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ON NONSMOOTH OPTIMALITY THEOREMS FOR ROBUST MULTIOBJECTIVE OPTIMIZATION PROBLEMS*

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Dedicated to Prof. Wataru Takahashi on his 70th birthday

Abstract. In this paper, we consider a nonsmooth multiobjective problem (UMP) with more than two locally Lipschitz objective functions and locally Lipschitz constraint functions in the face of data uncertainty, which is called a robust multiobjective optimization problem, and then associate (UMP) with its robust counterpart (RMP) by the worst case approach and define three kinds of solutions for (RMP), that is, robust efficient solution, weakly robust efficient solution and properly robust efficient solution, which can be regarded as ones for (UMP) defined by the worst case approach. We prove nonsmooth optimality theorems for weakly robust efficient solutions and properly robust efficient solutions for (UMP).

1. Introduction

Let X be a Banach space, and let functions $f_i, g_j\colon X \to \mathbb{R}$, $i = 1,\dots,p$, $j = 1,\dots,m$ be given. Consider the following multiobjective optimization problem with inequality constraints:

$$\text{(MP)} \qquad \text{Minimize} \quad (f_1(x),\dots,f_p(x)) \qquad \text{subject to} \quad g_j(x) \le 0,\ j = 1,\dots,m.$$

This problem in the face of data uncertainty in the constraints can be written by the following multiobjective optimization problem:

$$\text{(UMP)} \qquad \text{Minimize} \quad (f_1(x,u_1),\dots,(f_p(x,u_p)) \qquad \text{subject to} \quad g_j(x,v_j) \le 0,\ j = 1,\dots,m,$$

where u_i and v_j are uncertain parameters, and $u_i \in \mathcal{U}_i$, $v_j \in \mathcal{V}_j$ for some sequentially compact topological spaces $\mathcal{U}_i$ and $\mathcal{V}_j$, $i = 1,\dots,p$, $j = 1,\dots,m$ and $f_i\colon X \times \mathcal{U}_i \to \mathbb{R}$, $g_j\colon X \times \mathcal{V}_j \to \mathbb{R}$, $i = 1,\dots,p$, $j = 1,\dots,m$ are functions.

In this paper, we treat the robust approach for (UMP), which is the worst case approach for (UMP). Now we associate with (UMP) its robust counterpart:

$$\text{(RMP)} \qquad \text{Minimize} \quad \Big(\max_{u_1\in\mathcal{U}_1} f_1(x,u_1),\dots,\max_{u_p\in\mathcal{U}_p} f_p(x,u_p)\Big) \qquad \text{subject to} \quad g_j(x,v_j) \le 0,\ \forall v_j \in \mathcal{V}_j,\ j = 1,\dots,m,$$

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where the uncertain objective functions and constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{U}_i$, $i = 1, \ldots, p$ and $\mathcal{V}_j$, $j = 1, \ldots, m$. The problem (RMP) can be understood as the robust case (the worst case) of (UMP). So, optimizing (UMP) with (RMP) can be regarded as the robust approach (worst approach) for (UMP).

When $p = 1$, the problem (MP) involves one objective function and so it turns to the following scalar optimization problem:

$$\text{(P)} \quad \min\{f_1(x) \mid g_i(x) \leq 0, \ i = 1, \ldots, m\}.$$

When (P) is in the face of data uncertainty, it can be captured by the problem

$$\text{(UP)} \quad \min\{f_1(x, u_1) \mid g_i(x, v_i) \leq 0, \ i = 1, \ldots, m\},$$

which has been intensively studied in [2]-[7], [15]-[18], [23, 24]. The robust counterpart of (UP) is as follows [2, 4, 17]:

$$\text{(RP)} \quad \min\Big\{ \max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) \mid g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\Big\}.$$

Recently, to find robust solutions which are less sensitive to small perturbations in variables, Deb and Gupta [13, 14] defined two kinds of robust solutions for multiobjective optimization problems; the emphasis of their robust multiobjective approaches is to find a robust frontier, instead of the Pareto frontier in the problems.

In this paper, using (RMP), we define three kind robust solutions for the problems, which are different from the ones of Deb and Gupta [13, 14], as follows: let us define the set of robust feasible solutions as follows:

$$\mathcal{C} := \{x \in X \mid g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \ldots, m\}.$$

$\bar{x} \in \mathcal{C}$ is said to be a robust efficient solution of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leqq \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad i = 1, \ldots, p,$$
$$\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) < \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \quad \text{for some } \ k.$$

$\bar{x} \in \mathcal{C}$ is called a weakly robust efficient solution of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad i = 1, \ldots, p.$$

$\bar{x} \in \mathcal{C}$ is said to be a properly robust efficient solution of (UMP) if it is an efficient robust solution of (UMP) and there is a number $M > 0$ such that for all $i \in \{1, \ldots, p\}$ and $x \in \mathcal{C}$ satisfying $\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)$, there exists an index $k \in \{1, \ldots, p\}$ such that $\max\limits_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) < \max\limits_{u_k \in \mathcal{U}_k} f_i(x, u_k)$ and moreover

$$\frac{\max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)}{\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) - \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k)} \leqq M.$$

Jeyakumar, Li and Lee [18] proved a KKT optimality theorem for (UP) when involved functions in (RP) are continuously differentiable. Kuroiwa and Lee [19, 20] studied scalarizations and optimality theorems for (UMP) when involved functions are convex. Lee and Son [23] obtained a KKT optimality theorem for (RP) when involved functions in (UP) are locally Lipschitz. The aim of this paper is to extend the KKT optimality theorem in [23] to a robust multiobjective optimization problem. We prove nonsmooth optimality theorems for weakly robust efficient solutions and properly robust efficient solutions for (UMP) when involved functions are locally Lipschitz.

