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MULTIPLIERS OF BANACH-VALUED FUNCTION SPACES ON LCA GROUP

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Dedicated to the 70th birthday of Professor W. Takahashi

Abstract. Let G be a locally compact abelian (LCA) group with Haar measure dt and dual group $\widehat{G}$. Let A be a commutative Banach algebra, X and Y Banach spaces with A-module. Denote by $L^1(G,A)$ the space of all Bochner integrable A-valued functions defined on G. It is a commutative Banach algebra under convolution . $L^p(G,X)$ denotes the space of all X-valued measurable functions defined on G whose X-norms are in usual L^p space, it is a Banach space for each p , $1 \le p < \infty$. In this paper, we characterize the multiplier operators of various Banach-valued functions defined on G to be function spaces. The homomorphism A-module multipliers of X into Y is established in the forms of A replaced by $L^1(G,A)$; X by $L^1(G,X)$; Y by $L^p(G,Y)$ and Y^* by $L^q(G,Y^*)$, $1/p + 1/q$ and $1 < p,\ q < \infty$ where the applications of the Radon-Nikodym property (RNP) in wide sense are concerned.

1. Introduction and preliminaries

At first, we give an overview on multipliers. The concept of multipliers come from the theory of Fourier series. As one describes a Fourier series of an integrable periodic function $f \in L^1[0,2\pi]$ by

$$f \sim \sum_{n\in Z} c_n e^{int} \text{ with Fourier Coefficient } : \ c_n = \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt,$$

he attempts to describe a bounded sequence $\{\phi_n\}$, namely multiplier function of $L^1[0,2\pi]$, such that $\phi_n c_n$ still be a Fourier coefficient of some $g \in L^1[0,2\pi]$ such that

$$g \sim \sum_{n\in Z} \phi_n c_n e^{int} \text{ with } \hat{g}(n) = \phi_n c_n = (\phi\cdot\hat{f})(n) = \phi(n)\cdot\hat{f}(n),\ n \in \mathbf{Z}.$$

This sequence $\phi = \{\phi_n\}$ deduces to a continuous linear operator T_ϕ mapping $f \in L^1[0,2\pi] = L^1(\mathbb{T})$ to $g \in L^1(\mathbb{T})$ with the property $\widehat{T_\phi f} = \phi\cdot\hat{f} = \hat{g}$, defined on the discrete additive group Z, the Fourier transform defined on the dual group $\widehat{\mathbb{T}} = Z$, an integer (discrete) group . The interval $[0,2\pi]$ presents a compact group $\mathbb{T}$ in real line $\mathbf{R}$, a locally compact Abelian group, where $\mathbb{T}$ is defined by

$$\mathbb{T} = \{e^{it} \mid 0 \le t \le 2\pi\} \simeq \{e^{i2\pi t} \mid 0 \le t \le 1\}.$$

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These idea turns to general locally compact abelian (LCA) group G with dual group $\widehat{G}$, that is the space of all functions defined on G into the unit circle of the complex space $\mathbb{C}$. That is,

$$\widehat{G} = \{\gamma : G \to \text{ unit circle of the complex field } \mathbb{C}\}$$

such that

$$\begin{aligned} &|\langle x, \gamma\rangle| = 1, \quad \text{for all } \gamma \in \widehat{G},\ x \in G, \text{ and} \\ &\langle x_1 + x_2, \gamma\rangle = \langle x_1, \gamma\rangle \cdot \langle x_2, \gamma\rangle,\ \langle x, \gamma_1 + \gamma_2\rangle = \langle x, \gamma_1\rangle \cdot \langle x, \gamma_2\rangle, \\ &\langle 0, \gamma\rangle = 1 = \langle x, 0\rangle, \text{ for any } x_1, x_2, x \in G, \text{ and } \gamma_1,\ \gamma_2,\ \gamma \in \widehat{G}. \end{aligned}$$

There is a well known fundamental theorem for studying on multipliers in harmonic analysis. Multipliers have also appeared in studying operator theory, stochastic processes, PDE, optimization theory and the mathematical economics, ... etc .

Now we state firstly to describe a well known fundamental theorem as following (cf. Theorem 0.1.1 in Larsen [15], and Wendel [19] for application of multipliers).

Theorem 1.1. *Let G be a LCA group with dual group $\widehat{G}$. Suppose that $T \in \mathcal{L}(L^1(G), L^1(G)) = \mathcal{L}(L^1(G))$, the space of all continuous linear (= bounded linear) operators on $L^1(G)$. Then the following statements are equivalent.*

(i) $T\tau_a = \tau_a T$ *for any* $a \in G$,
where τ_a *is a translation operator* : $\tau_a f(t) = f(ta^{-1}) = f(t-a)$.
This T is called an **invariant** *operator.*

(ii) $T(f * g) = Tf * g = f * Tg$, *"$*$" denotes convolution of f, $g \in L^1(G)$.*
where $f * g(t) = \int_G f(t-s)g(s)ds = \int_G f(ts^{-1})g(s)ds$.
T commutes with the algebra operation (convolution "$*$" in $L^1(G)$),
this T is called a **multiplier** *of* $L^1(G)$.

(iii) $\exists !$ *a* $\mu \in M_b(G)$ *(bounded regular Borel measure space) such that*
$Tf = \mu * f$, *for all* $f \in L^1(G)$, *and so* $\widehat{Tf} = \hat{\mu}\hat{f}$,
where $\hat{f}(\gamma) = \int_G \overline{\langle t, \gamma\rangle} f(t)dt$, *and* $\hat{\mu}(\gamma) = \int_G \overline{\langle t, \gamma\rangle} d\mu(t)$.
$\hat{\mu}$ *is the* **Fourier Stieltjes** *transform of* $\mu \in M_b(G)$.

(iv) $\exists !$ *a bounded function* $\phi \in \widehat{M_b(G)} \equiv B(\widehat{G}) \subset C_b(\widehat{G})$ *such that* $\widehat{Tf} = \phi \cdot \hat{f}$,
for any $f \in L^1(G)$.

