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MULTIPLICITY OF SOLUTIONS FOR AN ANISOTROPIC PROBLEM IN ORLICZ-SOBOLEV SPACES

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his sixty-fifth anniversary

Abstract. We study a nonlinear anisotropic eigenvalue problem with homogeneous Dirichlet boundary condition involving an inhomogeneous and elliptic differential operator on a smooth bounded domain $\Omega \subset \mathbb{R}^n$. For that problem we prove the existence of a continuous family of eigenvalues. Moreover, each eigenvalue has at least two corresponding nonnegative eigenfunctions in a suitable anisotropic Orlicz-Sobolev space. Our main result is proved using as main tools the Mountain Pass Theorem and the Direct Method in Calculus of Variation.

1. Introduction

Equations involving inhomogeneous elliptic operators have been extensively studied in the literature in the framework of Orlicz-Sobolev spaces. With no hope of being complete, let us mention the pioneering works by Donaldson [8] and Gossez [18, 19] and some more recent advances by Clément et al. [5, 6], Fukagai et al. [13], Fukagai and Narukawa [15], Garcia-Huidobro et al. [16], Mihăilescu and Rădulescu [26], Bocea and Mihăilescu [4] etc. In all the above studies the inhomogeneous differential operator involved has the form

$$\operatorname{div}\left(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right). \tag{1.1}$$

The motivation in studying equations in which inhomogeneous differential operators of type (1.1) are involved comes from the fact that they can model with sufficient accuracy problems involving nonhomogenities. Setting

$$\Phi(t) := \int_0^t \phi(s)\, ds\,,$$

then the Euler-Lagrange functional associated to operator (1.1) is given by $\int \Phi(|\nabla u|)\, dx$. We point out a few examples from physics where such kind of nonlinearities can occur: *nonlinear elasticity*, when $\Phi(t) = (1+t^2)^\gamma - 1$, with $\gamma > 1/2$ (see, e.g., Fukagai and Narukawa [14]); *plasticity*, when $\Phi(t) = t^\alpha(\log(1+t))^\beta$, with $\alpha \geq 1$, $\beta > 0$ (see, e.g., Fuchs and Li [11]); *generalized Newtonian fluids*, when $\Phi(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta\, ds$, with $\alpha \in [0,1]$, $\beta > 0$ (see, e.g., Fuchs and Osmolovski [12]). In a similar context, we recall that recently Mihăilescu and Moroşanu [23]

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proposed a model of a *Hencky-type material* in $\mathbb{R}^2$ that involves a Hooke-type law in which a nonlinear and nonhomogeneous map relates the stress tensor with the strain tensor and leads to a system of partial differential equations involving inhomogeneous differential operators.

Motivated in part by the above advances, in this paper, we are interested in analyzing equations involving the following inhomogeneous and anisotropic differential operator

$$\sum_{i=1}^{n} \partial_{x_i} \left(\varphi_i \left(\partial_{x_i} u\right)\right), \tag{1.2}$$

where φ_i are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. The above-mentioned operator enables the study of equations with more complicated nonlinearities than (1.1) since the differential operator (1.2) allows a distinct behavior for partial derivatives in various directions. Problems involving this operator were also considered in [24].

To be more concrete, here we analyze the existence of nontrivial solutions of the following anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \partial_{x_i} \left(\varphi_i \left(\partial_{x_i} u\right)\right) = \lambda(u^{\alpha(x)-1}(x) - u^{\beta(x)-1}(x)), & \text{for } x \in \Omega, \\ u(x) = 0, & \text{for } x \in \partial\Omega, \\ u(x) \geq 0, & \text{for } x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive real number and the functions $\varphi_1, ..., \varphi_n$ are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and α, β are continuous nonnegative functions satisfying certain conditions that will be specified later in the paper.

Since problem (1.3) is anisotropic the classical Orlicz-Sobolev space where problems involving operator (1.1) were analyzed is not adequate for this new situation and consequently the appropriate function space where we will study problem (1.3) will be a generalization of the classical Orlicz-Sobolev space. Besides this fact, the term in the right hand side of (1.3) is inhomogeneous and it requires the variable exponent Lebesgue space setting.

The paper is organized as follows. In Section 2 we give the definitions of the variable exponent Lebesgue spaces, the Orlicz-Sobolev spaces and their generalization to the anisotropic case. Section 3 is devoted to specifying the assumptions on $\varphi_1, ..., \varphi_n, \alpha, \beta$. We also give the definition of the weak solution of problem (1.3) and we establish an existence and multiplicity result on problem (1.3). In Section 4 we prove this result by showing that problem (1.3) has at least two distinct nontrivial and nonnegative weak solutions in a suitable Orlicz-Sobolev type space.

2. Preliminary results

Through this section $\Omega \subset \mathbb{R}^n$ stands for a bounded domain.

2.1. **Variable exponent Lebesgue spaces.** For each continuous function $p : \overline{\Omega} \to (1, \infty)$, we define *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

On this space we introduce the *Luxemburg norm* by the formula

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

In this manner variable exponent Lebesgue spaces appear as a natural generalization of the classical Lebesgue spaces by replacing the fix exponent p with an exponent function $p(\cdot)$. Some of the properties and the results on the classical Lebesgue spaces readily generalize to the variable exponent framework. Thus, the variable exponent Lebesgue spaces endowed with the Luxemburg norm resemble the classical Lebesgue spaces in many respects: they are Banach spaces [20, Theorem 2.5], the Hölder's inequality holds [20, Theorem 2.1], they are separable [20, Corrolary 2.12], they are reflexive if and only if $1 < \inf_\Omega p \le \sup_\Omega p < \infty$ [20, Corollary 2.7] and continuous functions are dense if $\sup_\Omega p < \infty$ [20, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [20, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents, so that $p_1(x) \le p_2(x)$ for almost every $x \in \Omega$, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

In spite of common properties, the variable exponent Lebesgue spaces are not rearrangement invariant, the p-mean continuity may fail and the interpolation is not useful since it is not possible to interpolate from constant exponents to variable exponents.

We denote by $L^{q(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for any $x \in \overline{\Omega}$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, the Hölder's type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{\inf_\Omega p} + \frac{1}{\inf_\Omega q}\right) |u|_{p(\cdot)} |v|_{q(\cdot)} \tag{2.1}$$

holds true.

An important role in manipulating the variable exponent Lebesgue spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

For a constant exponent p, between the modular and the L^p-norm can be established an equality, that is $\rho_p(u) = \|u\|^p_{L^p(\Omega)}$. In the case of the variable exponents this simple relation does not remain valid but we can still deduce some useful inequalities between the modular and the Luxemburg norm. For each variable exponent p define

$$p_- = \inf_\Omega p \qquad p_+ = \sup_\Omega p\,.$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$ and $\sup_\Omega p < \infty$, then the following relations hold true

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|^{p_-}_{p(\cdot)} \le \rho_{p(\cdot)}(u) \le |u|^{p_+}_{p(\cdot)}\,, \tag{2.2}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|^{p_+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p_-}_{p(\cdot)}\,, \tag{2.3}$$

$$|u_n - u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0\,. \tag{2.4}$$

We refer to [9, 20, 7, 27] for further properties of variable exponent Lebesgue spaces.

