Journal of Nonlinear and Convex Analysis
Volume 21, Number 1, 2020, 1–19

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

VARIOUS EXAMPLES OF THE KKM SPACES

SEHIE PARK

Abstract. Recall that a partial KKM space is a topological space satisfying an abstract form of the well-known KKM theorem, and a KKM space is a partial KKM space satisfying the 'open-valued' version of the form. Recently, we have found several new examples of such spaces. As a continuation of the preceding two works, we are going to introduce old and new examples of the KKM spaces, which play a major role in applications of the KKM theory. Finally, we state why we should use triples $(X, D; \Gamma)$ for abstract convex spaces by giving further examples, and introduce an example of a partial KKM space which is not a KKM space.

1. Introduction

In order to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of the well-known G-convex spaces and adequate to establish the KKM theory.

In our previous works [36, 42, 44, 45], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces.

In our recent work [51], we introduced a logical origin of the Brouwer fixed point theorem, Sperner's weak combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem being Ky Fan's 1952 lemma. Also we noted that these three theorems are mutually equivalent and have nearly one hundred equivalent formulations and several thousand applications.

Moreover, in a consequent work [52], we introduced our multimap classes in the frame of the KKM theory; that is, the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$. There we collected the basic properties of such multimap classes and some mutual relations among them in general topological spaces or our abstract convex spaces.

Recall that a partial KKM space is a topological space satisfying an abstract form of the KKM theorem, and a KKM space is the one also satisfying the 'open-valued' version of the form. Recently, we have found several new examples of KKM spaces. As a continuation of the preceding two works, we are going to show old and new examples of the KKM spaces, which play the major role in applications of the KKM theory. Some historical remarks are also added.

This article is organized as follows: Section 2 deals with minimum amount of preliminaries on our abstract convex spaces including the KKM spaces. In Section

2010 *Mathematics Subject Classification.* 47H10, 49J53, 54C60, 54H25, 90A14, 90C76, 91A13, 91A10.

Key words and phrases. Convex space, H-space, G-convex space, ϕ_A-space, KKM-space, hyperconvex space, $\mathbb{R}$-tree, Horvath's convex space, $\mathbb{B}$-space, extended long line L^*.

3, we introduce a metatheorem on (partial) KKM spaces. This metatheorem is concerned with the common properties of all partial KKM spaces. Section 4 deals with known examples of classical KKM spaces including convex spaces, H-spaces, and G-convex spaces. In Section 5, we give basic theory of ϕ_A-spaces as a sample of KKM spaces. Section 6 concerns with relatively new examples of KKM spaces mainly founded by ourselves. Finally, in Section 7, we state why we should use triples $(X, D; \Gamma)$ for abstract convex spaces instead of pairs $(X; \Gamma)$ by giving further examples, and introduce an example of a partial KKM space due to Kulpa and Szymanski in 2014 which is not a KKM space.

2. Abstract convex spaces

For sets X and Y, a multimap (or a multifunction or simply a map) $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y.

For the concepts on our abstract convex spaces, KKM spaces and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$, we follow [45, 46] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D \rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D \rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E of E.

Definition 2.3. A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Here we give a system of abstract convex spaces $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ \Longrightarrow \text{H-space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ \Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

Later the foundations or elements of the KKM theory on abstract convex spaces were studied in [36, 41, 42, 44, 45, 47], where it was shown that many important results hold for partial KKM spaces.

For an abstract convex space $(E \supset D; \Gamma)$, an extended real-valued function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp. *quasiconvex*] whenever $\{x \in E \mid f(x) > r\}$ [resp. $\{x \in E \mid f(x) < r\}$] is Γ-convex for any $r \in \mathbb{R}$.

3. A Metatheorem on KKM spaces

In the KKM theory, it is routine to reformulate the KKM principle to the following equivalent forms [44, 45]:

Fan type matching property
Another intersection property
Geometric or section properties
Fan-Browder type fixed point property
Existence theorem of maximal elements
Analytic formulations, analytic alternatives
Minimax inequality, and others

Any of these statements characterizes KKM spaces, and any of closed versions of them characterizes partial KKM spaces; see [45].

For example, the Fan-Browder type fixed point theorem is used for the following:

Theorem 3.1. *An abstract convex space* $(X, D; \Gamma)$ *is a KKM space if and only if for any maps* $S : D \multimap X,\ T : X \multimap X$ *satisfying*

(1) $S(z)$ *is open [resp. closed] for each* $z \in D$;
(2) *for each* $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; *and*
(3) $X = \bigcup_{z\in M} S(z)$ *for some* $M \in \langle D\rangle$,

T *has a fixed point* $x_0 \in X$; *that is* $x_0 \in T(x_0)$.

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type. From this, we can deduce the following:

Theorem 3.2. *Let* $(X, D; \Gamma)$ *be a partial KKM space,* K *a nonempty compact subset of* X, *and* $G : D \multimap X$ *a map such that*

(1) $\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ *[that is,* G *is transfer closed-valued];*
(2) $\overline{G}$ *is a KKM map; and*
(3) *either*
   (i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D\rangle$; *or*
   (ii) *for each* $N \in \langle D\rangle$, *there exists a compact* Γ*-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

From this theorem we can deduce its equivalent formulations of the following forms for partial KKM spaces [41, 45]:

Theorems of Sperner and Alexandroff-Pasynkoff
Fan type matching theorem
Tarafdar type intersection theorem
Geometric or section properties
Fan-Browder type fixed point theorems
Maximal element theorems
Analytic alternatives
Fan type minimax inequalities
Variational inequalities
Horvath type fixed point theorem
Browder type coincidence theorem
von Neumann type minimax theorem
Nash type equilibrium theorem
Analytic alternatives (a basis of various equilibrium problems)
Fan type minimax inequalities
Variational inequalities, and others

Further applications of our theory on partial KKM spaces are given as follows [45, 46]:

Best approximations (under certain restrictions)
von Neumann type intersection theorem
Nash type equilibrium theorem
Himmelberg fixed point theorem for KKM spaces
Weakly KKM maps [37]

Consequently, we have the following as is suggested in [44]:

Metatheorem. *For any partial KKM space, all theorems mentioned in this section hold.*

4. Known subclasses of KKM spaces

4.1. Convex spaces

Definition 4.1. Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if $\mathrm{co}\, D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Park [26]. Note that (X, D) can be represented by $(X, D; \Gamma)$ where $\Gamma : \langle D \rangle \to X$ is the convex hull operator.