2. Preliminaries

In this section, we fix notation and give preliminary results for next sections. Let a function $f\colon X \to \mathbb{R}$ be given. We shall suppose that f is locally Lipschitz, that is, for each $x \in X$, there exist an open neighborhood U and a constant $L > 0$ such that for all y and z in U,

$$|f(y) - f(z)| \leq L\|y - z\|.$$

Definition 2.1. For each $d \in X$, the *generalized directional derivative* of f at x in the direction d, denoted $f^\circ(x; d)$, is given by

$$f^\circ(x;d) \;=\; \limsup_{h\to 0,\ t\to 0+} \frac{f(x+h+td) - f(x+h)}{t}.$$

We also denote the usual *one-sided directional derivative* of f at x by $f'(x; d)$. Thus

$$f'(x;d) = \lim_{t\to 0+} \frac{f(x+td) - f(x)}{t},$$

whenever this limit exists.

In the sequel, X^* denotes the (continuous) dual space of X and $\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^*. The norm of an element ξ of X^*, denoted $\|\xi\|^*$, is given by

$$\|\xi\|^* := \sup\{\langle \xi, d\rangle \mid d \in X, \|d\| \leq 1\}.$$

However, all statements involving a topology on X^* are with respect to the weak* topology, unless otherwise stated.

Definition 2.2. The *generalized gradient* of f at x, denoted by $\partial f(x)$, is the (nonempty) set of all ξ in X^* satisfying the following condition:

$$f^\circ(x; d) \geq \langle \xi, d\rangle \quad \text{for all} \quad d \in X.$$

We summarize some fundamental results in the calculus of generalized gradients (for more details, see [8]-[11], [21]):

(1) $\partial f(x)$ is a nonempty, convex, weak* compact subset of X^*
(2) The function $d \mapsto f^\circ(x; d)$ is convex.
(3) For every d in X, one has

$$f^\circ(x; d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial f(x)\}.$$

Let $\mathcal{V}$ be a sequentially compact topological space and let $g\colon X \times \mathcal{V} \to \mathbb{R}$ be a given function. Now, we will assume that the following conditions hold:

(C1) $g(x, v)$ is upper semicontinuous in (x, v).

(C2) g is locally Lipschitz in x, uniformly for v in $\mathcal{V}$, that is, for each $x \in X$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in \mathcal{V}$,

$$|g(y,v) - g(z,v)| \leq L\|y - z\|.$$

(C3) $g_x^\circ(x, v; \cdot) = g_x'(x, v; \cdot)$, the derivatives being with respect to x.

(C4) the generalized gradient $\partial_x g(x, v)$ with respect to x is weak* upper semicontinuous in (x, v).

Remark 2.3. In a suitable setting, conditions (C2), (C3), and (C4) will follow if the function g is convex in x and continuous in v. These conditions on the function g also hold when the derivative $\nabla_x g(x, v)$ with respect to x exists and is continuous in (x, v).

We define a function $\psi \colon X \to \mathbb{R}$ via

$$\psi(x) := \max\{g(x, v) \mid v \in \mathcal{V}\},$$

and we observe that our conditions (C1)-(C2) imply that ψ is defined and finite (with the maximum defining ψ attained) on X.

$$\mathcal{V}(x) := \{v \in \mathcal{V} \mid g(x, v) = \psi(x)\}.$$

It is easy to see that $\mathcal{V}(x)$ is nonempty and closed for each x in X.

The following lemma, which is a nonsmooth version of Danskin's theorem [12] for max-functions, makes connection between the functions $\psi'(x; d)$ and $g_x^\circ(x, v; d)$.

Lemma 2.4. *Under the conditions* (C1)-(C4)*, the usual one-sided directional derivative* $\psi'(x; d)$ *exists, and satisfies*

$$\begin{aligned}\psi'(x;d) = \psi^\circ(x;d) &= \max\{g_x^\circ(x, v; d) \mid v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d\rangle \mid \xi \in \partial_x g(x, v),\ v \in \mathcal{V}(x)\}.\end{aligned}$$

Proof. See [10, Theorem 2] (see also [8, Theorem 2.1], [12]). □

The following result will be useful in our later analysis.

Lemma 2.5 ([23])**.** *In addition to the basic conditions* (C1)-(C4)*, suppose that* $\mathcal{V}$ *is convex, and that* $g(x, \cdot)$ *is concave on* $\mathcal{V}$*, for each* $x \in U$*. Then the following statements hold:*

(i) *The set* $\mathcal{V}(x)$ *is convex and sequentially compact.*

(ii) *The set*

$$\partial_x g(x, \mathcal{V}(x)) := \{\xi \mid \exists v \in \mathcal{V}(x) \ \ \textit{such that} \ \ \xi \in \partial_x g(x, v)\}$$

is convex and weak compact.*

(iii) $\partial\psi(x) = \{\xi \mid \exists v \in \mathcal{V}(x) \ \ \textit{such that} \ \ \xi \in \partial_x g(x, v)\}$.

3. Necessary optimality theorems

Let X be a Banach space. Recall the robust counterpart (RMP) of (UMP):

$$\text{(RMP)}\qquad \text{Minimize}\qquad \Big(\max_{u_1\in\mathcal{U}_1} f_1(x,u_1),\ldots,\max_{u_p\in\mathcal{U}_p} f_p(x,u_p)\Big)$$
$$\text{subject to}\qquad g_j(x,v_j)\leq 0,\ \forall v_j\in\mathcal{V}_j,\ j=1,\ldots,m.$$

We assume that $f_i\colon X\times\mathcal{U}_i\to\mathbb{R}$, $i=1,\ldots,p$ are locally Lipschitz functions, and $g_j\colon X\times\mathcal{V}_j\to\mathbb{R}$, $j=1,\ldots,m$ are locally Lipschitz functions, and that $\mathcal{U}_i$, $i=1,\ldots,p$ and $\mathcal{V}_j$, $j=1,\ldots,m$ are sequentially compact topological spaces.