Any bounded linear operator $T \in \mathcal{L}(L^1(G))$ satisfies one of (i) $\sim$ (iv) is called a **multiplier** *of $L^1(G)$, and denoted by $\mathfrak{M}(L^1(G))$ the space of all multipliers of $L^1(G)$. $\mathfrak{M}(L^1(G))$ is a closed subalgebra of the operator algebra $\mathcal{L}(L^1(G))$.*

Essential Remarks

Harmonic analysis on LCA group G with dual group $\widehat{G}$ has the following useful concepts. (cf. the book: *Fourier Analysis on Groups* by W. Rudin.)

(1) The **Fourier transform** of $f \in L^1(G)$ is given by

$$\hat{f}(\gamma) = \int_G \overline{\langle t, \gamma\rangle} f(t)dt = \int_G \langle -t, \gamma\rangle f(t)dt.$$

and $\hat{f} \in \widehat{L^1(G)} = A(\widehat{G}) \subsetneqq C_0(\widehat{G})$, where $C_0(\widehat{G})$ is the space of all continuous functions vanishing at infinity of $\widehat{G}$. $A(\widehat{G})$ is a Banach algebra under pointwise product with norm $\| \hat{f} \|_\infty \leq \| f \|_1$. $A(\widehat{G})$ is dense in $C_0(\widehat{G})$ of first Category. The algebra $A(\widehat{G})$ is a Fourier algebra in pointwise product and shows that $A(\widehat{G}) \cong L^1(G)$ provided $A(\widehat{G})$ use the norm $\| \hat{f} \|_A = \| f \|_1$.

This shows that the Fourier transform of $f \in L^1(G)$ is uniquely defined.

$C_0(G)$ is a Banach algebra under pointwise product in the uniform norm $\| f \|_\infty = \sup_{t\in G} |f(t)|$ for $f \in C_0(G)$.

(2) Let $M_b(G)$ be the space of all bounded regular Borel measures on G.

Then for $\mu \in M_b(G)$, the **Fourier-Stieljes transforms** of μ is given by

$$\hat{\mu}(\gamma) = \int_G \overline{\langle \gamma, t\rangle} d\mu(t), \text{ and } \hat{\mu} \in \widehat{M_b}(\widehat{G}) \equiv B(\widehat{G}) \subsetneqq C_b(\widehat{G}),$$

where $C_b(\widehat{G})$ is the space of all bounded continuous functions on $\widehat{G}$, and $B(\widehat{G})$ is the space of all Fourier - Stieljes transform for $M_b(G)$. If we denote the space of all **invariant operators** by $(L^1(G), L^1(G))$, and the space of all multiplier operators of $L^1(G))$ by $\mathfrak{M}(L^1(G))$. Then by (i) (ii) and (iii) of Theorem 1.1, we get the following isometrically isomorphic relations.

$$\mathfrak{M}(L^1(G)) \cong (L^1(G), L^1(G)) \cong Hom_{L^1(G)}(L^1(G), L^1(G)) \cong M_b(G). \tag{1.1}$$

The last "$\cong$" follows from (iii) of Theorem 1.1 ,it shows that $\exists !\ \mu \in M_b(G)$ such that

$$Tf = \mu * f,\ for\ all\ f \in L^1(G) \text{ and } \| T \| = \| \mu \|,$$

where $\| \mu \|$ is the norm of the bounded regular Borel measure algebra $M_b(G)$.

2. Multipliers on a commutative Banach algebra

From Theorem 1.1, one sees that the multiplies of $L^1(G)$ can be regarded as various types in representations. It can also be extended to multiplies of $L^p(G)$ for each p, $1 < p < \infty$, by using (i) only, or to multiplies of general Banach algebra by using (ii) only. In the study of multiplier theory, it is essentially to identify (or characterize) the multiplier space which is a subalgebra of bounded linear operators as a function space.

In order to develop the multipliers in the expression (1.1) to vector-valued function spaces on G, we process to this section. Throughout we let A be a semi-simple commutative Banach algebra, a Banach space X is said to be **A-module** if $AX \subseteq X$ and $\| ax \|_X \leq \| a \|_A \| x \|_X$. An A-module Banach space X is said to be **essential A-module** if $AX = X$.

Theorem 2.1. *If A has a bounded approximate identity $\{e_\alpha\}$, that is, there exists a positive number K such that $\| e_\alpha \|_A \leq K$ for all α, then any A-module Banach space is essential A-module.*

Proof. If a Banach algebra A has a bounded approximate identity $\{e_\alpha\} \subseteq A$ and X is an A-module without order since A is semi-simple. Thus

$$e_\alpha x \in X \text{ and } \| e_\alpha x \|_X \leq K \| x \|_X \text{ for all } x \in X.$$

It follows that for any $x \in X$, we can show that

$$\| e_\alpha x - x \|_X \to 0 \text{ (and } \| x e_\alpha - x \|_X \to 0)$$

as the limit is taken over all α. This shows that AX is dense in X and so $AX = X$. The proof is completed. □

A continuous linear operator $T \in \mathcal{L}(A)$ is said to be a **multiplier** of A if

$$T(a \cdot b) = Ta \cdot b = a \cdot Tb \textit{ for all } a, b \in A.$$

Example 1. Let G be a LCA group. Then

(1) $L^1(G)$ is a commutative Banach algebra under convolution product, and $L^1(G)$ has a bounded approximate identify of norm 1.

(2) The Banach spaces $L^p(G)$, $1 \leq p < \infty$, and $C_0(G)$ are essential $L^1(G)$-module Banach spaces.

Proof. (1) The proof of commutative Banach algebra is easy. While $L^1(G)$ has a bounded approximate identity of norm 1, we can take a neighbourhood system $\{V_\alpha\}$ of open subsets at the origin $\theta \in G$ and define a system $\{e_\alpha\}$ of functions e_α by

$$e_\alpha = \frac{\chi_{V_\alpha}}{|V_\alpha|} \text{ (any open set } V \text{ in } G \text{ has Haar measure } |V| > 0)$$

with Haar measure $|V_\alpha|$ and the characteristic function χ_{V_α} of V_α.

