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CATEGORY THEOREMS FOR MULTIPLIERS OF $A^P(G)$ $(1 \leq p)$

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Abstract. Let G be an infinite noncompact locally compact Abelian (LCA) group with dual group in Pontrjagin sense (that is $\widehat{\widehat{G}} = G$). Consider the space

$$A^p(G) = \left\{ f \in L^1(G) \text{ with Fourier transform } \widehat{f} \in L^p(\widehat{G}) \right\} \quad \text{for } 1 \leq p < \infty.$$

We equip $A^p(G)$ with the norm $\|f\|^p = \|f\|_1 + \left\|\widehat{f}\right\|_p$ for $f \in A^p(G)$, $1 \leq p$, which is equivalent to the norm $\max\left\{\|f\|_1, \left\|\widehat{f}\right\|_p\right\}$. Then $A^p(G)$ is a semisimple commutative Banach algebra under convolution product with norm $\|\cdot\|^p$. In this article, we split the index interval $[1, \infty)$ to be two disjoint intervals $[1, 2]$ and $(2, \infty)$ and then employ the extended Fourier transform operator T $(= T_p)$ which maps

$$\{f \in A^p(G), 1 \leq p \leq 2\} \quad \longrightarrow \quad \left\{\widehat{f} \in L^q(\widehat{G}), 2 < q < \infty\right\},$$

with relation $\frac{1}{p} + \frac{1}{q} = 1$. This T is a bounded linear operator ($T \in \mathcal{L}(A^p(G))$), satisfying the condition of multipliers on convolution algebra $A^p(G)$, $1 \leq p \leq 2$. Consequently, category is applicable.

1. Introduction and preliminaries

A multiplier concept come from Fourier Series. If a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt = f(t) \tag{1.1}$$

is convergent uniformly to a periodic function $f(t) \in L^1(\mathbb{R})$ with period 2π, then multiply both sides of (1.1) by $\cos nx$ (or $\sin nx$) and use termwise integration on (1.1), one could get

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \text{ and } b_n \\ &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad \text{for} \quad n = 0, 1, 2, \cdots\cdots . \end{aligned}$$

Using Euler's formula

$$e^{int} = \cos nt + i \sin nt,$$

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the series (1.1) becomes an infinite complex series

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}, \text{ with coefficients } c_n = \widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,\ n \in \mathbb{Z}. \tag{1.2}$$

The expression (1.2) is a Fourier series of $f \in L^1[0, 2\pi]$ with Fourier coefficients C_n.

Q. *For a bounded sequence $\varphi = (\varphi_n)$, does there exist a function $g \in L^1[0, 2\pi]$ such that*

$$\widehat{g}(n) = \varphi_n c_n = \left(\varphi \cdot \widehat{f}\right)(n),\ n \in \mathbb{Z}?$$

The answer is positive. Namely, such sequence is a multiplier of $L^1(\mathbb{R})$.

Remark 1.1. The multiplier idea is extended from the line $\mathbb{R}$ to a general LCA (locally compact Abelian) group, and the integration on G is regarded to a fixed Haar measure on G, and expressed the integration of $f(x)$ by

$$\int_G f(x)dx.$$

For a real number r bigger than or equal to 1, let $L^r(G)$ denote the set of all measurable functions f such that

$$\int_G |f(x)|^r\, dx < \infty.$$

Remark 1.2. The multiplier function φ for the 2π periodic function $f \in L^1(\mathbb{R})$ defines a bounded linear operator T_φ which maps $f \in L^1(G)$ into $g \in L^1(G)$ by a continuous linear mapping $T_\varphi \in \mathcal{L}(L^1(G), L^1(G))$ such that

$$g = T_\varphi f \ \text{ with } \widehat{g}(\widehat{x}) = \widehat{T_\varphi f}\,(\widehat{x}) = \widehat{Tf}\,(\widehat{x}) \quad \text{for } \widehat{x} \in \widehat{G},$$

where $f \in L^1(G)$ has a Fourier transform defined by

$$\widehat{f}(\widehat{x}) = \int_G \overline{\langle x, \widehat{x}\rangle} f(x)dx \quad \text{for } \widehat{x} \in \widehat{G},$$

where $\widehat{G}$ is the dual group of G, and is defined by the set of all functions $\widehat{x}$ map $x \in G$ to $\langle x, \widehat{x}\rangle$ on the unit circle $|\langle x, \widehat{x}\rangle| = 1$ of complex field $\mathbb{C}$.