We recall the set of robust feasible solutions of (UMP):

$$\mathcal{C}:=\{x\in X\mid g_j(x,v_j)\leq 0,\ \forall v_j\in\mathcal{V}_j,\ j=1,\ldots,m\}.$$

Define $\phi_i(x):=\max_{u_i\in\mathcal{U}_i} f_i(x,u_i)$ for each $i=1,\ldots,p$ and $\psi_j(x):=\max_{v_j\in\mathcal{V}_j} g_j(x,v_j)$ for each $j=1,\ldots,m$. Then if f_i and g_j satisfy the conditions (C1) and (C2), $\phi_i,\psi_j\colon X\to\mathbb{R}$, $i=1,\ldots,p$ and $j=1,\ldots,m$, are locally Lipschitz functions.

Let $x\in\mathcal{C}$ and let us decompose $J:=\{1,\ldots,m\}$ into two index sets $J=J_1(x)\cup J_2(x)$, where $J_1(x)=\{j\in J\mid \psi_j(x)=0\}$ and $J_2(x)=J\setminus J_1(x)$. We put for each $i=1,\ldots,p$,

$$\mathcal{U}_i(x):=\{u_i\in\mathcal{U}_i\mid f_i(x,u_i)=\phi_i(x)\},$$

and for each $j\in J_1(x)$,

$$\mathcal{V}_j(x):=\{v_j\in\mathcal{V}_j\mid g_j(x,v_j)=\psi_i(x)\}.$$

Now we give a necessary optimality theorem for weakly robust efficient solutions for (UMP):

Theorem 3.1. *Assume that f_i, $i=1,\ldots,p$ and g_j, $j=1,\ldots,m$ satisfy the conditions* (C1) *and* (C2)*. If $x^*\in\mathcal{C}$ is a weakly robust efficient solution of* (UMP)*, then there exist $\mu_i\geqq 0$, $i=1,\ldots,p$, $\lambda_j\geqq 0$, $j\in J_1(x^*)$, not all zero, such that*

$$\sum_{i=1}^{p}\mu_i\phi_i^{\circ}(x^*;d)+\sum_{j\in J_1(x^*)}\lambda_j\psi_j^{\circ}(x^*;d)\geqq 0 \text{ for all } d\in X.$$

Proof. Suppose that there exists $d\in X$ such that

$$\begin{cases}\phi_i^{\circ}(x^*;d)<0, & i=1,\ldots,p,\\ \psi_j^{\circ}(x^*;d)<0, & j\in J_1(x^*).\end{cases}$$

Then we have for all $i=1,\ldots,p$,

$$\begin{aligned}\limsup_{t\to 0+}\frac{\phi_i(x^*+td)-\phi_i(x^*)}{t} &= \inf_{\bar\delta_i>0}\sup_{0<t<\bar\delta_i}\frac{\phi_i(x^*+td)-\phi_i(x^*)}{t}\\ &\leqq \inf_{\substack{\epsilon>0\\ \bar\delta_i>0}}\sup_{\substack{\|h\|<\epsilon\\ 0<t<\bar\delta_i}}\frac{\phi_i(x^*+h+td)-\phi_i(x^*+h)}{t}\\ &= \limsup_{h\to 0}\sup_{t\to 0+}\frac{\phi_i(x^*+h+td)-\phi_i(x^*+h)}{t}\end{aligned}$$

$$= \quad \phi_i^\circ(x^*;d) < 0.$$

So, we have

$$\limsup_{t\to 0+} \frac{\phi_i(x^*+td)-\phi_i(x^*)}{t} = \inf_{\bar{\delta}_i>0} \sup_{0<t<\bar{\delta}_i} \frac{\phi_i(x^*+td)-\phi_i(x^*)}{t} < 0.$$

Hence, there exist $\bar{\delta}_i^* > 0$, $i = 1,\dots,p$ such that for all $t \in (0,\bar{\delta}_i^*)$, $\phi_i(x^*+td) < \phi_i(x^*)$.

On the other hand, let $j \in J_1(x^*)$ be any fixed. Then, we have

$$\begin{aligned}
\limsup_{t\to 0+} \frac{\psi_j(x^*+td)-\psi_j(x^*)}{t} &= \inf_{\tilde{\delta}_j>0} \sup_{0<t<\tilde{\delta}_j} \frac{\psi_j(x^*+td)-\psi_j(x^*=h)}{t} \\
&\leqq \inf_{\substack{\epsilon>0 \\ \tilde{\delta}_j>0}} \sup_{\substack{\|h\|<\epsilon \\ 0<t<\tilde{\delta}_j}} \frac{\psi_j(x^*+h+td)-\psi_j(x^*+h)}{t} \\
&= \limsup_{h\to 0 \; t\to 0+} \frac{\psi_j(x^*+h+td)-\psi_j(x^*)}{t} \\
&= \psi_j^\circ(x^*;d) < 0.
\end{aligned}$$

So, we have

$$\limsup_{t\to 0+} \frac{\psi_j(x^*+td)-\psi_j(x^*)}{t} = \inf_{\tilde{\delta}_j>0} \sup_{0<t<\tilde{\delta}_j} \frac{\psi_j(x^*+td)-\psi_j(x^*)}{t} < 0.$$

Hence, there exist $\tilde{\delta}_j^* > 0$, $j \in J_1(x^*)$ such that for all $t \in (0,\tilde{\delta}_j^*)$, $\psi_j(x^*+td) < \psi_j(x^*) = 0$.