If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [23].

Example 4.2. (1) The original KKM theorem in 1929 is for a triple $(\Delta_n \supset V; \mathrm{co})$, where V is the set of vertices of an n-simplex Δ_n and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ is the convex hull operator. This triple can be regarded as $(\Delta_n, N; \Gamma)$, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_A := \mathrm{co}\{e_i \mid i \in A\}$ for each $A \subset N$.

(2) Fan's celebrated KKM lemma in 1961 is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of a t.v.s. E. He assumed the superfluous Hausdorffness of E.

The above examples are origins of our G-convex spaces $(X, D; \Gamma)$. It should be noted that many authors' KKM type theorems for a pair $(X; \Gamma)$ can not generalize the original KKM theorem or Fan's KKM lemma.

(3) A *convexity space* $(E, \mathcal{C})$ in the classical sense consists of a non-empty set E and a family $\mathcal{C}$ of subsets of E such that E itself belongs to $\mathcal{C}$ and any intersection of a subfamily of $\mathcal{C}$ also belongs to $\mathcal{C}$. See Sortan in 1984; where 283 references appear. The $\mathcal{C}$-*convex hull* of any subset $X \subset E$ is denoted and defined by $\mathrm{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. X is said to be $\mathcal{C}$-*convex* whenever $X{=}\mathrm{Co}_{\mathcal{C}} X$. If we define a multifunction $\Gamma : \langle E \rangle \multimap E$ by $\Gamma_A{:=}\mathrm{Co}_{\mathcal{C}} A$ for each $A \in \langle E \rangle$, then $(E, \mathcal{C})$ is an abstract convex space $(E; \Gamma)$ with any topology on E.

(4) Every nonempty convex subset X of a topological vector space is a convex space with respect to any nonempty subset D of X, and the converse is known to be not true.

4.2. H-spaces

Definition 4.3. A triple $(X, D; \Gamma)$ is called an H-*space* by Park [27] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [11] or an H-space by Bardaro and Ceppitelli [1].

Example 4.4. Any convex space X due to Lassonde is an H-space $(X; \Gamma)$ by putting $\Gamma_A{=}\ \mathrm{co}\, A$, the convex hull of $A \in \langle X \rangle$. Similarly, our convex spaces $(X, D; \Gamma)$ becomes H-spaces. Other examples of $(X; \Gamma)$ are any pseudo-convex space [10], any homeomorphic image of a convex space, any contractible space, and so on; see Bardaro and Ceppitelli [1] and Horvath [11].

Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces, and are examples of compact H-spaces without having the fixed point property. Every n-simplex Δ_n is an H-space $(\Delta_n, D; \Gamma)$, where D is the set of vertices and $\Gamma_A{=}\mathrm{co}\, A$ for $A \in \langle D \rangle$.

4.3. G-convex spaces

Definition 4.5. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \mathrm{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X, \Gamma) = (X, X; \Gamma)$.

There is a lot of examples of G-convex spaces; see [54, 55].

Example 4.6. (1) If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then (X, Γ) becomes a convex space due to Lassonde [23].

(2) If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, than (X, Γ) becomes a c-space (or an H-space) due to Horvath [11].

(3) For other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, and so on. For the literature, see Park and Kim [54, 55]. Later, we found a number of new examples of G-convex spaces. Especially, any continuous image of a G-convex space is a G-convex space; and any almost convex subset of a t.v.s. (see Himmelberg [9]) is a G-convex space.

(4) Later examples of G-convex spaces were given in [31] as follows: L-spaces and B'-simplicial convexity of Ben-El-Mechaiekh et al. [3], Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

Remark 4.7. (1) G-convex spaces are actually same to the so-called ϕ_A-spaces in the next section, and these are all KKM spaces.

(2) Most of the families of G-convex spaces mentioned above have some concrete examples. Some of relatively new ones will be shown in Section 6.

(3) In our previous work [45], we introduced basic results in the KKM theory of abstract convex spaces and KKM maps. Such results are applied to several variants of the concepts of G-convex spaces and KKM type maps. We studied the nature of such variants and criticized other authors' later 'generalizations' of our previous results.

5. ϕ_A-SPACES

Since the appearance of G-convex spaces in 1993, many authors have tried to imitate, modify, or generalize the concept and published a large number of papers. In fact, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplices. Such example are L-spaces due to Ben-El-Mechaiekh et al., spaces having property (H) due to Huang, FC-spaces due to Ding, convexity structures satisfying the H-condition by Xiang et al., M-spaces and L-spaces due to González et al., and others. Some authors claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. Some authors also tried to generalize the KKM principle for their own settings. They introduced various types of generalized KKM maps; for example, generalized KKM maps on L-spaces, generalized R-KKM maps, and many other artificial terminology. Some of them tried to rewrite certain results on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. In 2007, we found that most of such spaces are subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$; see Park [34, 35, 39]. Since then, the spaces became one of the main theme of the KKM theory; see [34, 35, 37, 39, 40, 43].

Definition 5.1. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle}) \text{ or } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, a subset C of X is said to be ϕ_A-*convex* with respect to a subset $D' \subset D$ if for each $B \in \langle D'\rangle$, we have $\operatorname{Im}\phi_B := \phi_B(\Delta_{|B|-1}) \subset C$.