$\{V_\alpha\}$ is a neighborhood system of $\theta \in G$ and ordered by $\beta \prec \alpha$ if $V_\alpha \subset V_\beta$. Then

$$\| e_\alpha \|_1 = \int_G \frac{\chi_{V_\alpha}}{|V_\alpha|} dt = \int_{V_\alpha} \frac{1}{|V_\alpha|} dt = 1 \text{ for all } \alpha.$$

(2) Since $L^1(G)$ has bounded approximately identity as shown in (1), for each p, $1 < p < \infty$, the spaces

$$L^p(G) = \left\{ f : G \to \mathbb{C} \;\Big|\; \left(\int_G (|f(t)|)^p dt \right)^{\frac{1}{p}} < \infty \right\},$$

and $C_0(G)$ = the space of all continuous functions vanishing at infinity on G, are essential $L^1(G)$-modules. That is

$$L^1(G) * L^p(G) = L^p(G), \textit{ for each } p, \; 1 < p < \infty \tag{2.1}$$

and

$$L^1(G) * C_0(G) = C_0(G). \tag{2.2}$$

□

In general if A has a bounded approximate identity, then B is essential A-module. We can construct a family $\{A \cdot B\}$ as a subspace of B by

$$S = \left\{ u = \sum_{i=1}^n a_i b_i \mid a_i \in A, \; b_i \in B, \; \sum_{i=1}^n \| a_i \|_A \| b_i \|_B < \infty, \forall n \in \mathbb{N} \right\}$$

where $\mathbb{N}$ *is the integer space.*

Define $|||u||| = \inf_{u\in S}\{\sum_i \| a_i \|_A\| b_i \|_B \mid u = \sum_i a_i b_i \in S\}$. It is easy to show that $||| \cdot |||$ is a norm, and $(S, ||| \cdot |||)$ is a closed subspace of $(B, \| \|_B)$. Since $\| u \|_B = \| a_i b_i \|_B \leq \| a_i \|_A \cdot \| b_i \|_B$ for all $a_i \in A,\ b_i \in B$, and for any finite sum $u = \sum_i^n a_i b_i,\ a_i \in A, b_i \in B$, we have

$$\| u \|_B \leq |||u|||_S = \inf\Big\{\sum_{i=1}^{n} \| a_i \|_A\| b_i \|_B \ \Big|\ u = \sum a_i b_i \in S\Big\}.$$

This shows that $S \subset B$, and the dual spaces of B and S turn to be $S^* \supset B^*$. Hence for $A \subset \mathfrak{M}(A)$, and $B^* \subset S^*$, one may check easily that

$$\mathfrak{M}(A) \cong S^* \text{ provided } A = B^*.$$

One can establish the following plausible theorem.

Theorem 2.2. *Let A be a commutative Banach algebra having a bounded approximate identity, and B an essential A-module with property $A = B^*$. Then the multiplier space $\mathfrak{M}(A)$ of A is isometrically isomorphic to S^*. On the other word, we can describe this theorem as the following interesting diagram to prove $\mathfrak{M}(A) \cong S^*$.*

$$\begin{array}{ccc} A & \subset & \mathfrak{M}(A) \\ \| \wr & \overset{?}{\Longrightarrow} & \| \wr \\ B^* & \subset & S^* \end{array}$$

Proof. Let $T \in \mathfrak{M}(A)$ correspond to a $\mu \in S^*$ by the following expression :
for any $u = \sum_i a_i b_i \in S \subset B$, the mapping : $T \to \mu$ defined by
(1) $\quad \langle u, \mu\rangle = \sum_i \langle b_i, Ta_i\rangle$ for $\sum_i a_i b_i = u \in S$.

We want to show that $T \to \mu$ defined by the above identity (1) is well defined. That is to show :

$$u = \sum_i a_i b_i = 0 \ \Rightarrow \langle u, \mu\rangle = \sum_i \langle b_i, Ta_i\rangle = 0.$$

Indeed for each α, let $Te_\alpha = h_\alpha$. We obtain

$$\begin{aligned} |\langle b_i, h_\alpha a_i\rangle - \langle b_i, Ta_i\rangle| &\leq \| h_\alpha a_i - Ta_i \|_A\| b_i \|_B \\ &\leq \| T \|\| e_\alpha a_i - a_i \|_A\| b_i \|_B \\ &\to 0, \ \text{ as the limit is taken over } \alpha. \end{aligned}$$

It yields that for $u = \sum_i a_i b_i \in S$, and $\mu \in S^*$, we have

$$\begin{aligned} \langle u, \mu\rangle = \Big|\sum_i \langle b_i, Ta_i\rangle\Big| &= \Big|\sum_i \langle b_i, Ta_i\rangle - \sum_i \langle b_i, h_\alpha a_i\rangle\Big| \\ &\leq \sum_i |\langle b_i, Ta_i - Te_\alpha a_i\rangle| \\ &\leq \sum_i \| b_i \|\| a_i - e_\alpha a_i \|\| T \| \\ &\to 0, \text{ as the limit is taken over all } \alpha. \end{aligned}$$

This shows that

$$u = \sum_i a_i b_i = 0 \Rightarrow \langle u, \mu \rangle = \sum_i \langle b_i, Ta_i \rangle = 0.$$

Next we would prove that the mapping $T \to \mu$ is an isometrically isomorphism map $\mathfrak{M}(A)$ onto S^*. Actually, for any $a \in A$, $\mu \in S^*$, define

$$t_a : \mu \in S^* \to B^*$$

$$by\ b \to \langle b, t_a\mu \rangle = \langle ab, \mu \rangle \ for\ all\ b\ \in\ B.$$

Evaluate $|\langle ab, \mu\rangle| = |\langle b, t_a\mu\rangle| \leq \| \mu \| \| a \|_A \| b \|_B$ yields $\| t_a\mu \| \leq \| \mu \| \| a \|_A$.
For $T \in \mathfrak{M}(A)$ is a continuous linear operator on A, if $A \cong B^*$, $a \in A$ corresponds a $t_a \in B^*$, says $t_a = Ta \in A$, such that for $b \in B$,

$$\langle b, Ta \rangle = \langle ab, \mu \rangle = \langle b, t_a\mu \rangle,$$

we obtain

(a) $$\| Ta \|_A \leq \| t_a\mu \| \leq \| \mu \| \| a \|_A \Longrightarrow \| T \| \leq \| \mu \| .$$