Moreover, since $\psi_j(x^*) < 0$, $j \in J_2(x^*)$ and ψ_j is continuous at x^*, there exist $\hat{\delta}_j^* > 0$, $j \in J_2(x^*)$ such that for all $t \in (0,\hat{\delta}_j^*)$, $\psi_j(x^*+td) < 0$. Let $\delta^* := \min\{\bar{\delta}^*, \tilde{\delta}^*, \hat{\delta}^*\}$, where $\bar{\delta}^* := \min_{j\in\{1,\dots,p\}} \bar{\delta}_i^*$, $\tilde{\delta}^* := \min_{i\in J_1(x^*)} \tilde{\delta}_j^*$ and $\hat{\delta}^* := \min_{i\in J_2(x^*)} \hat{\delta}_j^*$. Then for all $t \in (0,\delta^*)$, $x^*+td \in \mathcal{C}$ and $\phi_i(x^*+td) < \phi_i(x^*)$, $i = 1,\dots,p$. This is a contradiction since x^* is a weakly robust efficient solution of (UMP). Hence

$$\begin{cases} \phi_i^\circ(x^*;d) < 0, & i = 1,\dots,p, \\ \psi_j^\circ(x^*;d) < 0, & \forall j \in J_1(x^*) \end{cases}$$

has no solution $d \in X$. Since the functions $d \mapsto \phi_i^\circ(x;d)$, $i = 1,\dots,p$ and $d \mapsto \psi_j^\circ(x;d)$, $j = 1,\dots,m$ are convex, it follows from Gordan alternative theorem in [25] that there exist $\mu_i \geqq 0$, $i = 1,\dots,p$, $\lambda_j \geqq 0$, $j \in J_1(x^*)$, not all zero, such that

$$\sum_{i=1}^p \mu_i \phi_i^\circ(x^*;d) + \sum_{j\in J_1(x^*)} \lambda_j \psi_j^\circ(x^*;d) \geqq 0 \text{ for all } d \in X.$$

□

Definition 3.2. We define an *Extended Nonsmooth Mangasarian-Fromovitz constraint qualification* (ENMFCQ) at $x \in \mathcal{C}$ as follows:

$$\exists d \in X \text{ such that } \; g_{jx}^\circ(x, v_j; d) < 0, \; \forall v_j \in \mathcal{V}_j(x), \; \forall j \in J_1(x),$$

where $g^{\circ}_{jx}(x, v_j; d)$ denotes the generalized directional derivative of g_j with respect to x.

The following result is a robust KKT necessary optimality theorem for (UMP), which is a multiobjective version of Theorem 3.3 in [23] and a multiobjective and nondiffferentiable version of Theorem 3.1 in [18].

Theorem 3.3. *Assume that f_i, $i = 1, \ldots, p$ and g_j, $j = 1, \ldots, m$ satisfy the conditions* (C1)-(C4)*. Suppose that for each $x \in X$, $f_i(x, \cdot)$, $i = 1, \ldots, p$, are concave on $\mathcal{U}_i$, $i = 1, \ldots, p$ and $g_j(x, \cdot)$ are concave on $\mathcal{V}_j$, $j = 1, \ldots, m$. Let $x^* \in \mathcal{C}$ be a weakly robust efficient solution of* (UMP)*. Then there exist $\mu_i \geqq 0$, $i = 1, \ldots, p$, $\lambda_j \geqq 0$, $j = 1, \ldots, m$, not all zero, and $u_i^* \in \mathcal{U}_i(x^*)$, $i = 1, \ldots, p$, $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that*

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*),$$
$$0 = \lambda_j g_j(x^*, v_j^*), \ j = 1, \ldots, m,$$

where $\partial_x f_i(x^*, u_i^*)$ and $\partial_x g_j(x^*, v_j^*)$ are subdifferentials of the convex functions $f_i(\cdot, u_i^*)$ at x^* and $g_j(\cdot, v_j^*)$ at x^*. Moreover, if we further assume that the Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) holds, then there exist $\mu_i \geqq 0$, $i = 1, \ldots, p$, not all zero, and $u_i^* \in \mathcal{U}_i(x^*)$, $i = 1, \ldots, p$, $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*),$$
$$0 = \lambda_j g_j(x^*, v_j^*), \ j = 1, \ldots, m.$$

Proof. Let $x^* \in \mathcal{C}$ be a weakly robust efficient solution of (RMP). Then by Theorem 3.1, there exist $\mu_i \geqq 0$, $i = 1, \ldots, p$, $\lambda_j \geqq 0$, $j \in J_1(x^*)$, not all zero, such that

$$\sum_{i=1}^{p} \mu_i \phi_i^{\circ}(x^*; d) + \sum_{j \in J_1(x^*)} \lambda_j \psi_j^{\circ}(x^*; d) \geqq 0 \text{ for all } d \in X.$$

Since $\phi_i^{\circ}(x^*; d) = \max\{\langle \xi_i, d \rangle : \xi_i \in \partial \phi_i(x^*)\}$, $i = 1, \ldots, p$ and $\psi_j^{\circ}(x^*; d) = \max\{\langle \zeta_j, d \rangle : \zeta_j \in \partial \psi_j(x^*)\}$, $j = 1, \ldots, m$, we have for all $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{p} \max\{\langle \xi_i, d \rangle : \xi_i \in \partial \phi_i(x^*)\} + \sum_{j \in J_1(x^*)} \max\{\langle \zeta_j, d \rangle : \zeta_j \in \partial \psi_j(x^*)\} \geqq 0.$$