By putting $\Gamma_A := \phi_A(\Delta_n)$, any ϕ_A-space becomes an abstract convex space, and we will show that it is a KKM space later.

Note that, when $X = D$, a ϕ_A-space is called an FC-space by Ding [7] or a simplicial space by Kulpa and Szymanski [21]. Later, a ϕ_A-space is called a GFC-space by Khanh et al. [17, 18].

We collect some known facts on ϕ_A-spaces as follows in [33, 38, 42, 46, 47, 52]:

Definition 5.2. For a ϕ_A-space $(X, D; \phi_A)$, a *KKM map* $T : D \multimap X$ is the one satisfying

$$\phi_A(\Delta_J) \subset T(J) \text{ for each } A \in \langle D\rangle \text{ and } J \in \langle A\rangle.$$

This definition contains many particular cases previously appeared.

Proposition 5.3. *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ is a KKM map on the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for all $A \in \langle D\rangle$ with $|A| = n + 1$.*

Proposition 5.4. *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ is a KKM map on a new abstract convex space $(X, D; \Gamma^T)$.*

The following is a KKM type theorem for ϕ_A-spaces, and it can be proved by following that for G-convex spaces:

Theorem 5.5. *For a ϕ_A-space $(X, D; \phi_A)$, let $G : D \multimap X$ be a KKM map with closed values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property.* (*More precisely, for each $A \in \langle D\rangle$ with $|A| = n + 1$, we have $\phi_A(\Delta_n) \cap \bigcap_{z\in A} G(z) \neq \emptyset$.*) *Further, if*

$$\bigcap_{z\in M} G(z) \text{ is compact for some } M \in \langle D\rangle$$

then we have $\bigcap_{z\in D} G(z) \neq \emptyset$.

Theorem 5.6. *For a ϕ_A-space $(X, D; \phi_A)$, let $G : D \multimap X$ be a KKM map with open values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property.*

In [37], we applied basic results in the KKM theory on abstract convex spaces and KKM maps to the class of ϕ_A-spaces. These spaces unified G-convex spaces and many similar variants. Consequently, fundamental theorems on ϕ_A-spaces were used to correct or improve results on the so-called R-KKM maps or L-convex spaces due to some particular authors.

Example 5.7. There are many examples of ϕ_A-spaces; see [34, 35, 36, 39, 40, 43, 47]. The following are some of them:

(1) Since an L-space of Ben-El-Mechaiekh et al. [3] is a G-convex space $(X;\Gamma)$, it is a ϕ_A-space.

(2) A topological space Y is said to have the property (H) whenever a continuous function $\varphi_N : \Delta_n \to Y$ exists for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$.

(3) A pair $(Y, \{\varphi_N\})$ is called an FC-space whenever Y is a topological space and a continuous function $\varphi_N : \Delta_n \to Y$ exists for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ (where, some elements of N may be the same).

(4) Similarly, a ψ_A-space $(X, D; \{\psi_A\}_{A\in\langle D\rangle})$ is such that, for each $A \in \langle D \rangle$ with its cardinality $|A| = n+1$, the function $\psi_A : [0,1]^n \to X$ is continuous. Such type of spaces were treated by Michael [25] and Llinares [24]; see also [31].

For each $n \geq 0$, consider a continuous function $g_n : \Delta_n \to [0,1]^n$ defined by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto (\lambda_0(u), \dots, \lambda_{n-1}(u))$$

for each $u \in \Delta_n$. Moreover, by putting $\phi_A := \psi_A \circ g_n$, a ψ_A-space becomes a ϕ_A-space.

(5) In a t.v.s. E, consider a neighborhood system $\mathcal{V}$ of the origin O. Let Y be an almost convex dense subset of a subset D of E. For any $V \in \mathcal{V}$ and each $A := \{x_0, x_1, \dots, x_n\} \in \langle D \rangle$, a subset $B := \{y_0, y_1, \dots, y_n\} \in \langle Y \rangle$ is determined such that $y_i - x_i \in V$ for each $i = 0, 1, \dots, n$ and $\operatorname{co} B \subset Y$. Define a continuous function $\phi_A : \Delta_n \to \operatorname{co} B$ by

$$\phi_A : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto \phi_A(u) := \sum_{i=0}^{n} \lambda_i(u) y_i$$

for each $u \in \Delta_n$. Then $(Y, D; \{\phi_A\}_{A\in\langle D\rangle})$ is a ϕ_A-space, and can be made into a G-convex space. Note that $Y \subset D$.

(6) Any G-convex space is a ϕ_A-space, and its converse holds:

Proposition 5.8. *A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Consequently, G-convex spaces and ϕ_A-spaces are essentially same.

Theorem 5.9. (i) *A KKM map $G : D \multimap X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.*

(ii) *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

The following is a KKM theorem for ϕ_A-spaces that can be proved by simply modifying the corresponding ones in [30, 32, 48]:

Theorem 5.10. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp. open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n)\cap\bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(1) $\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Theorem 5.10 means that ϕ_A-spaces are KKM spaces.

Remark 5.11. (1) Let $X = \Delta_n$, D be the set of vertices of Δ_n, and $\Gamma :=$ co be the convex hull operator, then Theorem 5.10 reduces to the original KKM theorem in 1929 and its 'open-valued' version.

(2) In case D is a nonempty subset of a (not necessarily Hausdorff) t.v.s. X, Theorem 5.10 extends Fan's celebrated KKM lemma in 1961.

(3) Any KKM type theorem for a $(X, \{\varphi_A\})$ type space can extend neither the original KKM theorem nor Fan's KKM lemma.

Here we give two KKM type theorems which are improved versions of corresponding ones in previous literature:

Theorem 5.12. *Let X be a topological space, D a nonempty set, and $G : D \multimap X$ a map such that*

(1) G is transfer closed-valued [that is, $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$];

(2) there exists $z^* \in D$ with $\overline{G(z^*)}$ compact.