2.2. **Orlicz-Sobolev spaces.** Orlicz-Sobolev spaces have been used in the last decades to model various phenomena. An overview of Orlicz-Sobolev spaces is given in the monographs by Rao and Ren [28]. The theory of Orlicz spaces has been well developed and widely used in various branches of mathematics, for example, potential theory, nonlinear partial differential equations, harmonic analysis, statistics and probability, Fourier analysis, stochastic analysis, interpolation (see [22], [28], [29]).

Next, we will introduce the Orlicz spaces. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We define

$$\Phi(t) = \int_0^t \varphi(s)\, ds$$

for any $t \geq 0$. We notice that Φ is a *Young function*, that is $\Phi(0) = 0$, Φ is continuous, Φ is convex and $\lim_{t\to\infty} \Phi(t) = \infty$. Moreover, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \Phi(t)/t = 0$ and $\lim_{t\to\infty} \Phi(t)/t = \infty$, then Φ is called a *N-function* (see [1, 2]). The function $\Phi^\star$ defined by

$$\Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds$$

for any $t \geq 0$, is called the *complementary function of* Φ and it satisfies

$$\Phi^\star(t) = \sup_{s\geq 0} \left(st - \Phi(s)\right)$$

for any $t \geq 0$. We observe that $\Phi^\star$ is a N-function, too. The following Young's inequality

$$st \leq \Phi(s) + \Phi^\star(t)$$

holds true for any $s,\ t \geq 0$.

Letting

$$\Phi^- := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \Phi^+ := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$$

we will assume that

$$1 < \Phi^- \leq \frac{t\varphi(t)}{\Phi(t)} \leq \Phi^+ < \infty \tag{2.5}$$

for any $t > 0$. Under this condition we can establish the following inequalities (see [13, Lemma A.2])

$$\gamma(\omega)\Phi(t) \leq \Phi(\omega t) \leq \zeta(\omega)\Phi(t) \quad \text{for any} \ \ \omega, t > 0, \tag{2.6}$$

where

$$\gamma(\omega) = \begin{cases} \omega^{\Phi^+}, & \text{if } \omega \in (0,1], \\ \omega^{\Phi^-}, & \text{if } \omega > 1, \end{cases} \qquad \text{and} \quad \zeta(\omega) = \begin{cases} \omega^{\Phi^-}, & \text{if } \omega \in (0,1], \\ \omega^{\Phi^+}, & \text{if } \omega > 1\,. \end{cases}$$

Under assumption (2.5), the function Φ satisfies the Δ_2-*condition*, that is

$$\Phi(2t) \le C\Phi(t), \quad \text{for any} \quad t > 0, \tag{2.7}$$

where C is a positive constant. Indeed, if we take $C = 2^{\Phi^+}$ we infer by (2.6) that the Δ_2-condition is fulfilled by Φ.

Furthermore, we assume that the function Φ satisfies the following condition

$$\text{the map} \quad [0,\infty) \ni t \longrightarrow \Phi(\sqrt{t}) \quad \text{is convex.} \tag{2.8}$$

Examples. We point out some examples of functions φ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, with the corresponding primitives Φ satisfying conditions (2.5) and (2.8) (see [6, Examples 1-3, page 243]):

(1) $\varphi(t) = |t|^{p-2}t$, $\Phi(t) = \dfrac{|t|^p}{p}$ with $p > 1$ and $\Phi^- = \Phi^+ = p$.

(2) $\varphi(t) = \log(1+|t|^r)|t|^{p-2}t$, $\Phi(t) = \log(1+|t|^r)\dfrac{|t|^p}{p} - \dfrac{r}{p}\displaystyle\int_0^{|t|} \frac{s^{p+r-1}}{1+s^r}\, ds$ with $p, r > 1$ and $\Phi^- = p$, $\Phi^+ = p + r$.

(3) $\varphi(t) = \dfrac{|t|^{p-2}t}{\log(1+|t|)}$ for $t \neq 0$, $\varphi(0) = 0$,
$\Phi(t) = \dfrac{|t|^p}{p\log(1+|t|)} + \dfrac{1}{p}\displaystyle\int_0^{|t|} \frac{s^p}{(1+s)(\log(1+s))^2}\, ds$ with $p > 2$ and
$\Phi^- = p-1$, $\Phi^+ = p = \liminf\limits_{t\to\infty} \dfrac{\log\Phi(t)}{\log t}$.

With function φ, its primitive Φ and the complementary function $\Phi^\star$ of Φ defined above, the *Orlicz space* $L_\Phi(\Omega)$ defined by N-function Φ (see [1, 2, 5]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{ \int_\Omega u(x)v(x)\, dx;\ \int_\Omega \Phi^\star(|v(x)|)\, dx \le 1 \right\} < \infty.$$

Then, the Orlicz space $L_\Phi(\Omega)$ endowed with the *Orlicz norm* $\|\cdot\|_{L_\Phi}$ is a Banach space and in addition, its Orlicz norm is equivalent to the *Luxemburg norm* defined by

$$\|u\|_\Phi := \inf\left\{ k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\, dx \le 1 \right\}.$$

Conditions (2.7) and (2.8) assure that the Orlicz space $L_\Phi(\Omega)$ is a uniformly convex space (see [26, Proposition 2.2]) and thus a reflexive Banach space.

In the case of Orlicz spaces, the Hölder's inequality (see [28, Inequality 4, page 79]) reads as follows:

$$\int_\Omega u(x)v(x)\, dx \le 2\|u\|_{L_\Phi}\|v\|_{L_{\Phi^\star}}$$

for any $u \in L_\Phi(\Omega)$ and $v \in L_{\Phi^\star}(\Omega)$.

On the other hand, for any $u \in L_\Phi(\Omega)$, the following inequalities hold true:

$$\|u\|_\Phi < 1 \Rightarrow \|u\|_\Phi^{\Phi^+} \le \int_\Omega \Phi(|u(x)|)\, dx \le \|u\|_\Phi^{\Phi^-}, \tag{2.9}$$

$$\|u\|_\Phi > 1 \Rightarrow \|u\|_\Phi^{\Phi^-} \leq \int_\Omega \Phi(|u(x)|)\,dx \leq \|u\|_\Phi^{\Phi^+}. \tag{2.10}$$

Remark 2.1. If $\Phi(t) = \dfrac{|t|^p}{p}$ with $p > 1$, the corresponding Orlicz space $L_\Phi(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$.