Hence for all $d \in \mathbb{R}^n$,

$$\max_{\substack{\xi_i \in \partial \phi_i(x^*) \\ \zeta_j \in \partial \psi_j(x^*)}} \Big\langle \sum_{i=1}^{p} \mu_i \xi_i + \sum_{j \in J_1(x^*)} \lambda_j \zeta_j, d \Big\rangle \geqq 0.$$

This is equivalent to

$$\inf_{d\in\mathbb{R}^n} \max_{\substack{\xi_i\in\partial\phi_i(x^*)\\ \zeta_j\in\partial\psi_j(x^*)}} \Big\langle \sum_{i=1}^{p}\mu_i\xi_i + \sum_{j\in J_1(x^*)}\lambda_j\zeta_j, d\Big\rangle \geqq 0.$$

Since the sets $\partial\phi_i(x^*)$ and $\partial\psi_j(x^*)$ are convex and weak* compact, by "lop-sided" minimax theorem [1],

$$\max_{\substack{\xi_i\in\partial\phi_i(x^*)\\ \zeta_j\in\partial\psi_j(x^*)}} \inf_{d\in\mathbb{R}^n} \Big\langle \sum_{i=1}^{p}\mu_i\xi_i + \sum_{j\in J_1(x^*)}\lambda_j\zeta_j, d\Big\rangle \geqq 0.$$

So, there exist $\xi_i \in \partial\phi_i(x^*)$, $i = 1,\dots,p$ and $\zeta_j \in \partial\psi_j(x^*)$, $j \in J_1(x^*)$ such that for all $d \in \mathbb{R}^n$,

$$\Big\langle \sum_{i=1}^{p}\mu_i\xi_i + \sum_{j\in J_1(x^*)}\lambda_j\zeta_j, d\Big\rangle \geqq 0.$$

Hence $\displaystyle\sum_{i=1}^{p}\mu_i\xi_i + \sum_{j\in J_1(x^*)}\lambda_j\zeta_j = 0$, and we have

$$0 \in \sum_{i=1}^{p}\mu_i\partial\phi_i(x^*) + \sum_{j\in J_1(x^*)}\lambda_j\partial\psi_j(x^*).$$

Thus, by letting $\lambda_j = 0$ for all $j \in J_2(x^*)$, we have

$$\begin{aligned} &0 \in \sum_{i=1}^{p}\mu_i\partial\phi_i(x^*) + \sum_{j=1}^{m}\lambda_j\partial\psi_j(x^*),\\ &0 = \lambda_j\phi_j(x^*),\ j = 1,\dots,m. \end{aligned}$$

On the other hand, it follows from Lemma 2.5 (iii) that

$$\begin{aligned} &\partial\phi_i(x^*) = \{\xi_i \mid \exists u_i \in \mathcal{U}_i(x^*) \ \text{ such that } \ \xi_i \in \partial_x f_i(x^*, u_i)\},\ i = 1,\dots,p,\\ &\partial\psi_j(x^*) = \{\zeta_j \mid \exists v_j \in \mathcal{V}_j(x^*) \ \text{ such that } \ \zeta_j \in \partial_x g_j(x^*, v_j)\},\ j = 1,\dots,m. \end{aligned}$$

Therefore there are $u_i^* \in \mathcal{U}_i(x^*)$ and $v_j^* \in \mathcal{V}_j(x^*)$ satisfying the following conditions

$$\begin{aligned} 0 &\in \sum_{i=1}^{p}\mu_i\partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m}\lambda_j\partial_x g_j(x^*, v_j^*),\\ 0 &= \lambda_j g_j(x^*, v_j^*), \quad j = 1,\dots,m. \end{aligned}$$

We now assume that the extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) at x^* holds. Then $\bar{\mu}_i$, $i = 1,\dots,p$, are not all zero. In fact, if it is not true, then $\lambda_j \geqq 0$, $j \in J_1(x^*)$, not all zero, and

$$0 \in \sum_{j\in J_1(x^*)}\lambda_j\partial_x g_j(x^*, v_j^*) = \partial_x\Big(\sum_{j\in J_1(x^*)}\lambda_j g_j(\cdot, v_j^*)\Big)(x^*).$$

(The equality is induced by (C3)) So, $\sum_{j\in J_1(x^*)}\lambda_j g^\circ_{jx}(x^*,v_j^*;d)\geqq 0$ for all $d\in\mathbb{R}^n$ which contradicts (ENMFCQ). Hence there exist $\mu_i\geqq 0$, $i=1,\ldots,p$, not all zero, and $u_i^*\in\mathcal{U}_i$, $i=1,\ldots,p$, $\lambda_j\geqq 0$ and $v_j^*\in\mathcal{V}_j$, $j=1,\ldots,m$ such that

$$0\in\sum_{i=1}^p\mu_i\partial_x f_i(x^*,u_i^*)+\sum_{j=1}^m\lambda_j\partial_x g_j(x^*,v_j^*),$$
$$0=\lambda_j g_j(x^*,v_j^*),\ j=1,\ldots,m.$$

□

From the proof of Theorem 3.3, we can obtain the following theorem for a convex multiobjective optimization problem:

Theorem 3.4. *Assume that $f_i(\cdot,\cdot)$, $i=1,\ldots,p$ and $g_j(\cdot,\cdot)$, $j=1,\ldots,m$ are continuous, and $f_i(\cdot,u_i)$, $i=1,\ldots,p$ and $g_j(\cdot,v_j)$, $j=1,\ldots,m$ are convex on X. Suppose that for each $x\in X$, $f_i(x,\cdot)$, $i=1,\ldots,p$, are concave on $\mathcal{U}_i$, $i=1,\ldots,p$ and $g_j(x,\cdot)$ are concave on $\mathcal{V}_j$, $j=1,\ldots,m$. Let $x^*\in\mathcal{C}$ be a weakly robust efficient solution of* (UMP). *If the robust Slater condition holds, that is, there exists $\hat{x}\in X$ such that $g_j(\hat{x},v_j)<0$, for any $v_j\in\mathcal{V}_j$, $j=1,\ldots,m$, then there exist $\mu_i\geqq 0$, $i=1,\ldots,p$, not all zero, and $u_i^*\in\mathcal{U}_i(x^*)$, $i=1,\ldots,p$, $\lambda_j\geqq 0$ and $v_j^*\in\mathcal{V}_j(x^*)$, $j=1,\ldots,m$ such that*

$$0\in\sum_{i=1}^p\mu_i\partial_x f_i(x^*,u_i^*)+\sum_{j=1}^m\lambda_j\partial_x g_j(x^*,v_j^*),$$
$$0=\lambda_j g_j(x^*,v_j^*),\ j=1,\ldots,m,$$

where $\partial_x f_i(x^*,u_i^*)$ and $\partial_x g_j(x^*,v_j^*)$ are subdifferentials of the convex functions $f_i(\cdot,u_i^*)$ at x^* and $g_j(\cdot,v_j^*)$ at x^*.

By Theorem 2.6 in [22], we can get the following lemma for properly robust efficient solutions of (UMP) :

Lemma 3.5. [22] *Let $X=\mathbb{R}^n$. $\bar{x}\in\mathcal{C}$ is a properly robust efficient solution of* (UMP) *if and only if there exist $M>0$ and $\hat{\mu}_i>0$, $i=1,\ldots,p$ such that*

$$\min_{x\in\mathcal{C}}\hat{f}(x)=\hat{f}(\bar{x})=0,$$

$$\textit{where } \hat{f}(x)=\sum_{i=1}^p\hat{\mu}_i\Big[\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)-\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)\Big]$$
$$+M\Big(\sum_{i=1}^p\hat{\mu}_i\Big)\max_{i\in\{1,\ldots,p\}}\Big[\max_{u_i\in\mathcal{U}_i}f_i(x,u_i)-\max_{u_i\in\mathcal{U}_i}f_i(\bar{x},u_i)\Big].$$

Using Lemma 3.5, we can obtain a necessary optimality theorem for a properly robust efficient solution of (UMP):

Theorem 3.6. *Let $X = \mathbb{R}^n$. Assume that f_i, $i = 1, \ldots, p$ and g_j, $j = 1, \ldots, m$ satisfy the conditions* (C1)-(C4). *Suppose that for each $x \in X$, $f_i(x, \cdot)$, $i = 1, \ldots, p$, are concave on $\mathcal{U}_i$, $i = 1, \ldots, p$ and $g_j(x, \cdot)$ are concave on $\mathcal{V}_j$, $j = 1, \ldots, m$. Let $x^* \in \mathcal{C}$ be a properly robust efficient solution of* (UMP). *Assume that the extended Nonsmooth Mangasarian-Fromovitz constraint qualification* (ENMFCQ) *holds. Then, there exist $\mu_i > 0$, $u_i^* \in \mathcal{U}_i(x^*)$ $i = 1, \ldots, p$, and $\lambda_j \geqq 0$, $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that*

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*)$$

and $\lambda_j g_j(x^*, v_j^*) = 0$, $j = 1, \ldots, m$.

Proof. Let $x^* \in \mathcal{C}$ be a properly robust efficient solution of (UMP). Then by Theorem 3.1, there exist $M > 0$ and $\hat{\mu}_i > 0$, $i = 1, \ldots, p$ such that

$$\min_{x \in \mathcal{C}} \hat{f}(x) = \hat{f}(x^*) = 0,$$

$$\text{where } \hat{f}(x) = \sum_{i=1}^{p} \hat{\mu}_i \Big[\max_{\hat{\mu}_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x^*, u_i) \Big]$$
$$+ M\Big(\sum_{i=1}^{p} \hat{\mu}_i\Big) \max_{i \in \{1,\ldots,p\}} \Big[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x^*, u_i) \Big].$$

Here $\hat{f}$ is a locally Lipchitz function. Let $\psi_j(x) := \max_{v_j \in \mathcal{V}_j} g_j(x, v_j)$. Then, by Remark 6.1.2 in [11], there exist $\lambda_0 \geqq 0$ and $\lambda_j \geqq 0$, $j = 1, \ldots, m$, not all zero, such that

$$0 \in \lambda_0 \partial \hat{f}(x^*) + \sum_{j=1}^{m} \lambda_j \partial \psi_j(x^*), \quad \lambda_j \psi_j(x^*) = 0, \ j = 1, \ldots, m.$$

By Lemma 2.5 (iii), there exist $\lambda_0 \geqq 0$, $\lambda_j \geqq 0$, not all zero, $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \lambda_0 \partial \hat{f}(x^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*), \quad \lambda_j g_j(x^*, v_j^*) = 0, \ j = 1, \ldots, m.$$