Then, there exists a G-convex space $(X, D; \Gamma)$ such that G is a KKM map if and only if $\bigcap_{z\in D} G(z) \neq \emptyset$.

Theorem 5.13. *For a ϕ_A-space $(Y, D; \{\phi_N\}_{N\in\langle D\rangle})$, let $T : D \to 2^Y$ be a map such that $T(z)$ is nonempty and closed for each $z \in D$.*

(i) *If T is a KKM map, then for each $N \in \langle D\rangle$ with $|N| = n + 1$,*

$$\phi_N(\Delta_n) \cap \bigcap_{x\in N} T(x) \neq \emptyset.$$

(ii) *If the family $\{T(z) : z \in D\}$ has the finite intersection property, then T is a KKM map.*

6. Old and new examples of KKM spaces

A large number of examples of G-convex spaces were already given in Section 4. In this section, we introduce some new important examples of KKM spaces chronologically, most of them are due to ourselves. For details of some of them, see [54].

(1) Hyperconvex metric spaces — In a metric space (M, d), for a point $x \in M$ and any $t > 0$, let the closed ball be

$$B(x, t) := \{y \in M \mid d(x, y) \leq t\}.$$

In 1956, Aronszajn and Panitchpakdi defined as follows:

Definition 6.1. A metric space (H, d) is called a *hyperconvex metric space* whenever, for any arbitrary collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls of H, $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ implies $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$.

A normed vector space X is not hyperconvex in general, and spaces $(\mathbb{R}^n, ||\cdot||_\infty)$, l^∞, L^∞, and $\mathbb{R}$-tree is hyperbolic.

Horvath [12] showed the following:

Lemma 6.2. *Any hyperconvex metric space H is a c-space $(H;\Gamma)$, where $\Gamma(A)$ is the intersection of all closed balls in H containing $A \in \langle H \rangle$.*

Therefore, we have the following by our KKM theory:

Theorem 6.3. *Every hyperconvex metric space is a KKM metric space, that is, a metric space satisfying the KKM principle.*

(2) Hyperbolic metric spaces — Kirk [19] in 1982 first considered a wide class of spaces including convex metric spaces of 'hyperbolic' type, and later Reich and Shafrir [57] introduced hyperbolic metric spaces, which is a particular c-space.

Definition 6.4. ([57]) Let (X, ρ) be a metric space and $\mathbb{R}$ the real line. We say that a map $c : \mathbb{R} \to X$ is a *metric embedding* of $\mathbb{R}$ into X if

$$\rho(c(s), c(t)) = |s - t|$$

for all real s and t. The image of a metric embedding is called a *metric line*. The image of a real interval $[a, b] := \{t \in \mathbb{R} \mid a \leq t \leq b\}$ under such a map is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines, such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment denoted by $[x, y]$ joining x and y. For each $0 \leq t \leq 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

This point z is denoted by $(1 - t)x \oplus ty$.

Definition 6.5. ([57]) We say that X, or more precisely (X, ρ, M), is a *hyperbolic metric space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y, z)$$

for all x, y and z in X.

Examples of such metric spaces are all normed vector spaces, all Hadamard manifolds, and Hilbert balls having hyperbolic distance. Arbitrary cartesian product of hyperbolic spaces is also hyperbolic.

(3) Transfer FS convex map — Tian [59] defined the following:

Definition 6.6. ([59]) Let X be a convex subset of a Hausdorff t.v.s. and $Y \subset X$ be nonempty. A multimap $F : Y \multimap X$ is said to be *transfer FS convex* whenever, for any $\{y_1, \dots, y_n\} \subset Y$, corresponding $\{x_1, \dots, x_n\} \subset X$ exists such that, for any $J \subset \{1, \dots, n\}$, the following holds:

$$\mathrm{co}\{x_j \mid j \in J\} \subset \bigcup_{j \in J} F(y_j).$$

Here by putting $\Gamma\{y_j \mid j \in J\} := \mathrm{co}\{x_j \mid j \in J\}$, $(X, Y; \Gamma)$ becomes an abstract convex space, and a transfer FS map becomes a KKM map. So our KKM theory is applicable.

Transfer FS convexity was obtained already by Chang and Zhang [6] in 1991 and extended later by Park and Lee [56] in 2001.

(4) Topological semilattices — Horvath and Llinares-Ciscar [14] show that any topological semilattice $(X, \leq)$ having path-connected intervals satisfies an order theoretic variant of the KKM principle. We showed that such topological semilattices become G-convex spaces; see Park [28].

A *semilattice* or, more exactly, a sup-*semilattice*, is a partially ordered set $(X, \leq)$ for which any pair (x, x') of elements has a lub $x \vee x'$. Any $A \in \langle X \rangle$ has a lub denoted by $\sup A$. If $x \leq x'$, then the set $[x, x'] = \{y \in X \mid x \leq y \leq x'\}$ is called an *order interval*. For details, see [14].

Lemma 6.7. *Any topological semilattice $(X, \leq)$ with path-connected intervals is a G-convex space. More precisely, let D be a nonempty subset of X and $\Gamma : \langle D \rangle \multimap X$ a map such that*

$$\Gamma_A = \Gamma(A) = \bigcup_{a \in A} [a, \ \sup A] \quad \textit{for} \quad A \in \langle D \rangle.$$

Then $(X, D; \Gamma)$ is a G-convex space.

(5) Hyperconvex metric spaces — Let A be a nonempty bounded subset of a metric space (M, d). Then we define the following as in Khamsi [16]:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap \{B \subset M \mid B \text{ is a closed ball in } M \text{ such that } A \subset B\}$.

(ii) $\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{BI}(A)\}$, i.e., $A \in \mathcal{A}(M)$ if and only if A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

(iii) A is called *subadmissible*, if for each $N \in \langle A \rangle$, $\mathrm{BI}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

We introduce new definitions:

Definition 6.8. An abstract convex space $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, $D \subset M$ is nonempty, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map such that $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle D \rangle$. A multimap $G : D \multimap M$ is a *KKM map* if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

A Γ-convex subset of $(M, D; \Gamma)$ is subadmissible and conversely.