Embeddings between Orlicz spaces defined by Young functions are characterized in terms of the following partial-ordering relation between functions. A function F is said to dominate a function G globally, respectively near infinity, if there exists a positive constant K such that

$$G(t) \leq F(Kt)$$

for any $t > 0$, respectively for any t grater than some positive real number. Now, we recall an useful embedding result (see [2, Theorem 8.12]):

Theorem 2.2. *The embedding*

$$L_F(\Omega) \hookrightarrow L_G(\Omega)$$

is continuous if and only if either
(a) *F dominates G globally, or*
(b) *F dominates G near infinity and the n-dimensional Lebesgue measure of Ω, denoted by $|\Omega|$, is finite.*

Using the above result we can immediately establish the following lemma:

Lemma 2.3. *The Orlicz space $L_\Phi(\Omega)$ is continuously embedded in the Lebesgue space $L^q(\Omega)$ with $1 < q \leq \Phi^-$.*

With Φ that satisfies conditions (2.5) and (2.8) we introduce the *Orlicz-Sobolev space* $W^{1,\Phi}(\Omega)$ by

$$W^{1,\Phi}(\Omega) = \Big\{u \in L_\Phi(\Omega);\ \partial_{x_j} u \in L_\Phi(\Omega),\ \ j \in \{1, ..., n\}\Big\}.$$

This is a Banach space with respect to the following norm (obtained as a sum of Luxemburg norms)

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

We denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^1(\Omega)$ with respect to norm of $W^{1,\Phi}(\Omega)$. Taking into account [18, Lemma 5.7], we can consider on the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ the equivalent norms $\||\nabla u|\|_\Phi$ or $\displaystyle\sum_{j=1}^n \|\partial_{x_j} u\|_\Phi$. Conditions (2.7) and (2.8) assure that the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is a uniformly convex space (see [26, Proposition 2.2]) and thus a reflexive Banach space. For more details see the books [1, 2, 27, 28] and papers [5, 6, 16, 18].

Finally, we introduce a natural generalization of the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ that will enable us to study with sufficient accuracy problem (1.3). We will assume that the functions $\varphi_1, ..., \varphi_n$ are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and the corresponding primitives $\Phi_1, ..., \Phi_n$ satisfy conditions (2.5) and (2.8). Let us denote by $\overrightarrow{\Phi} : [0,\infty) \to \mathbb{R}^n$ the vectorial function $\overrightarrow{\Phi} = (\Phi_1, ..., \Phi_n)$. We define $W_0^{1,\overrightarrow{\Phi}}(\Omega)$, the *anisotropic Orlicz-Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{\Phi}} := \sum_{i=1}^n \|\partial_{x_i} u\|_{\Phi_i}.$$

The anisotropic Orlicz-Sobolev space $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ endowed with the above defined norm is a reflexive Banach space.

We denote by $\overrightarrow{\Phi}^+$, $\overrightarrow{\Phi}^-$ in $\mathbb{R}^n$ the vectors

$$\overrightarrow{\Phi}^+ = (\Phi_1^+, ..., \Phi_n^+), \quad \overrightarrow{\Phi}^- = (\Phi_1^-, ..., \Phi_n^-),$$

and by $\Phi_+^+, \Phi_+^-, \Phi_-^-$ the positive real numbers

$$\Phi_+^+ = \max\{\Phi_1^+, ..., \Phi_n^+\}, \quad \Phi_+^- = \max\{\Phi_1^-, ..., \Phi_n^-\}, \quad \Phi_-^- = \min\{\Phi_1^-, ..., \Phi_n^-\}.$$

Below we assume that

$$\sum_{i=1}^n \frac{1}{\Phi_i^-} > 1 \tag{2.11}$$

and we introduce $\Phi_\circ^- \in \mathbb{R}^+$ and $\Phi_* \in \mathbb{R}^+$ defined by

$$\Phi_\circ^- = \frac{n}{\displaystyle\sum_{i=1}^n \frac{1}{\Phi_i^-} - 1}, \quad \Phi_* = \max\{\Phi_+^-, \Phi_\circ^-\}.$$

We end this section by recalling an important result concerning the compactness embedding of the anisotropic Orlicz-Sobolev space $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ into the variable exponent Lebesgue space $L^{q(\cdot)}(\Omega)$ (see [24, Lemma 1]):

Theorem 2.4. *For any $q \in C(\overline{\Omega})$ that satisfies*

$$1 < q(x) < \Phi_* \quad \textit{for all} \ \ x \in \overline{\Omega}, \tag{2.12}$$

the embedding

$$W_0^{1,\overrightarrow{\Phi}}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact.

3. The main result

In this paper we investigate the problem (1.3) when functions $\varphi_1, ..., \varphi_n$ are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and the corresponding primitives $\Phi_1, ..., \Phi_n$ satisfy conditions (2.5) and (2.8) and $\alpha, \beta : \overline{\Omega} \to \mathbb{R}$ are continuous functions which satisfy

$$1 < \beta^- \le \beta(x) \le \alpha(x) \le \alpha^+ < \Phi_-^- \le \Phi_+^+ < \Phi_* \ \text{ for } \ x \in \Omega, \tag{3.1}$$

where $\beta^- := \inf_\Omega \beta$, $\alpha^+ := \sup_\Omega \alpha$ and, in addition, there exists at least a point $x_0 \in \Omega$ such that

$$\beta(x_0) < \alpha(x_0).$$

For each $i \in \{1, ..., n\}$, we introduce $a_i : [0, \infty) \to \mathbb{R}$ given by

$$a_i(t) = \begin{cases} \dfrac{\varphi_i(t)}{t}, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Taking into account that the functions φ_i are odd, then we deduce that $\varphi_i(t) = a_i(|t|)t$ for any $t \in \mathbb{R}$.

Concerning problem (1.3), we seek solutions to problem (1.3) belonging to the space $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ in the sense below.

Definition 3.1. We say that $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ is a *weak solution* to problem (1.3) if $u \geq 0$ almost everywhere in Ω and

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(|\partial_{x_i} u|)\, \partial_{x_i} u\, \partial_{x_i} v - \lambda \left(u^{\alpha(x)-1}\, v - u^{\beta(x)-1}\, v \right) \right\} dx = 0,$$

for all $v \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Moreover, we say that λ is an *eigenvalue* of problem (1.3) if the corresponding weak solution u defined above is not trivial.

The main result of this paper regarding the existence and the multiplicity of weak solutions is given by the following theorem.

Theorem 3.2. *There exists a positive constant μ such that any $\lambda \in \big[\mu, \infty\big)$ is an eigenvalue of problem (1.3). Moreover, problem (1.3) has at least two distinct nonnegative and nontrivial weak solutions for each $\lambda \geq \mu$.*

Remark. We point out the fact that Theorem 3.2 extends to the case of anisotropic Orlicz spaces the result obtained in [25, Theorem 1] in a similar context. Note that the term in the right hand side in problem (1.3) is more general than the one considered in equation (8) from [25] and requires a more careful analysis of the problem. On the other hand, Theorem 3.2 supplements the result obtained in [30, Theorem 3.1] where an anisotropic problem involving variable exponents is analyzed.