Assume to the contrary that $\lambda_0 = 0$. Then

$$0 \in \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

So, there exist $\bar{\zeta}_j \in \partial_x g_j(x^*, v_j^*)$, $j = 1, \ldots, m$ such that

$$0 = \sum_{j=1}^{m} \lambda_j \bar{\zeta}_j.$$

Since for any $d \in X$,, $g_j^\circ(x, v_j^*; d) = \max\{\langle \zeta_j, d\rangle \mid \zeta_j \in \partial_x g_j(x, v_j^*)\}$, $j = 1, \ldots, m$, then for any $d \in X$,

$$0 = \sum_{j=1}^{m} \lambda_j \langle \bar{\zeta}_j, d\rangle \leqq \sum_{j=1}^{m} \lambda_j g_j^\circ(x^*, v_j^*; d).$$

which contradict (ENMFCQ). So, λ_0 can not be 0. Hence we may assume without loss of generality that $\lambda_0 = 1$. Thus, there exist $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \partial \hat{f}(x^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

Let $\phi_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(x^*, u_i)$, $i = 1, \ldots, p$. Then $\phi_i(x^*) = 0$, $i = 1, \ldots, p$. So, we have

$$\hat{f}(x) = \sum_{i=1}^{p} \hat{\mu}_i \phi_i(x) + M\Big(\sum_{i=1}^{p} \hat{\mu}_i\Big) \max_{i \in \{1,\ldots,p\}} \phi_i(x).$$

Hence, there exist $\lambda_j \geqq 0$, $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$\begin{aligned} 0 \in &\partial \hat{f}(x^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*) \\ =&\partial\Big(\sum_{i=1}^{p} \hat{\mu}_i \phi_i(x) + M\Big(\sum_{i=1}^{p} \hat{\mu}_i\Big) \max_{i \in \{1,\ldots,p\}} \phi_i(x)\Big) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*) \\ \subset& \sum_{i=1}^{p} \hat{\mu}_i \partial \phi_i(x) + M\Big(\sum_{i=1}^{p} \hat{\mu}_i\Big) \mathrm{co}\{\partial \phi_i(x^*) \mid i = 1, \ldots, p\} + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*). \end{aligned}$$

Thus, there exist $\beta_i \geqq 0$, $i = 1, \ldots, p$, $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i=1}^{p} \hat{\mu}_i \partial \phi_i(x^*) + M\Big(\sum_{i=1}^{p} \mu_i\Big) \sum_{i=1}^{p} \beta_i \partial \phi_i(x^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

Since $\partial \phi_i(x^*)$, $i = 1, \ldots, p$ are convex, there exist $\mu_i > 0$, $i = 1, \ldots, p$, and $\lambda_j \geqq 0$, $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial \phi_i(x^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

By Lemma 2.5 (iii), there exist $\mu_i > 0$, $u_i^* \in \mathcal{U}_i(x^*)$, $i = 1, \ldots, p$, $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

Thus there exist $\mu_i > 0$, $u_i^* \in \mathcal{U}_i$ $i = 1, \ldots, p$, $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j$, $j = 1, \ldots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*)$$

and $\lambda_j g_j(x^*, v_j^*) = 0$, $j = 1, \ldots, m$. So the theorem holds. □

From the proof of Theorem 3.6, we can obtain the following theorem for a convex multiobjective optimization problem:

Theorem 3.7. *Let $X = \mathbb{R}^n$. Assume that $f_i(\cdot,\cdot)$, $i = 1,\dots,p$ and $g_j(\cdot,\cdot)$, $i = 1,\dots,m$ are continuous, and $f_i(\cdot,u_i)$, $i = 1,\dots,p$ and $g_j(\cdot,v_j)$, $i = 1,\dots,m$ are convex on X. Suppose that for each $x \in X$, $f_i(x,\cdot)$, $i = 1,\dots,p$ are concave on $\mathcal{U}_i$, $i = 1,\dots,p$ and $g_j(x,\cdot)$ are concave on $\mathcal{V}_j$, $j = 1,\dots,m$. Let $x^* \in \mathcal{C}$ be a properly robust efficient solution of* (UMP). *If the robust Slater condition holds, that is, there exists $\hat{x} \in X$ such that $g_j(\hat{x},v_j) < 0$, for any $v_j \in \mathcal{V}_j$, $j = 1,\dots,m$, then there exist $\mu_i > 0$, $i = 1,\dots,p$, not all zero, $u_i^* \in \mathcal{U}_i(x^*)$, $i = 1,\dots,p$, $\lambda_j \geqq 0$ and $v_j^* \in \mathcal{V}_j(x^*)$, $j = 1,\dots,m$ such that*

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*,u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*,v_j^*),$$
$$0 = \lambda_j g_j(x^*,v_j^*),\ j = 1,\dots,m,$$

where $\partial_x f_i(x^*,u_i^*)$ and $\partial_x g_j(x^*,v_j^*)$ are subdifferentials of the convex functions $f_i(\cdot,u_i^*)$ at x^* and $g_j(\cdot,v_j^*)$ at x^*.

Example 3.8. Let $x := (x_1,x_2) \in \mathbb{R}^2$, $(u_1,u_2) \in \mathcal{U}_1 \times \mathcal{U}_2 := [0,1] \times [0,1]$ and $v := (v_1,v_2) \in \mathcal{V} := \{v \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \leqq 1\}$. Consider the functions

$$\begin{aligned}(f_1(x_1,x_2,u_1,u_2), f_2(x_1,x_2,u_1,u_2)) &:= (u_1x_1, u_2x_2),\\ g(x_1,x_2,v_1,v_2) &:= x_1v_1 + x_2v_2 - 1.\end{aligned}$$

Let

$$\mathcal{C} := \{(x_1,x_2) \in \mathbb{R}^2 \mid \psi(x_1,x_2) \le 0\},$$

where

$$\psi(x) := \max_{(v_1,v_2)\in\mathcal{V}} g(x_1,x_2,v_1,v_2) = \max_{(v_1,v_2)\in\mathcal{V}} \langle (x_1,x_2),(v_1,v_2)\rangle - 1 = \sqrt{x_1^2 + x_2^2} - 1.$$