The following is due to Khamsi [16] and Yuan [61]:

Definition 6.9. Let (M, d) be a metric space. A subset $S \subset M$ is said to be *finitely metrically closed* [resp. *finitely metrically open*] if for each $F \in \mathcal{A}(M)$, the set $F \cap S = \mathrm{BI}(F) \cap S$ is closed [resp. open]. Note that $\mathrm{BI}(F)$ is always defined and belongs to $\mathcal{A}(M)$. Thus if S is closed [resp. open] in M, it is obviously finitely metrically closed [resp. open].

The following is also due to Khamsi [16] and Yuan [61]:

Theorem 6.10. [KKM-Map Principle] *Let H be a hyperconvex metric space, X an arbitrary subset of H, and $G : X \multimap H$ a KKM map such that each $G(x)$ is finitely metrically closed* [*resp. finitely metrically open*]. *Then the family $\{G(x) \mid x \in X\}$ has the finite intersection property.*

This shows that any hyperconvex metric space having *finitely metric topology* is a KKM space. Hence such space satisfies all results in Section 3 and [45].

In view of the above theorem, we have the following due to Khamsi [16, Theorem 4]:

Theorem 6.11. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

(6) E-convex spaces — Youness [60] introduces the E-convex set and the E-convex map as follows:

Definition 6.12. A set $M \subset \mathbb{R}^n$ is said to be *E-convex* whenever a function $E : \mathbb{R}^n \to \mathbb{R}^n$ exists such that $(1-\lambda)E(x) + \lambda E(y) \in M$ holds for any $x, y \in M$ and $0 < \lambda < 1$.

There is an E-convex set that is not convex; see Youness [60]. An E-convex set is an abstract convex space, and hence our KKM theory is applicable to it.

In fact, let D be a nonempty subset of M. By defining $\Gamma : \langle D \rangle \to M$ as

$$\Gamma\{x_0, \ldots, x_n\} = \operatorname{co} E\{x_0, \ldots, x_n\} = \{\Sigma_{i=0}^n \lambda_i E(x_i) \mid 0 \leq \lambda_i \leq 1,\ \Sigma_{i=0}^n \lambda_i = 1\}$$

for each $A := \{x_0, x_1, \ldots, x_n\} \in \langle D \rangle$, $(M, D; \Gamma)$ becomes an abstract convex space.

Such Γ has convex values and clearly a continuous map $\phi_A : \Delta_n \to \Gamma_A$ exists. Therefore, $(M, D; \Gamma)$ becomes an H-space and ϕ_A-space, that is, a KKM space.

(7) Bayoumi's KKM spaces — Let $0 < p \leq 1$. Recall the definitions given by Bayoumi [2]:

Definition 6.13. (*p-convex set*) A set A in a vector space V is said to be *p-convex* if, for any $x, y \in A$, $s, t \geq 0$, we have

$$(1-t)^{1/p} x + t^{1/p} y \in A, \quad \text{whenever } 0 \leq t \leq 1.$$

Definition 6.14. (*p-convex hull*) If X is a topological vector space and $A \subset X$, the closed p-convex hull of A denoted by $\overline{C}_p(A)$ is the smallest closed p-convex set containing A.

Definition 6.15. (*p-convex combination*) Let A be p-convex and $x_1, \ldots, x_n \in A$, and $t_i \geq 0$, $\sum_1^n t_i^p = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$. If $\sum_1^n |t_i|^p \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 6.16. (*locally p-convex space*) A topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way.

Using these concepts, in [8], definitions of almost p-convex sets and the p-convexly almost fixed point property are introduced as generalizations of almost convex sets (due to Himmelberg [9]) and the almost fixed point property, resp.

Now we have a new KKM space as in [50]:

Lemma 6.17. *Suppose that X is a subset of a topological vector space E and D is a nonempty subset of X such that $C_p(D) \subset X$. Let $\Gamma_N := C_p(N)$ for each $N \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a ϕ_A-space.*

(8) Γ-convex spaces — Zafarani [62] introduce Γ-convex spaces as generalizations of our G-convex spaces. This concept is actually our abstract convex spaces without assuming any topology. We recognized the existence of Zafarani's paper after we developed sufficiently rich investigations on our abstract convex spaces.

(9) ℝ-tree — Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [20]. Later it was known that $(H, X; \Gamma)$ is a KKM space; see Park [45].

(10) Horvath's convex space — According to Horvath [13], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see Horvath [13].

(11) 𝔹-spaces — Briec and Horvath [4] introduced $\mathbb{B}$-spaces, which are also KKM spaces [4, Corollary 2.2].

(12) Connected linearly ordered spaces — Such a space $(X, \leq)$ can be made into an abstract convex topological space $(X, D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A, \ \max A] = \{x \in X \mid \min A \leq x \leq \max A\}$ for each $A \in \langle D \rangle$. Further, it is a KKM space; see Park [33, Theorem 5(i)] and [41].

(13) Extended long line L^* — The set L^* can be made into a KKM space $(L^*, D; \Gamma)$. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology; see Park [41].

Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D \rangle \multimap L^*$ be the one as in the above (12). But L^* is not a G-convex space. In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^* \rangle$ and $\Delta_1 = [0, 1]$, there does not exist a continuous function

$\phi_A : [0,1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not G-convex.

(14) R-KKM spaces of Sankar Raj and Somasundaram — The authors [58] introduced an R-KKM map $T : A \to 2^B$ for two nonempty subsets A, B of a normed space X, the sufficient condition for which the set $\bigcap\{T(x) \mid x \in A\}$ is nonempty. Applying such intersection theorem, they show an extended version of the Fan-Browder fixed point theorem, in a normed linear space setting, by providing an existence of the best proximity point.