4. Proof of Theorem 3.2

We start by pointing out a useful result formulated in the following lemma. The proof of the result can be carried out in a standard manner (see, e.g. [17, Lemma 7.6]) and consequently we will omit it.

Lemma 4.1. *If $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ then $u_+, u_-, |u| \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ and*

$$\partial_{x_i} u_+ = \begin{cases} 0, & \text{if } u \leq 0, \\ \partial_{x_i} u, & \text{if } u > 0, \end{cases} \qquad \partial_{x_i} u_- = \begin{cases} 0, & \text{if } u \geq 0, \\ \partial_{x_i} u, & \text{if } u < 0, \end{cases}$$

$$\partial_{x_i} |u| = \begin{cases} \partial_{x_i} u, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \\ -\partial_{x_i} u, & \text{if } u < 0, \end{cases}$$

where $u_\pm(x) = \max\{\pm u(x), 0\}$ for all $x \in \Omega$.

In order to prove the main result, we associate to problem (1.3) the corresponding Euler-Lagrange functional $J_\lambda : W_0^{1,\overrightarrow{\Phi}}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \int_\Omega \sum_{i=1}^{n} \Phi_i(|\partial_{x_i} u|)\, dx - \lambda \int_\Omega \frac{u_+^{\alpha(x)}}{\alpha(x)}\, dx + \lambda \int_\Omega \frac{u_+^{\beta(x)}}{\beta(x)}\, dx, \tag{4.1}$$

where $u_+(x) = \max\{u(x), 0\},\ x \in \Omega$. Lemma 4.1 and standard arguments assure that $J_\lambda \in C^1(W_0^{1,\overrightarrow{\Phi}}(\Omega), \mathbb{R})$ and its Fréchet derivative is given by

$$\begin{aligned}\langle J_\lambda'(u), v\rangle = &\int_\Omega \sum_{i=1}^{n} a_i(|\partial_{x_i} u|)\, \partial_{x_i} u\, \partial_{x_i} v\ \ dx - \lambda \int_\Omega u_+^{\alpha(x)-1}\, v\, dx + \\ &\lambda \int_\Omega u_+^{\beta(x)-1}\, v\, dx,\end{aligned} \tag{4.2}$$

for all $u,\ v \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$.

Lemma 4.2. *Any critical point of the functional J_λ is nonnegative.*

Proof. Let u be a critical point of J_λ. We prove that $u \geq 0$ in Ω. Taking into consideration the conclusion of Lemma 4.1, we get

$$\begin{aligned} 0 &= \langle J_\lambda'(u), u_-\rangle \\ &= \int_\Omega \sum_{i=1}^{n} a_i(|\partial_{x_i}\, u|)\partial_{x_i} u\, \partial_{x_i} u_-\, dx - \lambda \int_\Omega u_+^{\alpha(x)-1} u_-\, dx + \\ &\quad \lambda \int_\Omega u_+^{\beta(x)-1} u_-\, dx \\ &= \int_\Omega \sum_{i=1}^{n} a_i(|\partial_{x_i}\, u|)\partial_{x_i} u\, \partial_{x_i} u_-\, dx \\ &= \int_\Omega \sum_{i=1}^{n} a_i(|\partial_{x_i}\, u_-|)|\partial_{x_i} u_-|^2\, dx \\ &\geq \int_\Omega \sum_{i=1}^{n} \varphi_i(|\partial_{x_i}\, u_-|)|\partial_{x_i} u_-|\, dx \\ &\geq \int_\Omega \sum_{i=1}^{n} \Phi_i^-\ \Phi_i(|\partial_{x_i}\, u_-|)\, dx \\ &\geq \Phi_-^- \int_\Omega \sum_{i=1}^{n} \Phi_i(|\partial_{x_i}\, u_-|)\, dx \end{aligned}$$

or

$$\int_\Omega \sum_{i=1}^{n} \Phi_i(|\partial_{x_i}\, u_-|)\, dx \leq 0. \tag{4.3}$$

We deduce that $\int_\Omega \Phi_i(|\partial_{x_i}\ u_-|)\ dx = 0$ for each $i = 1, ..., n$. This combined with (2.9) yields $\|\partial_{x_i}\ u_-\|_{\Phi_i} = 0$ for each $i = 1, ..., n$, or $\|u_-\|_{\overrightarrow{\Phi}} = 0$ which means that $u \geq 0$. □

Remark 4.3. By Lemma 4.2 it follows that the nontrivial critical points of functional J_λ are nonnegative weak solutions of problem (1.3).

Based on the above remark, we deduce that in order to seek the nonnegative and nontrivial weak solutions of problem (1.3) it is enough to find nontrivial critical points of functional J_λ. Thus, we can use the critical point theory in proving Theorem 3.2.

From now on, we consider the functional $I : W_0^{1,\overrightarrow{\Phi}}(\Omega) \to \mathbb{R}$ defined as

$$I(u) = \int_\Omega \sum_{i=1}^{n} \Phi_i(|\partial_{x_i}\ u|)\ dx$$

for every $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Standard arguments assure that I is well-defined on $W_0^{1,\overrightarrow{\Phi}}(\Omega)$, $I \in C^1(W_0^{1,\overrightarrow{\Phi}}(\Omega), \mathbb{R})$ and the Fréchet derivative is given by

$$\langle I'(u), v\rangle = \int_\Omega \sum_{i=1}^{n} a_i(|\partial_{x_i} u|)\ \partial_{x_i} u\ \partial_{x_i} v\ \ dx$$

for all $u, v \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$.

Lemma 4.4. *The functional I is weakly lower semicontinuous.*

Proof. The conclusion of this lemma is obvious since we deal with a functional I which is continuous and convex on the Banach space $W_0^{1,\overrightarrow{\Phi}}(\Omega)$. □

Lemma 4.5. *There exists a positive constant $\mathcal{S}$ such that the inequality*

$$\sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i}\ u|)\ dx \geq \mathcal{S} \int_\Omega |u|^{\Phi_-^-}\ dx$$

holds true for all $u \in S := \{v \in W_0^{1,\overrightarrow{\Phi}}(\Omega);\ \ \|v\|_{\overrightarrow{\Phi}} > n\ \}$.