In studying multiplier theory, there is a basic theorem (cf. Lai, Lee and Liu [6, Theorem 1.1]). Simply we say that a bounded linear operator $T \in \mathcal{L}(L^1(G))$ is a multiplier operator if any one of the following equivalent statements (i)-(iii) holds (cf. [6, Theorem 1.1]):

(i) T commutes with the translation operator ρ_a, that is $T\rho_a = \rho_a T$ with $\rho_a f(x) = f(x - a)$ for any $a \in G$ and $f \in L^1(G)$;

(ii) T commutes with the algebra product convolution of $L^1(G)$, that is

$$T(f * g) = Tf * g = f * Tg \quad \text{for any } f, g \in L^1(G);$$

(iii) There exists a unique $\mu \in \mathscr{M}(G)$, the space of bounded regular measures of G, such that $\mu * f = Tf$ for any $f \in L^1(G)$, and $\|\mu\| = \|T\|$.

Hence definition of the multiplier is various by (iii) we often use the symbol space $\mathfrak{M}(L^1(G)) \cong \mathscr{M}(G)$, where $\cong$ denotes the isometrically isomorphic.

Remark 1.3. Let G be a LCA group with dual group $\widehat{G}$. If $\mathbb{T}$ and $\mathfrak{D}$ are respectly the compact and discrete subgroups of G, then

$$G/\mathbb{T} = \mathfrak{D} = \widehat{\mathbb{T}}$$

and

$$G/\mathfrak{D} = \mathbb{T} = \widehat{\mathfrak{D}}.$$

In particular, if $\mathbb{R} = G$, $\mathbb{T} = [0, 2\pi]$ and $\mathfrak{D} = \mathbb{Z}$, then

$$\mathbb{R}/\mathbb{T} = \mathbb{Z} = \widehat{\mathbb{T}},$$

$$\mathbb{R}/\mathbb{Z} = \mathbb{T}([0, 2\pi]) = \mathbb{Z}$$

and $|\langle n, t\rangle| = \left|e^{int}\right| = 1$ denotes the unit circle with Euler's formula

$$\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = 1.$$

The terminology "multiplier" has also appeared in studying operator theory, P.D.E. stochastic process, optimization analysis (theory), and mathematical economic. For example, in minimax programming problems, one known the Lagrangian multipliers has many useful idea, it integrates the objective function as well as the constraints (inequality/equality) to be a global equation and then to solve the problems in applied mathematics.

In next section we will consider functions consistent two variables in LCA group G as well as the dual group $\widehat{G}$ in acountment.

2. The space $A^P(G)$ for $p \ge 1$

Let G be an infinite noncompact LCA group with dual group $\widehat{G}$ in the Pontrjagin sense (i.e. $\widehat{\widehat{G}} = G$) through out. Recall the space

$$A^p(G) = \left\{ f \in L^1(G) \text{ with Fourier transform } \widehat{f} \in L^p(\widehat{G}) \right\} \quad \text{for } 1 \le p,$$

and supply the norm

$$\|f\|^p = \|f\|_1 + \left\|\widehat{f}\right\|_p \quad \text{for } f \in A^p(G),$$

which norm $\|f\|^p$ is equivalent to the max-norm

$$\max\left\{ \|f\|_1, \left\|\widehat{f}\right\|_p \right\} \quad \text{for } f \in A^p(G).$$

By the uniqueness theorem of Fourier transforms of $f \in L^1(G)$, one sees that for each p, $1 \le p < \infty$, $A^p(G)$ is a semisimple $L^1(G)$-subalgebra under the norm $\|\cdot\|^p$ and convolution product " $*$ ". Since the group algebra $L^1(G)$ has a bounded approximate identity of norm $\|\cdot\|_1 = 1$, and so $A^p(G)$ is an essentially $L^1(G)$-Banach module. Indeed (cf. Lai et al. [6, Theorem 2.1]), we have

$$L^1(G) * A^p(G) = A^p(G)$$

and

$$\|f * g\|^p \le \|f\|_1 \|g\|^p, \quad \forall f \in L^1(G) \text{ and } g \in A^p(G).$$

It is remarkable that the approximate identity for $A^p(G)$ comes from $L^1(G)$ approximate identity with the norm $\|.\|_1 = 1$, but the approximate identity of $A^p(G)$ could not have $A^p(G)$-uniform bounded for each p, $1 \le p < \infty$ (see Lai [3, p.574])(cf. also Larsen [7, p.216]). Moreover, the system of $A^p(G)$-algebras, $1 \le p < \infty$, forms an ascending chain of dense ideals with respect to the index $p \ge 1$ in $L^1(G)$. The next theorem is immediately.