In [49], we show that, when we introduce the finitely generated topology on X, if (A, B) is a proximal pair and we let $\Gamma\{x_1, \dots, x_n\} := \text{co}\{y_1, \dots, y_n\}$, then we show that the abstract convex space $(B, A; \Gamma)$ becomes a partial KKM space. Moreover, since Γ has convex values, $(B, A; \Gamma)$ becomes an H-space and hence a KKM space. So it has many equivalent properties and the Fan-Browder type fixed point theorem is one of them.

(15) KKM spaces of Chaipunya and Kumam — The authors [5] study applications of intersection theorems related non-self KKM maps on Hadamard manifolds. Their results improve corresponding ones of Sanka Raj and Somasundaram [58]. They apply their KKM lemma to the Browder fixed point theorem for nonself maps and the solvability of generalized equilibrium problems.

The pair (A, B) set up by two given nonempty subsets A and B of a metric space (S, d) is called a *proximal pair* if to each point $(x, y) \in A \times B$, there corresponds a point $(\overline{x}, \overline{y}) \in A \times B$ such that

$$d(x, \overline{y}) = d(\overline{x}, y) = \text{dist}(A, B),$$

where $\text{dist}(A, B) := \inf\{d(x, y) \mid x \in A,\ y \in B\}$. In addition, if both A and B are convex, we say that (A, B) is a *convex proximal pair*.

And then, the authors assumed that M is a Hadamard manifold with the geodesic distance d. Given a point $x \in M$ and two nonempty subsets $A, B \subset M$, they write $d(x, A) := \inf_{z\in A} d(x, z)$. For any nonempty subset $A \subset M$, denoted by $\text{co}(A)$ the geodesically convex hull of A, i.e., the smallest geodesically convex set containing A. Note that the geodesically convex hull of any finite subset is compact.

Definition 6.18 ([5])**.** Let (A, B) be a proximal pair in a Hadamard manifold M. A nonself map $T : A \multimap B$ is said to be KKM if for each finite subset $D := \{x_1, x_2, \dots, x_m\} \subset A$, there is a subset $E := \{y_1, y_2, \dots, y_m\} \subset B$ such that $d(x_i, y_i) = \text{dist}(A, B), \forall i \in \{1, 2, \dots, m\}$, and

$$\text{co}(\{y_i \mid i \in I\}) \subset T(\{x_i \mid i \in I\})$$

for every $\emptyset \neq I \subset \{1, 2, \dots, m\}$.

Theorem 6.19 ([5])**.** *Suppose that (A, B) is a proximal pair in a Hadamard manifold M and $T : A \multimap B$ is a KKM map with nonempty closed values. Then, the family $\{T(x) \mid x \in A\}$ has the finite intersection property.*

Theorem 6.20. ([5]) *Suppose that (A, B) is a proximal pair in a Hadamard manifold M and $T : A \multimap B$ is a KKM map with nonempty closed values. If $T(x_0)$ is compact at some $x_0 \in A$, then the intersection $\bigcap\{T(x) \mid x \in A\}$ is nonempty.*

Here B seems to be better assumed to be geodesically convex.

In [5], an example of a partial KKM space $(B, A; \Gamma)$ is given and, A and B have no inclusion relation. Note that, the KKM non-selfmap $T : A \multimap B$ is a generalized KKM map in the sense of Chang and Zhang [6]; see also [56].

Finally in this section, we add a partial KKM space:

(16) Vector lattices of Kawasaki et al. — The authors study several fixed point theorems on vector lattices with units having certain topologies introduced by Kawasaki himself in 2009. Their main result ([15], Theorem 1) is essentially the following:

Theorem 6.21. *Let X be a Hausdorff Archimedean vector lattice with unit having the Kawasaki topology, Y a compact subset of X, and Z a convex subset of Y. Suppose that two multimaps $S : Z \multimap Y$, $T : Y \multimap Y$ satisfy*
(1) *$S(z)$ is open for each $z \in Z$;*
(2) *for each $y \in Y$, $\emptyset \neq \mathrm{co}_\Gamma S^-(y) \subset T^-(y)$.*
Then T has a fixed point $x_0 \in Y$; that is $x_0 \in T(x_0)$.

Note that $(Y \supset Z; \mathrm{co})$ is a partial KKM space in view of the Theorem 3.1 characterizing such spaces. Therefore the vector lattice in Theorem 6.21 satisfies a large number of statements in [44, 45] and Section 3.

7. Further examples on abstract convex spaces

In this section, we state why we should use triples $(X, D; \Gamma)$ for abstract convex spaces instead of pairs $(X; \Gamma)$ by recalling several examples of G-convex spaces, and we introduce an example of a partial KKM space due to Kulpa and Szymanski in 2014 which is not a KKM space.

(I) The original definition of G-convex space $(X, D; \Gamma)$ due to Park and H. Kim [53–55] assumed $X \supset D$ and the following isotonicity condition:

$$(\alpha) \qquad \text{if } A, B \in \langle D \rangle \text{ and } A \subset B, \text{ then } \Gamma_A \subset \Gamma_B$$

This isotonicity was removed since 1998, and the restriction $X \supset D$ was erased since 1999; see our articles after 2000. However most of useful examples of G-convex spaces seems to satisfy (α), some examples not satisfying (α) seem to be artificial:

Example 7.1. Let $\Delta_3 = \mathrm{co}\, V$ and $V = \{e_0, e_1, e_2, e_3\}$.

(1) As seen in the original KKM theorem, $(\Delta_3, V; \mathrm{co})$ is a G-convex space, where $\mathrm{co} : \langle V \rangle \multimap \Delta_3$ is the closure operation. Note that (α) holds in this example.