Proof. Let $u \in S$ be fixed. Thus, $\|u\|_{\overrightarrow{\Phi}} > n$ and this fact implies that there exists $j \in \{1, ..., n\}$ such that $\|\partial_{x_j} u\|_{\Phi_j} > 1$. Using (2.10) we obtain

$$\int_\Omega \Phi_j(|\partial_{x_j} u|)\ dx \geq \|\partial_{x_j} u\|_{\Phi_j}^{\Phi_j^-} \geq \|\partial_{x_j} u\|_{\Phi_j}^{\Phi_-^-}\ . \tag{4.4}$$

Besides this, since $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ we infer that $\partial_{x_j} u \in L_{\Phi_j}(\Omega)$. Since $\Phi_-^- \leq \Phi_j^-$ then, by Lemma 2.3, the Orlicz space $L_{\Phi_j}(\Omega)$ is continuously embedded in the Lebesgue space $L^{\Phi_-^-}(\Omega)$, that means there exists a positive constant $C_j > 0$ such that

$$|\partial_{x_j} u|_{L^{\Phi_-^-}(\Omega)} \leq C_j \|\partial_{x_j} u\|_{\Phi_j}. \tag{4.5}$$

Relation (11) in [10, page 722] allows to take a positive constant $D_j > 0$ such that

$$|\partial_{x_j} u|_{L^{\Phi_-^-}(\Omega)} \geq D_j\ |u|_{L^{\Phi_-^-}(\Omega)}. \tag{4.6}$$

Therefore, inequalities (4.4), (4.5) and (4.6) imply the existence of a positive constant $\mathcal{S} := \min_{j\in\{1,\ldots,n\}} (D_j/C_j)^{\Phi_-^-}$ such that

$$\sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\,dx \geq \mathcal{S} \int_\Omega |u|^{\Phi_-^-}\,dx.$$

Thus, Lemma 4.5 is proved. □

Remark 4.6. A consequence of Lemma 4.5 is the fact that there exists a positive real number

$$\lambda^\star := \inf_{u\in S} \frac{\sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\,dx}{\int_\Omega |u|^{\Phi_-^-}\,dx}, \tag{4.7}$$

where $S := \{v \in W_0^{1,\overrightarrow{\Phi}}(\Omega);\ \|v\|_{\overrightarrow{\Phi}} > n\ \}$.

Lemma 4.7. *The functional J_λ is weakly lower semicontinuous, coercive and bounded from below.*

Proof. Taking into account the result of Lemma 4.4 we deduce that functional I is weakly lower semicontinuous. For justifying that J_λ is weakly lower semicontinuous, we consider a sequence $\{u_k\} \subset W_0^{1,\overrightarrow{\Phi}}(\Omega)$ which converges weakly to u in $W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Using that I is weakly lower semicontinuous we have

$$I(u) \leq \liminf_{k\to\infty}\ I(u_k). \tag{4.8}$$

Further, since (3.1) is fulfilled, $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ is continuously and compactly embedded in $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$ (by Theorem 2.4) and, consequently, we deduce that $\{(u_k)_+\}$ converges strongly to u_+ in $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$. Combining these two strong convergences with relation (4.8), we obtain

$$J_\lambda(u) \leq \liminf_{k\to\infty}\ J_\lambda(u_k),$$

which means that the functional J_λ is weakly lower semicontinuous.

By conditions $1 < \beta^- \leq \beta(x) \leq \alpha(x) \leq \alpha^+ < \Phi_-^-$ for every $x \in \Omega$ (see (3.1)), we infer that

$$\lim_{s\to\infty} \frac{\dfrac{s^{\alpha(x)}}{\alpha(x)} - \dfrac{s^{\beta(x)}}{\beta(x)}}{s^{\Phi_-^-}} = 0$$

for every $x \in \Omega$. Hence, there exists a positive constant D_λ such that

$$\lambda\left(\frac{s^{\alpha(x)}}{\alpha(x)} - \frac{s^{\beta(x)}}{\beta(x)}\right) \leq \frac{\lambda^\star}{2}\, s^{\Phi_-^-} + D_\lambda$$

for any $\lambda > 0$, $s \geq 0$ and $x \in \Omega$, where $\lambda^\star$ is defined in (4.7). In view of the above inequality, for any $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|u\|_{\overrightarrow{\Phi}} > n$, we find

$$\begin{aligned} J_\lambda(u) &= \sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\ dx - \lambda \int_\Omega \left(\frac{1}{\alpha(x)}\ u_+^{\alpha(x)} - \frac{1}{\beta(x)}\ u_+^{\beta(x)} \right) dx \\ &\geq \sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\ dx - \lambda \int_\Omega \left(\frac{1}{\alpha(x)}\ |u|^{\alpha(x)} - \frac{1}{\beta(x)}\ u_+^{\beta(x)} \right) dx \\ &\geq \sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\ dx - \frac{\lambda^\star}{2} \int_\Omega |u|^{\Phi_-^-}\ dx - D_\lambda\ |\Omega| \\ &\geq \frac{1}{2} \sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\ dx - D_\lambda\ |\Omega|. \end{aligned}$$

In order to go further, we denote by

$$\kappa_{i,u} := \begin{cases} \Phi_+^+, & \text{if } \|\partial_{x_i} u\|_{\Phi_i} < 1, \\ \Phi_-^-, & \text{if } \|\partial_{x_i} u\|_{\Phi_i} > 1, \end{cases}$$

for each $i \in \{1, ..., n\}$ and each $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|u\|_{\overrightarrow{\Phi}} > n$. Inequalities (2.9) and (2.10) yield

$$\begin{aligned} \sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} u|)\ dx &\geq \sum_{i=1}^{n} \|\partial_{x_i} u\|_{\Phi_i}^{\kappa_{i,u}} \\ &\geq \sum_{i=1}^{n} \|\partial_{x_i} u\|_{\Phi_i}^{\Phi_-^-} - \sum_{\{i;\ \kappa_{i,u} = \Phi_+^+\}} (\|\partial_{x_i} u\|_{\Phi_i}^{\Phi_-^-} - \|\partial_{x_i} u\|_{\Phi_i}^{\Phi_+^+}) \\ &\geq \frac{1}{n^{\Phi_-^- - 1}} \|u\|_{\overrightarrow{\Phi}}^{\Phi_-^-} - n \end{aligned}$$

for each $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|u\|_{\overrightarrow{\Phi}} > n$. Collecting the above pieces of information, we obtain that

$$J_\lambda(u) \geq \frac{1}{2n^{\Phi_-^- - 1}}\ \|u\|_{\overrightarrow{\Phi}}^{\Phi_-^-} - \frac{n}{2} - D_\lambda\ |\Omega|$$

for each $u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|u\|_{\overrightarrow{\Phi}} > n$. This inequality leads to the fact that J_λ is coercive and bounded from below. The proof of Lemma 4.7 is completed. □

By Lemma 4.7 and [31, Theorem 1.2] *we conclude that there exists* $v_1 \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ **a global minimizer of the functional** J_λ**.**

Lemma 4.8. *There exists a positive real number* μ *such that*

$$\inf_{u \in W_0^{1,\overrightarrow{\Phi}}(\Omega)} J_\lambda(u) < 0$$

for each $\lambda \geq \mu$.