Since $T \in \mathfrak{M}(A)$, from $u = \sum a_i b_i \in S$, $\mu \in S^*$, we have

$$|\langle u, \mu \rangle| \ = \ \Big| \sum_i \langle b_i, Ta_i \rangle \Big| \ \leq \ \sum_i \| Ta_i \|_A \| b_i \|_B$$

$$\Longrightarrow |\langle u, \mu \rangle| \ \leq \ \inf_{u \in S} \| T \| \sum_i \| a_i \|_A \| b_i \|_B \leq \| T \| \| u \|_S .$$

Thus,

(b) $$\| \mu \| \leq \| T \|, \text{ for all } u = \sum_i a_i b_i \in S.$$

By (a) and (b) we obtain that $\| \mu \| = \| T \|$. □

Remark. In Theorem 2.1 , we assume that the Banach algebra A has a bounded approximate identity . But we notice that not every Banach algebra has a bounded approximate identity . For example, It is well known that the space

$$A^p(G) = \{ f \in L^1(G) \mid \text{the Fourier transform } \hat{f} \in L^p(\hat{G}), 1 \leq p < \infty \}$$

with norm $\| f \|_{A^p} = \| f \|_1 + \| \hat{f} \|_p$, is a commutative Banach algebra under convolution product. This Banach algebra $A^p(G)$ has no uniform ($\| \cdot \|_{A^p}$) bounded approximate identity in $A^p(G)$. We state this algebra as the following example.

Example 2 (cf Lai [6] , p.574, and Larsen [15])). Prove that $A^p(G)$, $1 \leq p < \infty$ is a commutative Banach algebra, and that $A^p(G)$ has no bounded approximate identity.

Proof. Let $f, g \in A^p(G)$. Then $\hat{f}, \hat{g} \in C_0(\hat{G}) \cap L^p(\hat{G})$, for $1 \leq p < \infty$.
It yields

$$\begin{aligned} \| f * g \|_{A^p} &= \| f * g \|_1 + \| \hat{f} \cdot \hat{g} \|_p \\ &= \| g * f \|_{A^p} \\ &\leq \| f \|_1 \| g \|_1 + \| \hat{f} \|_\infty \| \hat{g} \|_p \\ &\leq \| f \|_1 (\| g \|_1 + \| \hat{g} \|_p) \ (since\ \ \| \hat{f} \|_\infty \leq \| f \|_1) \end{aligned}$$

$$\begin{aligned} &= (\| f \|_1 + \| \hat{f} \|_p) \| g \|_{A^p} \\ &= \| f \|_{A^p} \| g \|_{A^p} \end{aligned}$$

Hence $A^p(G)$ is a commutative Banach algebra.
Let $\{e_\alpha\}$ be a bounded approximate identity in $L^1(G)$ with Fourier transform $\hat{e}_\alpha$ of e_α having compact support in $\hat{G}$ for each α, then $\hat{e}_\alpha \in L^p(\hat{G})$ deduces that the approximate identity $\{e_\alpha\} \subset A^p(G) \subset L^1(G)$ can not uniform bounded in $A^p(G)$.

Indeed, for any $f \in A^P(G)$,

$$\begin{aligned} \| e_\alpha * f - f \|_{A^p} &= \| e_\alpha * f - f \|_1 + \| \hat{e}_\alpha \hat{f} - \hat{f} \|_p \\ &\leq \| e_\alpha * f - f \|_1 + \| \hat{f} \|_\infty \| \hat{e}_\alpha - 1 \|_p \\ &\longrightarrow 0, \text{ as limit taken over all } \alpha. \end{aligned}$$

This implies that $\hat{e}_\alpha \to 1$ uniformly on a compact set $K \subset \hat{G}$. It follows that for an $\epsilon > 0$,

$$\int_K (|\hat{e}_\alpha(\gamma)| - 1)^p d\gamma > -\epsilon \implies \int_K |\hat{e}_\alpha(\gamma)|^p d\gamma > \mathfrak{m}(K) - \epsilon$$

where $\mathfrak{m}(K)$ is the measure of K and ϵ is small, it yields

$$\| \hat{e}_\alpha \|_p > \Big(\frac{\mathfrak{m}(K)}{2}\Big)^{\frac{1}{p}},$$

but $\mathfrak{m}(K)$ may be large enough. Hence $\| \hat{e}_\alpha \|_p$ is not uniform bounded on α, and so $\| e_\alpha \|_{A^p}$ is not uniformly bounded in $A^p(G)$. □

3. Multipliers of Banach module homomorphism

Recall that A is a commutative Banach algebra and that X and Y are A-module Banach spaces. A bounded linear operator $T \in \mathcal{L}(X, Y)$ satisfying

$$T(ax) = a(Tx) \text{ for all } a \in A,\ x \in X, \tag{3.1}$$

is called a **multiplier** of X to Y under A-module. The space of such multipliers is A-module homomorphisms from X to Y and is denoted by

$$\mathfrak{M}_A(X, Y) = Hom_A(X, Y) = \{T \in \mathcal{L}(X, Y) |\ T(ax) = a(Tx), a \in A, x \in X\}. \tag{3.2}$$

It is a closed subalgebra of $\mathcal{L}(X, Y)$, the space of all bounded linear mappings of X into Y. In particular, if $A = X = Y = L^1(G)$, then the multiplier space $\mathfrak{M}(L^1(G))$ coincides with the expression of isometrically isomorphic relations "$\cong$" as in (1.1).

$$\mathfrak{M}(L^1(G)) = Hom_{L^1(G)}(L^1(G), L^1(G)) \cong (L^1(G), L^1(G)) \cong M_b(G). \tag{3.3}$$

where $(E(G), F(G))$ stands for the space of all invariant operators commute with translation operator τ_a on the function spaces of $E(G)$ to $F(G)$. The isometrically isomorphism " $\cong$ " in (3.3) is the same as (1.1) which is followed from (i) (ii) (iii) of Theorem 1.1.