(2) For the G-convex space $(\Delta_3, V; \Gamma)$, let $\Gamma\{e_0, e_1\} := \mathrm{co}\{e_0, e_1, e_2\}$, and let $\Gamma(N) := \mathrm{co}\, N$ for any other $N \in \langle V \rangle$. Then Γ does not satisfy the isotonicity (α).

We give another example showing the necessity of using a triple $(X, D; \Gamma)$ instead of a pair $(E; \Gamma)$:

Example 7.2. (1) The well-known Sperner theorem and Alexandorff-Pasynkoff theorem on $n + 1$ closed sets covering an n-simplex were derived by applying the KKM theorem to the triple $(\Delta_n, V; \mathrm{co})$. No proofs of such theorems using a pair $(E; \Gamma)$ appeared yet.

(2) In Shapley's generalization of the KKM theorem, a triple $(\Delta_n, N; \Gamma)$ appears, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_S := \Delta^S = \mathrm{co}\,\{e_i \mid i \in S\}$ for each $S \in \langle N \rangle$; see [29] and the references therein.

(3) Let $\mathcal{C} := \mathcal{C}[0,1]$ be the class of continuous real functions defined on $[0,1]$, and $\mathcal{P} := \mathcal{P}[0,1]$ be the subclass of all polynomials $p(x)$, $x \in [0,1]$, having real coefficients. Choose an $\varepsilon > 0$ and, for each $f \in \mathcal{C}$, choose a $p_f \in \mathcal{P}$ in ε-neighborhood of f, we have $\max_{x\in[0,1]}|f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle \mathcal{C} \rangle \multimap \mathcal{P}$ be defined by $\Gamma_A := \mathrm{co}\,\{p_{f_i}\}_{i=0}^n \in \mathcal{P}$ for each $A = \{f_i\}_{i=0}^n \in \langle \mathcal{C} \rangle$. Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear mapping satisfying $e_i \mapsto p_{f_i}$ for each i. Then, $(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$ becomes a G-convex space satisfying the condition $(*)$ and $X \subsetneq D$.

(4) Similarly, by choosing a proper subset D of $\mathcal{C}$, we obtain G-convex space $(X, D; \Gamma)$ satisfying $X \nsubseteq D$ or $X \nsupseteq D$.

Such examples are why we did not assume any inclusion relation between X and D in the definition of G-convex spaces.

(5) Since there are many forms of the Stone-Weierstrass approximation theorem, we can make many examples similar to 3. or 4.

(II) For a quite long time, there was a question that whether the class of partial KKM spaces properly contain that of KKM spaces. At last Kulpa and Szymanski [22] found an example of a partial KKM space that is not a KKM space as follows:

Example 7.3. ([22]) Define an abstract convex space $([0,1]; \Gamma)$ by defining $\Gamma : \langle [0,1] \rangle \multimap [0,1]$ as follows: for $0 < p < 0.5 < q < 1$, we define

$$\Gamma(\{p\}) = \{p\}, \quad \Gamma(\{q\}) = \{q\}, \quad \Gamma(\{p,q\}) = [0,1] \setminus \{0.5\},$$

and define $\Gamma(A) = [0,1]$ for other $A \in \langle [0,1] \rangle$.

Then, $([0,1]; \Gamma)$ is a partial KKM space, but not a KKM space.

Added in Proof. Further examples of KKM spaces are obtained our later paper entitled *Extending the realm of Horvath spaces*, J. Nonlinear Convex Anal. **20**(8) (2019), 1609–1621.

References

[1] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484–490.

[2] A. Bayoumi, *Foundations of Complex Analysis in Nonlocally Convex Spaces — Function Theory without Convexity Condition,* Elsevier, 2003.

[3] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–151.

[4] W. Briec and C. Horvath, *Nash points, Ky Fan inequality and equilibria of abstract economies in Max-Plus and $\mathbb{B}$-convexity*, J. Math. Anal. Appl. **341** (2008), 188–189.

[5] P. Chaipunya and P. Kumam, *Nonself KKM maps and corresponding theorems in Hadamard manifolds*, Appl. Gen. Topol. **16** (2015), 37-44.

[6] S.-S. Chang and Y. Zhang, *Generalized KKM theorem and variational inequalities*, J. Math. Anal. Appl. **159** (1991), 208–223.

[7] X. P. Ding, *Maximal element theorems in product FC-spaces and generalized games*, J. Math. Anal. Appl. **305** (2005), 29–42.

[8] L. Gholizadeh, E. Karapinar and M. Roohi, *Some fixed point theorems in locally p-convex spaces*, Fixed Point Theory Appl. (2013), 2013:312, 10pp.