Proof. Since $\beta(x_0) < \alpha(x_0)$ there exists a small neighborhood $\Omega_1 \subset \Omega$ of x_0 such that $\beta(x) < \alpha(x)$ for all $x \in \Omega_1$. Let $\omega_1 \subset \Omega_1$ be a compact subset, large enough and $v_0 \in W_0^{1,\overrightarrow{\Phi}}(\Omega_1)$ be such that $v_0(x) = 3/2$ in ω_1 and $0 \leq v_0(x) \leq 3/2$ in $\Omega_1 \setminus \omega_1$. Then $\frac{v_0^{\alpha(x)}}{\alpha(x)} - \frac{v_0^{\beta(x)}}{\beta(x)} > 0$ for all $x \in \omega_1$ and we have

$$\begin{aligned}\int_{\Omega_1} \left(\frac{v_0(x)^{\alpha(x)}}{\alpha(x)} - \frac{v_0(x)^{\beta(x)}}{\beta(x)} \right) dx \;&\geq\; \int_{\omega_1} \frac{v_0^{\alpha(x)}}{\alpha(x)}\, dx - \int_{\omega_1} \frac{v_0^{\alpha(x)}}{\beta(x)}\, dx - \\ &\quad \frac{1}{\beta^+} \int_{\Omega_1 \setminus \omega_1} v_0(x)^{\beta(x)}\, dx \\ &\geq \int_{\omega_1} \left(\frac{v_0^{\alpha(x)}}{\alpha(x)} - \frac{v_0^{\beta(x)}}{\beta(x)} \right) dx - \\ &\quad \frac{1}{\beta^+} \left(\frac{3}{2} \right)^{\beta^+} |\Omega_1 \setminus \omega_1| \,.\end{aligned}$$

Undoubtedly, the last line in the above inequality is positive provided that $|\Omega_1 \setminus \omega_1|$ is sufficiently small which is achieved for $\omega_1 \subset \Omega_1$ *large enough.* It follows that there exists $\mu > 0$ such that $J_\lambda(v_0) < 0$ for any $\lambda \geq \mu$. □

The above lemma shows that v_1 is nontrivial.

Further, we fix $\lambda \geq \mu$ and define the function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$h(x,t) = \begin{cases} 0, & \text{if } \ t < 0, \\ t^{\alpha(x)-1} - t^{\beta(x)-1}, & \text{if } \ 0 \leq t \leq v_1(x), \\ v_1^{\alpha(x)-1}(x) - v_1^{\beta(x)-1}(x), & \text{if } \ t > v_1(x), \end{cases}$$

where v_1 is the global minimizer of J_λ. We also introduce the function $H : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$H(x,t) = \int_0^t \ h(x,s) \ ds,$$

which is the primitive of the function h with respect to its second variable.

Next, we consider the functional $K_\lambda : W_0^{1,\overrightarrow{\Phi}}(\Omega) \to \mathbb{R}$ defined by

$$K_\lambda(v) = \int_\Omega \sum_{i=1}^n \Phi(|\partial_{x_i} v|) \ dx - \lambda \int_\Omega \ H(x,v) \ dx.$$

Standard arguments assure that $K_\lambda \in C^1(W_0^{1,\overrightarrow{\Phi}}(\Omega), \mathbb{R})$ and its Frechét derivative is given by

$$\langle K_\lambda'(v), w \rangle = \int_\Omega \sum_{i=1}^n a_i(|\partial_{x_i} v|)\, \partial_{x_i} v\, \partial_{x_i} w \ \ dx - \lambda \int_\Omega \ h(x,v)\ w \ \ dx,$$

for any $v, w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$.

Remark 4.9. We point out that any critical point of the functional K_λ is nonnegative in Ω. The proof is similar with the one considered in the case of J_λ.

Lemma 4.10. *A critical point v of the functional K_λ satisfies $v \leq v_1$ in Ω.*

Proof. Taking into account that v is a critical point of K_λ and v_1 is a a critical point of J_λ we have

$$\begin{aligned} 0 &= \langle K'_\lambda(v), (v-v_1)_+\rangle - \langle J'_\lambda(v_1), (v-v_1)_+\rangle \\ &= \int_\Omega \sum_{i=1}^n [a_i(|\partial_{x_i} v|)\partial_{x_i} v - a_i(|\partial_{x_i} v_1|)\partial_{x_i} v_1]\, \partial_{x_i}(v-v_1)_+ dx - \\ &\quad \lambda \int_\Omega \left[h(x,v) - \left(v_1^{\alpha(x)-1} - v_1^{\beta(x)-1}\right)\right](v-v_1)_+ dx \\ &= \int_{[v>v_1]} \sum_{i=1}^n [a_i(|\partial_{x_i} v|)\, \partial_{x_i} v - a_i(|\partial_{x_i} v_1|)\, \partial_{x_i} v_1]\, \partial_{x_i}(v-v_1) dx, \end{aligned}$$

or

$$\int_{[v>v_1]} \sum_{i=1}^n [a_i(|\partial_{x_i} v|)\, \partial_{x_i} v - a_i(|\partial_{x_i} v_1|)\, \partial_{x_i} v_1]\, (\partial_{x_i} v - \partial_{x_i} v_1)\, dx = 0. \tag{4.9}$$

Since for each $i \in \{1,...,n\}$ the functions Φ_i are continuous, increasing, $\Phi_i(0) = 0$ and satisfy condition (2.8), we can apply [21, Theorem 2.1] and we obtain that the following inequality

$$\begin{aligned} \frac{1}{2}\int_\Omega \left[\Phi_i(|\partial_{x_i} v|) + \Phi_i(|\partial_{x_i} w|)\right] dx \geq & \int_\Omega \Phi_i\left(\left|\frac{\partial_{x_i} v + \partial_{x_i} w}{2}\right|\right) dx + \\ & \int_\Omega \Phi_i\left(\left|\frac{\partial_{x_i} v - \partial_{x_i} w}{2}\right|\right) dx \end{aligned} \tag{4.10}$$

holds true for any $v,\ w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ and $i \in \{1,...,n\}$.

Moreover, by the fact that the functions Φ_i are convex, it follows that

$$\begin{aligned} \Phi_i(|\partial_{x_i} v(x)|) &\leq \Phi_i\left(\left|\frac{\partial_{x_i} v(x) + \partial_{x_i} w(x)}{2}\right|\right) + \\ &\quad \frac{1}{2}\, a_i(|\partial_{x_i} v|)\, \partial_{x_i}\ v\ (\partial_{x_i} v(x) - \partial_{x_i} w(x)) \end{aligned}$$

and

$$\begin{aligned} \Phi_i(|\partial_{x_i} w(x)|) &\leq \Phi_i\left(\left|\frac{\partial_{x_i} w(x) + \partial_{x_i} v(x)}{2}\right|\right) + \\ &\quad \frac{1}{2}\, a_i(|\partial_{x_i} w|)\, \partial_{x_i}\ w\ (\partial_{x_i} w(x) - \partial_{x_i} v(x)) \end{aligned}$$

hold true for any $v,\ w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$, $x \in \Omega$ and $i \in \{1,...,n\}$. Upon adding the last two inequalities and integrating over Ω, we get