Theorem 2.1. *If $f \in A^p(G)$, $1 \le p$, the Fourier transform $\widehat{f} \in C_0(\widehat{G}) \cap L^p(\widehat{G})$ is a bounded continuous function in $L^p(\widehat{G})$ vanishing at infinity of $\widehat{G}$, then $\widehat{f} \in L^r(\widehat{G})$ for any $r > p \ge 1$*

Proof of Theorem 2.1. At first we note that $f \in A^p(G), 1 \le p < \infty$, one sees easily that $\widehat{f} \in C_0(\widehat{G})$ and $\widehat{f} \in L^p(\widehat{G})$, so that $\widehat{f} \in C_0(\widehat{G}) \cap L^p(\widehat{G})$, and that $\widehat{f}$ is a bounded continuous function in $L^p(\widehat{G})$ vanishing at infinity of $\widehat{G}$. Hence $\widehat{f} \in L^r(\widehat{G})$ for any $r > p \ge 1$, but $\widehat{f} \in L^r(\widehat{G})$ does not imply $f \in A^r(G)$. The proof is completed. □

By Theorem 2.1, we can split the index set $1 \le p < \infty$ (of $A^p(G)$) to be two disjoint intervals $1 \le p \le 2$ and $2 < q < \infty$ with relationship $\frac{1}{p} + \frac{1}{q} = 1$ and obtain the systems of functions

$$\{A^p(G) : 1 \le p \le 2\} \quad \text{and,} \quad \left\{A^q(\widehat{G}) : 2 < q < \infty\right\} \tag{2.1}$$

Moreover,

$$A^1(G) = \lim_{1 \le p < 2} A^p(G) = \bigcup_{1 \le p \le 2} A^p(G) \supset L^2(G). \tag{2.2}$$

Then the Fourier transform operator $\widehat{T_p}$ from:

$$\text{domain } D_T^p(G) = \{f \in A^p(G),\ 1 \le p \le 2\} \text{ into}$$
$$\text{range } R^q_{T_p} = \left\{\widehat{T_p} f \in L^q(\widehat{G}),\ 2 < q < \infty \subset \widehat{G}\right\}$$
$$\text{with } \frac{1}{p} + \frac{1}{q} = 1 \text{ such that } \left\|\widehat{T_p} f\right\|_q \le K_q \|f\|_p\,,$$

has the property (motivated from the paper of Hewitt [2]). Many other properties for Fourier transform operators of classes $L^p(G)$, $1 \le p \le 2$, will be presented in next section. Moreover, (2.2) represents the closure of such ascending chain of dense ideals with respect to $p \ge 1$ in $L^1(G)$.

3. Fourier transform operators for classes $L^p(G), 1 \le p \le 2$

Let G be an infinite noncompact LCA group with dual group $\widehat{G}$ in the Pontrjagin sense (i.e. $\widehat{\widehat{G}} = G$). For convenience, we denote the Fourier transforms on $L^1(G)$ and on $L^2(G)$ respectively by

$$\widehat{T_1} : L^1(G) \to C_0(\widehat{G})$$

$$\text{with} \quad \widehat{T_1} f(\widehat{x}) = \int_G \langle -x, \widehat{x} \rangle f(x) dx \quad \text{for } f \in L^1(G), \tag{3.1}$$

and satisfying $\left\|\widehat{T_1}f\right\|_\infty \leq \|f\|_1$

$$\widehat{T_2} : L^2(G) \to L^2(\widehat{G}) \quad \text{with} \quad \left\|\widehat{T_2}f\right\|_2 = \|f\|_2 \quad \text{for } f \in L^2(G) \tag{3.2}$$

If $f \in C_c(G)$, the Fourier transform operator $\widehat{T}$ $(= T_p)$ is defined by

$$\widehat{T}f(\widehat{x}) = \int_G \langle -x, \widehat{x}\rangle\, f(x)dx \quad \text{ for } \widehat{x} \in C_c(G)$$

$$\text{and } \widehat{T_1}f = \widehat{T_2}f = \widehat{T}f \quad \text{for all } f \in C_c(G) \;\; (\subset L^p(G))\,. \tag{3.3}$$

A. Weil [10, pp.116-117] had shown by using the convexity theorem of M. Riesz that the Fourier transform $\widehat{T}$ in (3.3) can be extended to a bounded linear operator T_p

$$\widehat{T_p} : (L^p(G),\, 1 \leq p \leq 2) \;\longrightarrow\; (L^q(\widehat{G}),\, 2 < q < \infty) \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \tag{3.4}$$

and hence has $\quad \widehat{T_p}f \in L^q(\widehat{G}) \cap C_0(\widehat{G}) \quad \text{for } f \in L^p(G)$

$$\text{and } \left\|\widehat{T_p}f\right\|_q \leq K_q \|f\|_p\,, \quad \text{for } 1 \leq p \leq 2.$$

where K_q is a positive number satisfying $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p \leq 2$ as $q \to \infty$ as well as $p \not> 2$.

