x_n - T_n x_n\| + (1-\gamma_n)\|T x_n - S x_n\| \\
&\to 0.
\end{aligned}$$

Similarly, we have that $\|x_n - T x_n\| \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. From Lemma 4.1, we have that v is a point of $F(S)\cap F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We have that $u, v \in F(S)\cap F(T)$. Suppose $u \neq v$. From $u, v \in F(S)\cap F(T)$, we know that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. Since E satisfies Opial's condition, we have that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{i\to\infty}\|x_{n_i} - u\| \\
&< \lim_{i\to\infty}\|x_{n_i} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\|
\end{aligned}$$

$$\begin{aligned} &= \lim_{j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{j\to\infty} \|x_{n_j} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|. \end{aligned}$$

This is a contradiction. Thus we must have $u = v$. This implies that $\{x_n\}$ converges weakly to a point of $F(S) \cap F(T)$. This completes the proof. □

Using Theorem 4.2, we obtain the following weak convergence theorems by Hojo and Takahashi [8] in a Banach space; see also [27].

Theorem 4.3 ([8])**.** *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Proof. Putting $S = T$ and $\gamma_n = \frac{1}{2}$ in Theorem 4.2, we obtain the desired result from Theorem 4.2. □

Using Theorem 4.2, we also obtain the following result.

Theorem 4.4. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha', \beta', \gamma', \delta' \in \mathbb{R}$. Let S and T be $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma,' \delta')$-extended generalized hybrid mappings of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$ and $\beta' \leq 0$ and $\gamma' \leq 0$, respectively. Let μ and λ be real numbers with $0 < \mu < 1$ and $0 < \lambda < 1$. Define a mapping $U : C \to C$ by*

$$U = \mu I + (1 - \mu)(\lambda S + (1 - \lambda)T).$$

If $F(S) \cap F(T) \neq \emptyset$, then for any $x \in C$, $U^n x$ converges weakly to an element $z \in F(S) \cap F(T)$.

Proof. Putting $\gamma_n = \lambda$ for all $n \in \mathbb{N}$ in Theorem 4.2, we have that

$$T_n = \lambda S + (1 - \lambda)T.$$

Furthermore, we have that for any $x_1 = x \in C$,

$$x_2 = Ux_1 = Ux, x_3 = U^2 x_1 = U^2 x \ldots.$$

Thus we have from Theorem 4.2 that $U^n x$ converges weakly to an element $z \in F(S) \cap F(T)$. This completes the proof. □

Open Problems

1. Can we prove Theorem 4.2 for finite or infinite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition?

2. Can we prove Theorem 4.2 for a uniformly convex Banach space with a Fréchet differentiable norm instead of a Banach space satisfying Opial's condition?

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