$$\begin{aligned} & \frac{1}{2}\int_\Omega [a_i(|\partial_{x_i} v|)\, \partial_{x_i} v - a_i(|\partial_{x_i} w|)\, \partial_{x_i} w]\, (\partial_{x_i} v - \partial_{x_i} w)\, dx \geq \\ & \int_\Omega \left[\Phi_i(|\partial_{x_i} v|) + \Phi_i(|\partial_{x_i} w|)\right] dx - 2\int_\Omega \Phi_i\left(\left|\frac{\partial_{x_i} v + \partial_{x_i} w}{2}\right|\right) dx \end{aligned}$$

for any $v,\ w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ and $i \in \{1, ..., n\}$. Combining that with (4.10) we find

$$\begin{aligned}&\int_\Omega [a_i(|\partial_{x_i} v|)\, \partial_{x_i} v - a_i(|\partial_{x_i} w|)\, \partial_{x_i} w]\, (\partial_{x_i} v - \partial_{x_i} w)\, dx \geq \\ &4 \int_\Omega \Phi_i \left(\left| \frac{\partial_{x_i} v - \partial_{x_i} w}{2} \right| \right) dx\end{aligned} \tag{4.11}$$

for any $v,\ w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ and $i \in \{1, ..., n\}$. Using (4.9) and summing inequality (4.11) from $i = 1$ to $i = n$ we obtain

$$\sum_{i=1}^{n} \int_{[v>v_1]} \Phi_i(|\partial_{x_i} v - \partial_{x_i} v_1|)\ \ dx = 0,$$

or,

$$\sum_{i=1}^{n} \int_\Omega \Phi_i(|\partial_{x_i} (v - v_1)_+|)\ \ dx = 0.$$

In view of inequalities (2.9) and (2.10) we find $\|(v - v_1)_+\|_{\overrightarrow{\Phi}} = 0$. As $v - v_1 \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ by the conclusion of Lemma 4.1 we also get $(v - v_1)_+ \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Consequently, $(v - v_1)_+ = 0$ in Ω, that is $v \leq v_1$ in Ω. The proof of Lemma 4.10 is complete. □

Next, we will investigate the existence of a critical point for the functional K_λ. We start by proving some important lemmas.

Lemma 4.11. *There exist two constants $\theta \in (0, \|v_1\|_{\overrightarrow{\Phi}})$ and $a > 0$ such that*

$$K_\lambda(v) \geq a \text{ for any } v \in W_0^{1,\overrightarrow{\Phi}}(\Omega) \text{ with } \|v\|_{\overrightarrow{\Phi}} = \theta.$$

Proof. Let $v \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|v\|_{\overrightarrow{\Phi}} < 1$ be arbitrary fixed. It is easy to see that

$$\frac{s^{\alpha(x)}}{\alpha(x)} - \frac{s^{\beta(x)}}{\beta(x)} \leq 0 \text{ for any } s \in [0, 1] \text{ and any } x \in \Omega.$$

We denote by $\Omega_3 := \{x \in \Omega;\ v(x) > \min\{1, v_1(x)\}\}$. If $x \in \Omega \setminus \Omega_3$ then $v(x) \leq v_1(x)$ and $v(x) \leq 1$ and we infer that

$$H(x, v) = \frac{v_+^{\alpha(x)}}{\alpha(x)} - \frac{v_+^{\beta(x)}}{\beta(x)} \leq 0.$$

When $x \in \Omega_3 \cap \{x \in \Omega;\ v_1(x) < v(x) < 1\}$ we get

$$H(x, v) = \frac{v_1^{\alpha(x)}}{\alpha(x)} - \frac{v_1^{\beta(x)}}{\beta(x)} + \left(v_1^{\alpha(x)-1} - v_1^{\beta(x)-1} \right) (v - v_1) \leq 0.$$

So, we deduce that $H(x, v) \leq 0$ on $(\Omega \setminus \Omega_3) \cup (\Omega_3 \cap \{x \in \Omega;\ v_1(x) < v(x) < 1\})$.

Hereinafter, we denote by $\Omega_3' := \Omega_3 \setminus \{x \in \Omega;\ v_1(x) < v(x) < 1\}$. If we take $w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ with $\|w\|_{\overrightarrow{\Phi}} < 1$, applying Jensen's inequality and taking into account

(2.9), we find

$$\frac{\|w\|_{\overrightarrow{\Phi}}^{\Phi_+^+}}{n^{\Phi_+^+-1}} = n\left(\frac{\sum_{i=1}^{n}\|\partial_{x_i}w\|_{\Phi_i}}{n}\right)^{\Phi_+^+} \leq \sum_{i=1}^{n}\|\partial_{x_i}w\|_{\Phi_i}^{\Phi_+^+} \leq \sum_{i=1}^{n}\|\partial_{x_i}w\|_{\Phi_i}^{\Phi_i^+} \leq \sum_{i=1}^{n}\int_\Omega \Phi_i(|\partial_{x_i}w|)\,dx. \tag{4.12}$$

Consequently, we obtain

$$K_\lambda(v) \geq \frac{\|v\|_{\overrightarrow{\Phi}}^{\Phi_+^+}}{n^{\Phi_+^+-1}} - \lambda\int_{\Omega_3'} H(x,v)\;dx \tag{4.13}$$

provided that $\|v\|_{\overrightarrow{\Phi}} < 1$.

Looking at the definition of the functional H, it follows that

$$\begin{aligned}
\lambda\int_{\Omega_3'} H(x,v)dx =&\lambda\int_{\Omega_3'\cap\,[v>v_1]}\left(\frac{v_1^{\alpha(x)}}{\alpha(x)} - \frac{v_1^{\beta(x)}}{\beta(x)}\right)dx+\\
&\lambda\int_{\Omega_3'\cap\,[v>v_1]}\left(v_1^{\alpha(x)-1} - v_1^{\beta(x)-1}\right)(v-v_1)dx+\\
&\lambda\int_{\Omega_3'\cap\,[v<v_1]}\left(\frac{v_+^{\alpha(x)}}{\alpha(x)} - \frac{v_+^{\beta(x)}}{\beta(x)}\right)\;dx\\
\leq&\frac{\lambda}{\alpha^-}\int_{\Omega_3'\cap\,[v>v_1]} v_1^{\alpha(x)}dx + \lambda\int_{\Omega_3'\cap[v>v_1]} v_1^{\alpha(x)-1}v\;dx+\\
&\frac{\lambda}{\alpha^-}\int_{\Omega_3'\cap\,[v<v_1]} v_+^{\alpha(x)}\;dx\\
\leq&\lambda\;\mathfrak{C}_1\int_{\Omega_3'} v_+^{\alpha(x)}\;dx,
\end{aligned} \tag{4.14}$$

where $\alpha^- = \inf_{x\in\Omega}\alpha(x)$ and $\mathfrak{C}_1$ is a positive constant.