[9] C. J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.
[10] C. Horvath, *Points fixes et coïncidences pour les applications multivoques sans convexité*, C.R. Acad. Sci. Paris **296** (1983), 403–406.
[11] C. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.
[12] C. D. Horvath, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.
[13] C. D. Horvath, *Topological convexities, selections and fixed points*, Topology Appl. **155** (2008), 830–850.
[14] C. D. Horvath and J. V. Llinares-Ciscar, *Maximal elements and fixed points for binary relations on topological ordered spaces*, J. Math. Econom. **25** (1996), 291–306.
[15] T. Kawasaki, M. Toyoda and T. Watanabe, *Takahashi's, Fan-Browder's and Schawder-Tychonoff's fixed point theorems in a vector lattice*, RIMS Kôkyûroku, Kyoto Univ. **1685** (2010), 221–230.
[16] M. A. Khamsi, *KKM and Ky Fan theorems in hyperconvex metric spaces*, J. Math. Anal. Appl. **204** (1996), 298–306.
[17] P. Q. Khanh and N.H. Quan, *Intersection theorems, coincidence theorems and maximal element theorems in GFC-spaces*, Optimization **59** (2010), 115–124.
[18] P. Q. Khanh, N. H. Quan and J.-C. Yao, *Generalized KKM-type theorems in GFC-spaces and applications*, Nonlinear Anal. **71** (2009), 1227–1234.
[19] W. A. Kirk, *Krasnoselskii's iteration process in hyperbolic space*, Numer. Funct. Anal Optim. **4** (1982), 371–381
[20] W. A. Kirk and B. Panyanak, *Best approximations in* $\mathbb{R}$*-trees*, Numer. Funct. Anal. Optimiz. **28** (2007), 681–690.
[21] W. Kulpa and A. Szymanski, *Applications of general infimum principles to fixed-point theory and game theory*, Set-valued Anal. **16** (2008), 375–398.
[22] W. Kulpa and A. Szymanski, *Some remarks on Park's abstract convex spaces*, Top. Meth. Nonlinear Anal. **44** (2014), 369–379.
[23] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.
[24] J.-V. Llinares, *Unified treatment of the problem of existence of maximal elements in binary relations: a characterization*, J. Math. Economics **29** (1998), 285–302.
[25] E. Michael, *Convex structures and continuous selections,* Canad. J. Math. **11** (1959), 556–575.
[26] S. Park, *Convex spaces and KKM families of subsets*, Bull. Korean Math. Soc. **27** (1990) 11-14.
[27] S. Park, *On the KKM type theorems on spaces having certain contractible subsets*, Kyungpook Math. J. **32** (1992), 607–628.
[28] S. Park, *Five episodes related to generalized convex spaces*, Nonlinear Funct. Anal. Appl. **2** (1997), 49–61.
[29] S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 187–222.
[30] S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comp. Appl. Math. **7** (2000), 1–28.
[31] S. Park, *New subclasses of generalized convex spaces*, Fixed Point Theory and Applications, Y. J. Cho (ed.), Nova Sci. Publ., New York, 2000, pp.91–98.
[32] S. Park, *Remarks on topologies of generalized convex spaces*, Nonlinear Funct. Anal. & Appl. **5** (2000), 67–79.
[33] S. Park, *Examples of* $\mathfrak{KC}$*-maps and* $\mathfrak{KO}$*-maps on abstract convex spaces*, Soochow J. Math. **33** (2007), 477–486.
[34] S. Park, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. National Inst. Math. Sci. **2** (2007), 35–47.
[35] S. Park, *Comments on some abstract convex spaces and the KKM maps*, Nonlinear Anal. Forum **12** (2007), 125–139.

[36] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45** (2008), 1–27.

[37] S. Park, *Comments on the KKM theory on ϕ_A-spaces*, PanAmerican Math. J. **18** (2008), 61–71.

[38] S. Park, *Remarks on weakly KKM maps in abstract convex spaces*, Inter. J. Math. Math. Sci., vol. **2008** (2008), Article ID 423596, 10pp. doi:10.1155/2008/423596.

[39] S. Park, *Comments on the KKM theory on ϕ_A-spaces*, PanAmerican Math. J. **18** (2008), 61–71.

[40] S. Park, *Comments on recent studies on abstract convex spaces*, Nonlinear Anal. Forum **13** (2008), 1–17.

[41] S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. TMA **69** (2008), 4352–4364.

[42] S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex Anal. **9** (2008), 331–350.

[43] S. Park, *Generalized convex spaces, L-spaces, and FC-spaces*, J. Glob. Optim. **45** (2009), 203–210.

[44] S. Park, *Remarks on the partial KKM principle*, Nonlinear Anal. Forum **14** (2009), 51–62.

[45] S. Park, *The KKM principle in abstract convex spaces : Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[46] S. Park, *Applications of some basic theorems in the KKM theory* [in: The series of papers on S. Park's Contribution to the Development of Fixed Point Theory and KKM Theory], Fixed Point Theory Appl. vol. 2011:98 doi:10.1186/1687-1812-2011-98.

[47] S. Park, *Abstract convex spaces, KKM spaces and ϕ_A-spaces*, Nonlinear Anal. Forum **17** (2012), 1–10.

[48] S. Park, *A review of the KKM theory on ϕ_A-spaces or GFC-spaces*, Advances in Fixed Point Theory **3** (2013), 353–382.

[49] S. Park, *New examples of KKM spaces*, Nonlinear Anal. Forum **21** (2016), 23–35.

[50] S. Park, *On the KKM theory of locally p-convex spaces*, RIMS Kôkyûroku, Kyoto Univ. **2011** (2016), 70–77.

[51] S. Park, *A panoramic view of the realm of Ky Fan's 1952 lemma*, in: Proc. NACA2017, Yokohama Publishers, Yokohama, 2019, pp. 247–260.

[52] S. Park, *On multimap classes in the KKM theory*, RIMS Kôkyûroku, Kyoto Univ. **2114** (2019), 73–81.

[53] Park, S. and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU **18** (1993), 1-21.

[54] S. Park and H. Kim, *Coincidence theorems for admissible multifunctions on generalized convex spaces*, J. Math. Anal. Appl. **197** (1996), 173–187.

[55] S. Park and H. Kim, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551–571.

[56] S. Park and W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces*, J. Nonlinear Convex Anal. **2** (2001), 157–166.

[57] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. TMA **15** (1990), 537–558.

[58] V. Sankar Raj and S. Somasundaram, *KKM-type theorems for best proximity points*, Appl. Math. Lett. **25** (2012), 496–499.

[59] G. Tian, *Necessary and sufficient conditions for maximization of a class of preference relations*, Rev. Econ. Studies **60** (1993), 949–958.

[60] E. A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl. **102** (1999), 439–450.

[61] G. X.-Z. Yuan, *The characterization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications*, J. Math. Anal. Appl. **235** (1999), 315–325.

[62] J. Zafarani, *KKM property in topological spaces*, Bull. Soc. Royale Sci. Liège **73** (2004), 171–185.

Manuscript received September 8, 2018
revised March 5, 2019

Sehie Park
The National Academy of Sciences, Republic of Korea; Seoul 06579 and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea
E-mail address: park35@snu.ac.kr; sehiepark@gmail.com