In general, the multiplier space $Hom_A(X, Y^*)$ was characterized by Rieffel [16] as the following dual space of the module tensor product $X \otimes_A Y$:

$$Hom_A(X, Y^*) \cong (X \otimes_A Y)^*, \tag{3.4}$$

where $\otimes_A$ denotes the A-module tensor product defined by

$$X \otimes_A Y = X \hat{\otimes}_\gamma Y / K.$$

Here K is the closed linear subspace of the complete projective tensor product space $X\hat{\otimes}_\gamma Y$ generating by elements:

$$ax \otimes y - x \otimes ay, \text{ for } a \in A, x \in X, y \in Y$$

and $\hat{\otimes}_\gamma$ is the completion of the algebra tensor $X \otimes Y$ under the largest reasonable cross norm γ. Here

$$X \otimes Y = \Big\{ u = \sum_i x_i \otimes y_i \;\Big|\; \sum_i \| x_i \|_X \| y_i \|_Y < \infty \Big\}$$

with norm

$$\gamma(u) \equiv |||u||| = \inf_u \sum_i \| x_i \otimes y_i \| = \inf_u \sum_i \| x_i \|_x \| y_i \|_y,$$

$\inf_u$ means that the infimum is taken by all representations of $u = \sum_i x_i \otimes y_i$ in $X \otimes Y$.

The reasonable crossnorm means that

$$u \in X \otimes Y,\ u = x \otimes y \text{ implies } \| u \| = \| x \otimes y \| = \| x \|_X \| y \|_Y;$$
$$\text{and } u = \sum_i x_i \otimes y_i,\ \| u \| = inf \sum_i \| x_i \|_X \| y_i \|_Y\ .$$

Note that a bounded linear operator $T \in Hom_A(X, Y^*)$ in (3.4) corresponding a continuous linear functional ψ on $X \otimes_A Y$ is given by

$$(Tx)(y) = \psi(x \otimes y) \text{ for all } x \in X, y \in Y.$$

Here $Hom_A(X, Y^*) = \mathcal{M}_A(X, Y^*)$ is the space of all A-module homomorphisms from X to Y^*, the topological dual of Y, that is , each $T \in Hom_A(X, Y^*)$satisfies

$$T(ax) = a(Tx) \text{ for all } a \in A,\ x \in X,\ Tx \in Y^*.$$

where T is a bounded linear operator from X to Y^*; $X \otimes_A Y$ denotes the A-module tensor product space of X and Y.

There are some known results in scalar-valued function space of $L^1(G)$-module by convolution. We state three typical $L^1(G)$-module multiplier problems as follows.

(i) $Hom_G(L^1(G), L^1(G)) \cong M_b(G)$, (by Theorem 1.1, (iii) $\Longleftrightarrow$ (ii)) where $Hom_G = Hom_{L^1(G)}$, and $M_b(G)$ is the space of all bounded regular Borel measures on G.

(ii) $Hom_G(L^1(G), L^p(G)) \cong (L^1(G) \otimes_G L^q(G))^* = (L^q(G))^* = L^p(G)$, for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ where $\otimes_G = \otimes_{L^1(G)}$.

(iii) $Hom_G(L^P(G), L^P(G)) \cong (L^p(G) \otimes_G L^q(G))^* \cong S_p(G)^*$, where $S_p(G)$ is a Banach algebra generated by

$$\Big\{ u = \sum_i^\infty f_i g_i : f_i \in L^p(G), g_i \in L^q(G), \sum_i^\infty \| f_i \|_p \| g_i \|_q < \infty \Big\}$$

under pointwise product and the norm is defined by (cf. Theorem 2.2 or Larsen [15])

$$|||u||| = \inf \Big\{ \sum_i^\infty \| f_i \|_p \| g_i \|_q\,;\ u = \sum_i^\infty f_i \cdot g_i \in S_p(G) \Big\}.$$

4. Some multiplier spaces of $L^1(G, A)$-module spaces

Assume that A is a commutative semisimple Banach algebra with a bounded approximate identity of norm 1. The space $L^1(G, A)$ is the set of all A-valued (Bochner) integrable functions defined on G. It is a Banach algebra under convolution product.

It is known that if two Banach algebras A and B have bounded approximate identities $\{a_\alpha\}$ and $\{b_\beta\}$, respectively, then $A\widehat{\otimes}_\gamma B$ has a bounded approximate identity $\{a_\alpha \otimes b_\beta\}_{(\alpha,\beta)}$ where $\widehat{\otimes}_\gamma$ is the completion of usual tensor product of Banach algebras with respect to the projective tensor norm γ.

It is easy to show that $L^1(G, A) = L^1(G)\,\widehat{\otimes}_\gamma\, A$. Thus if A has a bounded approximate identity, then so does $L^1(G, A)$. Hence $L^1(G, X)$ is an essential $L^1(G, A)$-module. Moreover, $L^p(G, X)$ for each p, $1 < p < \infty$ and $C_0(G, X)$ are also essential $L^1(G, A)$-module Banach spaces.

We would characterize the multipliers of Banach space / algebra-valued function spaces defined on locally compact Abelian group G, which is closely related in scalar-valued function on G. Mainly, there are three statements for a bounded linear operator $T \in \mathcal{L}(L^1(G, A))$ as in Theorem 1.1 (i), (ii) and (iii) by changing $L^1(G)$ to $L^1(G, A)$ as the following (i), (ii) and (iii):

(i) $T\tau_a = \tau_a T$ for any $a \in G$, T is an invariant operator.
That is, $T \in (L^1(G, A), L^1(G, A))$.
(ii) $T(f * g) = Tf * g = f * Tg$, T is a multiplier of $L^1(G, A)$, $f, g \in L^1(G, A)$.
(iii) $\exists !\ \mu \in M_b(G, A)$ such that $Tf = \mu * f$ for all $f \in L^1(G, A)$.