Further, we choose a real number r such that $1 < \Phi_+^+ < r < \Phi_*$. According to Theorem 2.4, $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ is continuously embedded in the classical Lebesgue space $L^r(\Omega)$. More exactly, there exists a constant $\mathfrak{C} > 0$ such that $|v|_{L^r(\Omega)} \leq \mathfrak{C}\;\|v\|_{\overrightarrow{\Phi}}$ for any $v \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Using these remarks and inequalities (4.14) we obtain

$$\begin{aligned}
\lambda\int_{\Omega_3'} H(x,v)dx &\leq \lambda\;\mathfrak{C}_1\int_{\Omega_3'} v_+^{\alpha(x)}\;dx\\
&\leq \lambda\;\mathfrak{C}_2\int_{\Omega_3'} v_+^{r}\;dx\\
&\leq \lambda\;\mathfrak{C}_3\;\|v\|_{\overrightarrow{\Phi}}^{r},
\end{aligned}$$

where $\mathfrak{C}_2$, $\mathfrak{C}_3$ are positive constants. The above estimates and inequality (4.13) yield

$$K_\lambda(v) \geq \left(\frac{1}{n^{\Phi_+^+ - 1}} - \lambda\, \mathfrak{C}_3 \, \|v\|_{\overrightarrow{\Phi}}^{r-\Phi_+^+} \right) \|v\|_{\overrightarrow{\Phi}}^{\Phi_+^+}.$$

Since r verifies $\Phi_+^+ < r$ then for a $\theta \in (0, \min\{1, \|v_1\|_{\overrightarrow{\Phi}}\})$ small enough we deduce that $K_\lambda(v) \geq a > 0$ with $\|v\|_{\overrightarrow{\Phi}} = \theta$. The proof of Lemma 4.11 is complete. $\square$

Lemma 4.12. *The functional K_λ is coercive.*

Proof. This proof can be carried out in a similar manner as the proof of Lemma 4.7 and for that reason we shall omit it. $\square$

Lemma 4.13. *The functional K_λ has a mountain-pass type critical point.*

Proof. In view of Lemma 4.11 and the Mountain Pass Theorem (see [3] with the variant given by [32, Theorem 1.15]), we obtain that there exists a sequence $\{v_k\} \subset W_0^{1,\overrightarrow{\Phi}}(\Omega)$ such that

$$K_\lambda(v_k) \to c > 0 \tag{4.15}$$

and

$$K'_\lambda(v_k) \to 0, \tag{4.16}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} K_\lambda(\gamma(t)) \geq a > 0\,,$$

with a given by Lemma 4.11 and

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,\overrightarrow{\Phi}}(\Omega)) : \ \gamma(0) = 0,\ \gamma(1) = v_1\}.$$

The sequence $\{v_k\}$ is bounded, as a consequence of Lemma 4.12 and (4.15). So, passing eventually to a subsequence of $\{v_k\}$, still denoted by $\{v_k\}$, we may assume that there exists $v_2 \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$ such that the subsequence $\{v_k\}$ converges weakly to v_2 in $W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Next, we verify that $\{v_k\}$ converges strongly to v_2 in $W_0^{1,\overrightarrow{\Phi}}(\Omega)$. By Theorem 2.4, the anisotropic Orlicz-Sobolev space $W_0^{1,\overrightarrow{\Phi}}(\Omega)$ is continuously and compactly embedded in the variable exponent Lebesgue spaces $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$. Thus, $\{v_k\}$ converges strongly to v_2 in $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$. On the other hand, relation (4.16) implies $\lim_{k\to\infty} \langle K'_\lambda(v_k), v_k - v_2 \rangle = 0$. Using all the above pieces of information we deduce by

$$\begin{aligned} \langle I'(v_k) - I'(v_2), v_k - v_2 \rangle &= \langle K'_\lambda(v_k) - K'_\lambda(v_2), v_k - v_2 \rangle + \\ &\quad \lambda \int_\Omega \big[h(x, v_k) - h(x, v_2)\big](v_k - v_2)\, dx \end{aligned}$$

that

$$\langle I'(v_k) - I'(v_2), v_k - v_2 \rangle = o(1)$$

or

$$\lim_{k\to\infty} \sum_{i=1}^{n} \int_\Omega \big[a_i(|\partial_{x_i} v_k|)\, \partial_{x_i} v_k - a_i(|\partial_{x_i} v_2|)\, \partial_{x_i} v_2\big] \left(\partial_{x_i} v_k - \partial_{x_i} v_2\right) \ dx = 0\,. \tag{4.17}$$

Taking into account that inequality (4.11) holds true for any $i \in \{1, ..., n\}$, we find

$$\lim_{k\to\infty} \sum_{i=1}^{n} \int_{\Omega} \Phi_i(|\partial_{x_i} v_k - \partial_{x_i} v_2|) \ dx = 0.$$

By (2.9) and (2.10) we infer that the sequence $\{v_k\}$ converges strongly to v_2 in $W_0^{1,\overrightarrow{\Phi}}(\Omega)$. After all, since $K_\lambda \in C^1(W_0^{1,\overrightarrow{\Phi}}(\Omega), \mathbb{R})$ and (4.15) and (4.16) hold true, we deduce that $K_\lambda(v_2) = c > 0$ and $\langle K'_\lambda(v_2), w\rangle = 0$ for any $w \in W_0^{1,\overrightarrow{\Phi}}(\Omega)$. Hence, v_2 is a critical point of K_λ. □

PROOF OF THEOREM 3.2. By Lemma 4.2, the global minimizer v_1 of the functional J_λ satisfies the property $v_1 \geq 0$ in Ω. Moreover, by Lemma 4.8 it follows that $J_\lambda(v_1) < 0$ for any $\lambda \geq \mu$ and therefore, v_1 is a nontrivial weak solution of problem (1.3) for any $\lambda \geq \mu$. Consequently, any $\lambda \geq \mu$ is an eigenvalue for problem (1.3).

To find the second nontrivial critical point of J_λ, our idea is to show that actually, the critical point v_2 of the functional K_λ obtained by a mountain-pass argument is a critical point of J_λ different from v_1.

Accordingly to Remark 4.9 and Lemma 4.10, we have $0 \leq v_2 \leq v_1$ in Ω, and therefore,

$$h(x, v_2) = v_2^{\alpha(x)-1} - v_2^{\beta(x)-1} \text{ and } H(x, v_2) = \frac{v_2^{\alpha(x)}}{\alpha(x)} - \frac{v_2^{\beta(x)}}{\beta(x)}.$$

It follows that

$$K_\lambda(v_2) = J_\lambda(v_2) \ \text{ and } \ K'_\lambda(v_2) = J'_\lambda(v_2),$$

for any $\lambda \geq \mu$, and consequently v_2 is a critical point of the functional J_λ for any $\lambda \geq \mu$. Additionally $J_\lambda(v_2) = c > 0 = J_\lambda(0) > J_\lambda(v_1)$, which implies that v_2 is nontrivial and it is distinct from v_1.

In conclusion, we have revealed that problem (1.3) has two distinct nonnegative and nontrivial weak solutions for λ large enough. The proof of our main result is complete.

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