Now, we consider the following multiobjective problem with uncertainty data:

$$\begin{aligned}\text{(UMP)}\quad &\min\ (f_1(x_1,x_2,u_1,u_2), f_2(x_1,x_2,u_1,u_2))\\ &\text{s.t.}\ \ g(x_1,x_2,v_1,v_2) \le 0,\ \forall (v_1,v_2) \in \mathcal{V},\end{aligned}$$

its robust counterpart:

$$\begin{aligned}\text{(RMP)}\quad &\min\ \Big(\max_{(u_1,u_2)\in\mathcal{U}_1\times\mathcal{U}_2} f_1(x_1,x_2,u_1,u_2), \max_{(u_1,u_2)\in\mathcal{U}_1\times\mathcal{U}_2} f_2(x_1,x_2,u_1,u_2)\Big)\\ &\text{s.t.}\ \ g(x_1,x_2,v_1,v_2) \le 0,\ \forall (v_1,v_2) \in \mathcal{V}.\end{aligned}$$

Then the set of properly robust efficient solutions of (UMP) is $\{(x_1,x_2) \mid x_1^2 + x_2^2 \leqq 1,\ x_1 \leqq 0,\ x_2 \leqq 0\}$. Moreover, the set of weakly robust efficient solutions of the problem is $\{(x_1,x_2) \mid x_1^2 + x_2^2 \leqq 1,\ x_1 \leqq 0 \text{ or } x_2 \leqq 0\}$. So, it is clear that $(x_1^*,x_2^*) = (1,0) \in \mathcal{C}$ is a weakly robust efficient solution of (UMP). We have $\mathcal{U}_1(x_1^*,x_2^*) \times \mathcal{U}_2(x_1^*,x_2^*) = \{(1,u_2^*) \mid u_2^* \in \mathcal{U}_2\}$ and $\mathcal{V}(x_1^*,x_2^*) = \{(1,0)\}$. Moreover, it is easy to check that (ENMFCQ) holds at (x_1^*,x_2^*) and if we let $\mu_1 = 0$, $\mu_2 = 1$, $\lambda_1 = 0$, $(u_1^*,u_2^*) = (1,0)$ and $(v_1^*,v_2^*) = (1,0)$, then we have

$$\mu_1\nabla_x f_1(x_1^*,x_2^*,u_1^*,u_2^*) + \mu_2\nabla_x f_2(x_1^*,x_2^*,u_1^*,u_2^*) + \lambda_1\nabla_x g(x_1^*,x_2^*,v_1^*,v_2^*) = 0,$$
$$\lambda_1 g(x_1^*,x_2^*,v_1^*,v_2^*) = 0.$$

Moreover, $(\bar{x}_1,\bar{x}_2) = (-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}) \in \mathcal{C}$ is a properly robust efficient solution of (UMP). We have $\mathcal{U}_1(\bar{x}_1,\bar{x}_2) \times \mathcal{U}_2(\bar{x}_1,\bar{x}_2) = \{(0,0)\}$ and $\mathcal{V}(\bar{x}_1,\bar{x}_2) = \{(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})\}$.

Moreover, it is easy to check that (ENMFCQ) holds at $(\bar{x}_1, \bar{x}_2)$ and if we let $\mu_1 = \mu_2 = 1$, $\lambda_1 = 0$, $(\bar{u}_1, \bar{u}_2) = (0, 0)$ and $(\bar{v}_1, \bar{v}_2) = \big(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\big)$, then we have

$$\mu_1 \nabla_x f_1(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) + \mu_2 \nabla_x f_2(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)) + \lambda_1 \nabla_x g(\bar{x}_1, \bar{x}_2, \bar{v}_1, \bar{v}_2) = 0,$$
$$\lambda_1 g(\bar{x}_1, \bar{x}_2, \bar{v}_1, \bar{v}_2) = 0.$$

Thus, Theorem 3.3 and 3.6 hold.

Moreover, $(\hat{x}_1, \hat{x}_2) = (0, 0) \in \mathcal{C}$ is a properly robust efficient solution of the problem (3.1). We have $\mathcal{U}_1(\hat{x}_1, \hat{x}_2) \times \mathcal{U}_2(\hat{x}_1, \hat{x}_2) = \mathcal{U}_1 \times \mathcal{U}_2$ and $\mathcal{V}(\hat{x}_1, \hat{x}_2) = \mathcal{V}$. But (ENMFCQ) does not holds at $(\hat{x}_1, \hat{x}_2)$. On the other hand, the Slater's condition holds at $(\hat{x}_1, \hat{x}_2)$, that is, $g(\hat{x}_1, \hat{x}_2, v_1, v_2) < 0$, for all $(v_1, v_2) \in \mathcal{V}$. Moreover, $f_1(\cdot, u_1, u_2)$, $f_2(\cdot, u_1, u_2)$ and $g(\cdot, v_1, v_2)$ are convex. If we let $\mu_1 = \mu_2 = 1$, $\lambda_1 = 0$, $(\hat{u}_1, \hat{u}_2) = (0, 0)$ and $(\hat{v}_1, \hat{v}_2) = (1, 0)$, then we have

$$\mu_1 \nabla_x f_1(\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2) + \mu_2 \nabla_x f_2(\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2) + \lambda_1 \nabla_x g(\hat{x}_1, \hat{x}_2, \hat{v}_1, \hat{v}_2) = 0,$$
$$\lambda_1 g(\hat{x}_1, \hat{x}_2, \hat{v}_1, \hat{v}_2) = 0.$$

Thus, Theorem 3.7 holds.

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