But in Banach vector A-valued space, not any bounded linear invariant operator is always a multiplier (cf. Tewari, et al. [18]). They showed as Theorem 4.1.

Theorem 4.1. *If A has $\dim A > 1$, there is a bounded linear invariant operator $T \in (L^1(G, A), L^1(G, A))$ which is not a multiplier of $L^1(G, A)$. That is,*

$$T \notin Hom_A(L^1(G, A), L^1(G, A)).$$

This theorem disprove Akinyele's results in [1] about the equivalent of multiplier and invariant operator on $L^1(G, A)$. Actually, $(i) \Rightarrow (ii)$ is false. The other implications: $(ii) \Rightarrow (i)$, $(ii) \Leftrightarrow (iii)$ and $(iii) \Rightarrow (i)$ are true.

In [18] Tewari, et al. established that

$$Hom_A(L^1(G, A), L^1(G, A)) \cong M(G, A) \text{ (that is, (ii) } \Leftrightarrow \text{ (iii))}$$

provided A has identity of norm 1.
This result is extended by Lai [10, Theorem 9] which we state as Theorem 4.2.

Theorem 4.2. *Let A be a commutative Banach algebra with identity of norm 1, and X an A-module Banach space. Then the following two statements are equivalent.*

(a) $T \in Hom_{L^1(G,A)}(L^1(G, A), L^1(G, X))$.
(b) $\exists !$ *X-valued vector measure $\mu \in M_b(G, X)$ such that*

$$Tf = f * \mu \textit{ for all } f \in L^1(G, A).$$

Moreover, by Theorem 4.2, we have the next relation:

$$(4.1) \quad (L^1(G, A), L^1(G, X)) \overset{\not\Rightarrow}{\Leftarrow} Hom_{L^1(G,A)}(L^1(G, A), L^1(G, X)) \cong M_b(G, X).$$

If $A = \mathbb{C}$ in (4.1), we deduce that

$$Hom_G(L^1(G), L^1(G, X)) \cong M_b(G, X). \text{ (where Hom}_G = \text{Hom}_{L^1(G)}) \tag{4.2}$$

If $X = A$ in Theorem 4.2, we then deduce

$$\mathcal{M}(L^1(G, A)) = Hom_{L^1(G,A)}(L^1(G, A), L^1(G, A)) \cong M_b(G, A). \tag{4.3}$$

We treat with on a more general form and prove that (cf. Lai [[10], Theorem 9])

$$Hom_{L^1(G,A)}(L^1(G, A), L^1(G, X)) \cong M_b(G, X). \tag{4.4}$$

Furthermore, we also prove that (cf. Lai [11])

$$Hom_{L^1(G,A)}(L^1(G, A), L^p(G, X)) \cong L^p(G, X) \text{ for } 1 < p < \infty, \tag{4.5}$$

provided both X^* and X^{**} have wide Radon Nikodym Propery (R.N.P. for bravity).

Complementary Interpretation for a Banach Space X has wide R.N.P..

Let Ω be a locally compact Hausdorff topological space. Let (Ω, Γ, μ) be a measure space with a fixed positive measure μ. Let X be a normed linear space.

Definition 4.3. In a finite measure space (Ω, Γ, μ), if each μ-continuous vector measure $\psi : \Gamma \to X$ of bounded variation, there exists a Bochner integrable function $g : \Omega \to X$ such that $\psi(E) = \int_E |g(t)|_X d\mu$, $E \in \Gamma$. The Banach space X is said to have the R.N.P. with respect to the positive measure μ.

Proposition 4.4. *The dual Banach space X^* of X has R.N.P. with respect to a finite measure E if and only if X is an Asplund space. That is every continuous real value convex functions defined on an open subset of X is Fréchèt differentiable on a dense subset G_δ in its domain. (The set $A \subset X$ is a G_δ set if A= countable intersection of open sets, $A = \bigcap U_\alpha$, U_α is open.)*

Consider a function space $L^p(\Omega, X)$ of all X-valued strongly measurable function f on Ω such that $\|f\|_{Xp} = (\int_\Omega |f(t)|_X^p d\mu(t))^{1/p} < +\infty$ for $1 \leq p < \infty$.

In the case $\Omega = G$, $\mu =$ Haar measure of G. Then the dual space of $L^p(G, X)$, $1 \leq p < \infty$, and the space $L^q(G, X^*)$, $1/p + 1/q = 1$, has the following relation stated as in the following Theorem.

Theorem 4.5 (cf. Lai [11], Theorem 2)**.** *For each p, $1 < p < \infty$, the duality expression holds*

$$L^p(G, X)^* \cong L^q(G, X^*), \ \frac{1}{p} + \frac{1}{q} = 1$$

if and only if X^* has the wide R.N.P. (see the definition later).

In harmonic analysis it takes (G, Γ, μ) instead of the measure space (Ω, Γ, μ) in which G is a LCA group with Haar measure μ. Thus we denote Γ a family of compact subsets of G. Then a Banach space X is said to have a wide R.N.P. with respect to (Haar) measure μ if for any $K \in \Gamma$, $\mu(K) < \infty$ has R.N.P. with respect to μ_K defined by $\mu_K(E) = \mu(K \cap E)$ for any measurable subset $E \subset G$. So that E is partitioned to be finite sum of disjoint characteristic functions.

Our main goal is to characterize

$$L^p(G, X)^* \cong L^q(G, X^*), 1/p + 1/q = 1$$

under the process of $C_c(G,X)$ and $C_c(G,X^*)$ are dense in $L^p(G,X)$ and $L^q(G,X^*)$ respectively, and employ these simple sums of step functions to approach the strongly (Bochnar) integrable function. Therefore in the characterization always by using $C_c(G,X)$ and $C_c(G,X^*)$ function spaces to dense in $L^p(G,X)$ and $L^q(G,X^*)$, respectively, and in case use the spaces $C_c(G,X)$ embeded isometrically isomorphism in $C_c(G,X^{**})$. Thus cause us to assume X^* and X^{**} have wide R.N.P., can be assumed that X and X^* have wide R.N.P. to improve X^* and X^{**} have wide R.N.P. regarded X is embeded in X^{**}. This improvement was indebted from Prof. T. S. Quo in The National University of Singapore.