Now recall the algebras $A^p(G)$, $1 \leq p < \infty$, defined by

$$A^p(G) = \left\{ f \in L^1(G) \text{ with Fourier transform } \widehat{f} \in L^p(\widehat{G}) \right\}.$$

We have known that

$$A^1(G) = \bigcup_{1 \leq p \leq 2} A^p(G) \subset L^2(G)$$

and consider the domain $D^p_T(G)$ and the range space $R^q_{T_p}$ of Fourier transform operator $\widehat{T_p}$ by

$$\widehat{T_p} : D^p_T(G) = \{f \in A^p(G),\, 1 \leq p \leq 2\} \text{ into } R^q_{T_p} = \left\{ \widehat{T_p}f \in L^q(\widehat{G}),\, 2 < q < \infty \right\}.$$

Thus $T_p(D^p_T) \subset D^p_{T_r}(G)$. Since $A^p(G)$ is an essential $L^1(G)$-module for $1 \leq p \leq 2$. Thus T_p can be extended to a bounded linear transform $\widehat{T_p}$ with domain $L^p(G)$ and the range $T_p(L^p(G))$ is dense in $L^q(\widehat{G})$ such that $\left\|\widehat{T_p}f\right\|_q \leq \|f\|_p$ for any $f \in L^p(G)$ $\quad (1 \leq p \leq 2)$.

Hence $T_p \in \mathfrak{L}(L^p(G), L^p(G)) = \mathfrak{M}(A^p(G)$ a multiplier space of $A^p(G)$, $1 \leq p < 2$. (E. Hewitt [2]).

Furthermore, one sees that $\widehat{T_p}$ is a one to one mapping. Indeed, for $f \in L^p(G)$ and $\varphi \in L^p(\widehat{G})$, one defines a bilinear form $\mathfrak{B}$ by

$$\mathfrak{B}(f,\varphi) = \int_G f \cdot T'_p(\varphi)dx = \int_{\widehat{G}} T_p(f)\varphi d\widehat{x}. \tag{3.5}$$

If $T_pf = 0$, $\mathfrak{B}(f,\varphi) = 0$ $\forall \varphi \in L^p(\widehat{G})$ deduces $f = 0$, this shows the one to one property of T_p, where $\widehat{T}_p : L^p(G) \to L^q(\widehat{G})$ and $T'_p : L^p(\widehat{G}) \to L^q(G)$ are well defined bounded linear mapping since $C_c(G)$ and $C_c(\widehat{G})$ are dense in $L^p(G)$ and $L^q(\widehat{G})$ respectively (cf. A. Weil [10, pp.116-117]).

4. On category theorem of multiplier $A^p(G)$ $1 < p < 2$

In [8], Larsen, Liu and Wang investigated the algebra

$$A^p(G) = \left\{ f \in L^1(G) \text{ with Fourier transform } \widehat{f} \in L^p(\widehat{G}) \right\}$$

and had shown that $L^1(G) \cap L^2(G) = A^2(G)$ and stated a plausible conjecture as follows:

$$\text{“}L^1(G) \cap L^p(G) = A^q(G) \quad \text{for } 1 < p < 2 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 \text{ is false.”}$$

In [4], Lai proved this conjecture (see Lai [4, Theorem 1].) It is based on the construction in Hewitt [2] to prove that for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, the set $L^1(G) \cap L^p(G)$ with norm $\|f\|_1 + \left\|\widehat{f}\right\|_p = \|f\|^q$ is a dense set in $A^q(\widehat{G})$ with respect to $A^q(\widehat{G})$-norm topology. Thus $L^1(G) \cap T(Ap(G)) \not\subset A^q(\widehat{G})$ for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that $A^p(G)$ has no multipliers for $p > 3$. Since $T(A^p(G)) \subset A^p(G)$, have not well defined. Hence we concluded only the next theorem.

Theorem 4.1. *The set $L^1(G) \cap L^p(G)$ is a dense set of 1st category in $A^q(\widehat{G})$ with respect to A^q topology. And the set of functions in $A^q(\widehat{G})$ which are not in $L^1(G) \cap L^p(G)$ is a dense set of the 2nd category.*

Proof. The reason of this come from the function f in $A^q(\widehat{G})$ which are not in $L^1(G) \cap L^p(G)$. This prove that $L^1(G) \cap L^p(G)$ is a dense set of 2nd category in $A^p(\widehat{G})$. □

Remark 4.2. For interesting readers, some open problems related to this paper can be found in [4] and [5].

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