Note that we do not discuss the space $L^\infty(G,X)$ but use $C_0(G,X)$ to instead of $L^\infty(G,X)$ since $C_c(G,X)$ is not dense in $L^\infty(G,X)$.

We shall now consider the other spaces of Banach (vector)- valued functions defined on G. The spaces given as the following are essential. They are easily to see the results given by each of the explanation.

(1) $L^1(G,A)$ denotes the set of all Bochner integrable A-valued functions defined on G. It is a commutative Banach algebra under convolution, and $L^1(G,A) \cong L^1(G)\,\widehat{\otimes}_\gamma\, A$ has a bounded approximate identity provided A has.

(2) $L^p(G,X)$ is the space of all X-valued measurable functions defined on G whose X-norm are usual L^p-space ($f=g$ in L^p means $f=g$ a.e.).
It is a Banach space for each p, $1 \le p < \infty$, which is essential $L^1(G,A)$-module.

(3) $C_0(G,X)$ denotes the space of all X-valued continuous functions defined on G vanishing at infinity. It is also an essential $L^1(G,A)$-module under supremum norm over G:

$$f \in C_o(G,X),\ \parallel f \parallel_{X,\infty} = \sup_{t\in G} \parallel f(t) \parallel_X .$$

(4) $C_0(G,A)$ denotes the space of all A-valued continuous functions defined on G vanishing at infinity. It is a commutative Banach algebra under pointwise product. It is also an $L^1(G,A)$-module Banach space by convolution.

Theorem 4.6. *By $L^1(G,A)$ is a commutative Banach algebra having a bounded approximate identity, then we have following identities*

(a) $L^1(G,A) * C_0(G,X) = C_0(G,X) \cong L^1(G,A)\ \hat{\otimes}_\gamma\ C_0(G,X)$.

(b) $L^1(G,A) * C_0(G,A) = C_0(G,A) \cong L^1(G,A)\ \hat{\otimes}_\gamma\ C_0(G,A)$.

Proof. (a) We only have to show that

$$C_0(G,X) \cong L^1(G,A)\ \hat{\otimes}_\gamma\ C_0(G,X).$$

Let $f \in L^1(G,A)$ and $F \in C_0(G,X)$. Define a bilinear map

$$B : (f,F) \in L^1(G,A) \times C_0(G,X) \longrightarrow f * F \in C_0(G,X).$$

Then the bilinear map B gives rise a bounded linear map of norm $\|B\| \le 1$ defined by $\|f * F\|_{X\,\infty} = \sup_{t\in G} |f * F(t)|_X \le \|f\|_{A1}\|F\|_{X\,\infty}$ where

$$|f * F(t)|_X \le \int_G |f(ts^{-1})|_A |F(s)|_X\, ds \le \sup_{t\in G} |F(t)|_X \int_G |f(s)|_A\, ds,$$

$$\|f * F\|_{X\,\infty} = \sup_{t\in G} |f * F(t)|_X \le \|f\|_{A1} \sup_{t\in G} |F(t)|_X = \|f\|_{A1}\|F\|_{X\,\infty},$$

It then yields the projective tensor product norm

$$\gamma(u) = \inf\Big\{ \sum_i \|f_i \otimes F_i\|_{X\,\infty} = \sum_i \|f_i\|_{A1}\|F_i\|_{X\,\infty} \ : \ u = \sum_i f_i \otimes F_i, f_i \in L^1(G,A), F_i \in C_0(G,X)\Big\}.$$

γ is the largest reasonable crossnorm of the projective tensor norm. The set $\{f * F \mid f \in L^1(G,A), F \in C_0(G,X)\} \subset C_0(G,X)$, and is dense in $C_0(G,X)$ under the γ-norm in $C_0(G,X)$. Hence $C_0(G,X) = L^1(G,A)\ \hat{\otimes}_\gamma\ C_0(G,X)$, proves (a). The proof of (b) can be taken $X = A$ in the expression (a) to get $L^1(G,A) * C_0(G,A) = C_0(G,A) \cong L^1(G,A)\ \hat{\otimes}_\gamma\ C_0(G,A)$. □

There are many researchers (cf. [1, 3–5], [9–14], [16–18]) study the multiplier spaces in various function spaces of Banach space or Banach algebra-valued functions defined on G. But there is a problem that in general , not every invariant continuous linear operator is always a multiplier of the concerned object. Precisely in the Banach (vector-) valued function space, an invariant operator $T \in (L^1(G,A), L^1(G,A))$ need not be a multiplier operator in $\mathfrak{M}(L^1(G,A))$.
That is, $T \notin Hom_A(L^1(G,A), L^1(G,A))$ provided dim $A > 1$ (cf. Tewari, Dutta and Vaidya [18]). Moreover, for each p, $1 \le p < \infty$, the space

$$L^p(G,X) = \{f : G \longrightarrow X \mid |f(t)|_X^p \in L^1(G)\}$$
where $|f(t)|_X$ denotes the norm of X,

is also an essential $L^1(G,A)$-module Banach space.

If $p = \infty$, the space $L^\infty(G,X)$ is defined by the same way with norm

$$\|f\|_{X,\infty} = ess\sup_{t\in G} |f(t)|_X, \ \ for\ f \in L^\infty(G,X).$$

In this case, we consider only a closed subspace $C_0(G,X)$ in $L^\infty(G,X)$. This space $C_0(G,X)$ is the X-valued continuous functions vanishing at infinity on G, and supply the norm as

$$\|f\|_{X,\infty} = \sup_{t\in G} |f(t)|_X \ for\ f \in C_o(G,X),$$

it is a Banach space. The dual space $C_0(G,X)^*$ is indentified by $M(G,X^*)$ in usual form, provided X^* has wide R.N.P. (cf. Lai [11]), and Tewari et al. [18]).

For $1 < p < \infty$, $1/p + 1/q = 1$, the dual space of $L^p(G,X)$ is isometrically isomorphic to $L^q(G,X^*)$ if and only if X^* has the wide R.N.P. (cf. Lai [11])

The following theorem is useful representation and is easy to derive.

Theorem 4.7.

$$L^1(G,A) = L^1(G)\,\widehat{\otimes}_\gamma\,A,\ \ C_0(G,A) \cong C_0(G)\,\widehat{\otimes}_\gamma\,A,$$

$$and\ L^p(G,X) = L^p(G)\,\widehat{\otimes}_\gamma\,X\ \ for\ 1 < p < \infty.$$

5. Multipliers of $L^1(G, A)$ to $L^1(G, X)$, $L^p(G, X)$ and $C_0(G, X)$

At first, we state the following Theorem for the characterization of the invariant operators.

Theorem 5.1. *Let X and Y be Banach spaces. Then the following statements are equivalent.*

(a) $T \in (L^1(G, Y), L^1(G, X))$ *is an invariant operator.*

(b) *There exists a unique continuous linear map* $L \in \mathcal{L}(Y, M_b(G, X))$ *such that* $T(f \otimes y) = f * L_y$ *for all* $f \in L^1(G)$, $y \in Y$.

Moreover, $(L^1(G, Y), L^1(G, X)) \cong \mathcal{L}(Y, M_b(G, X))$.

Proof. (a) ⇒ (b) Let $T \in (L^1(G, Y), L^1(G, X))$ and $y \in Y$. We define

$$T_y : L^1(G) \longrightarrow L^1(G, X) \text{ by } T_y f = T(fy) \text{ for all } f \in L^1(G).$$

It is clear that T_y is translation invariant whenever T is.
So that $T_y \in (L^1(G), L^1(G, X))$. Applying Theorem 1.1. (iii), we see that T_y is a multiplier. That is,

$$T_y \in Hom_{L^1(G)}(L^1(G), L^1(G, X)),$$

and hence there is a unique $\mu_y \in M_b(G, X)$ such that

$$T_y f = f * \mu_y \text{ for all } f \in L^1(G, A), \text{ and } \|T_y\| = \|\mu_y\|.$$

Note that, $\|T_y\| \leq \|y\|_Y \|T\|$.
(b) ⇒ (a) Conversely, if $L \in \mathcal{L}(Y, M_b(G, X))$, we define a mapping

$$T^1_L : \ L^1(G) \times Y \longrightarrow L^1(G, X) \text{ by } T^1_L(f, y) = f * L(y) \text{ for all } f \in L^1(G), \ y \in Y.$$

Then T^1_L is a bilinear continuous operator, and by the universal property of tensor product, there exists a linear map T_L, from Theorem 4.7: $L^1(G) \widehat{\otimes}_\gamma Y = L^1(G, Y)$,

$$T_L : L^1(G) \widehat{\otimes}_\gamma Y = L^1(G, Y) \longrightarrow L^1(G, X)$$

such that

$$T_L(f \otimes y) = f * L(y) \text{ for all } f \in L^1(G), \ y \in Y$$

and satisfying

$$\|T_L\| \leq \|L\|.$$

This T_L is a translation invariant operator since

$$\begin{aligned} \tau_s T_L(f \otimes y) &= \tau_s T(fy) \\ &= \tau_s(f * L_y) \\ &= \tau_s f * L(y) \\ &= T_L(\tau_s f(y)) \\ &= T_L \tau_s(fy) \text{ for all } s \in G, \ y \in Y, \ f \in L^1(G). \end{aligned}$$

Hence $T_L \in (L^1(G, Y), L^1(G, X))$. By the first paragraph in the proof, we obtain $\|T_L\| = \|L\|$. Finally, the one-one correspondence between $(L^1(G, Y), L^1(G, X))$ and $\mathcal{L}(Y, M_b(G, X))$ is obvious. Therefore we obtain

$$(L^1(G, Y), L^1(G, X)) \cong \mathcal{L}(Y, M_b(G, X)),$$

and the proof is completed. □

Theorem 5.2. *Let A be a commutative semi-simple Banach algebra (not necessarily with identity) and X a Banach A-module. Then*

$$Hom_{L^1(G,A)}(L^1(G,A), L^1(G,X)) \cong Hom_A(A, M_b(G,X)). \tag{5.1}$$

Proof. It is known that a multiplier operator $T \in Hom_{L^1(G,A)}(L^1(G,A), L^1(G,X))$ is an invariant operator $T \in (L^1(G,A), L^1(G,X))$.
Then for any $t \in G$ and $f, g \in L^1(G,A)$,

$$g * (T\tau_t)f = T(g * \tau_t(f)) = \tau_t(g) * Tf = \tau_t(g * Tf) = g * \tau_t(Tf),$$

and hence

$$L^1(G,A) * (T\tau_t - \tau_t T)f = \{0\} \text{ for all } t \in G \text{ and } f \in L^1(G,A).$$

The reason is guaranteed from semi-simple of A and then the Banach algebra $L^1(G,A)$ without order, as well as $L^1(G,X)$ order free. It follows that T is invariant, that is, $T \in (L^1(G,A), L^1(G,X))$. Taking $Y = A$ in Theorem 5.1 , it yields (5.1). □

6. Remark

Finally we remake that an invariant operator

$$T \in (L^1(G,A), L^1(G,A)) \nRightarrow T \in \mathfrak{M}(L^1(G,A)).$$

We may ask that what is the necessary and sufficient condition for which

$$(L^1(G,A), L^1(G,A)) = Hom_{L^1(G)}(L^1(G,A), L^1(G,A))?$$

That is, whether $\mathfrak{M}(L^1(G,A)) \cong (L^1(G,A), L^1(G,A))$?

Precisely, one could consult Lai/Chang [13], Theorem 5.2, which we stated as Theorem 6.1.

Theorem 6.1. *Let A be a commutative Banach algebra with identity of norm 1. X be a unit linked, order-free, Banach-module and A a faithful representation on X, then each invariant operator $T : L^1(G,A) \to F(G,X)$ is a multiplier if and only if $A \cong C$. Here $F(G,X) = L^p(G,X)$ for each p, $1 \le p < \infty$, or $F(G,X) = C_0(G,X)$.*

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