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APPLICATION OF ELZAKI FRACTIONAL VARIATIONAL ITERATION METHOD FOR SOLVING FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. This article introduces a new method that combines the Elzaki transform with the variational iteration method for solving fractional delay differential equations (FDDEs) that involve proportional delay. The fractional derivative is considered in the Liouville-Caputo sense. The study explores the existence and uniqueness of solutions for FDDEs. Three numerical examples are provided to demonstrate the effectiveness and efficiency of the proposed method. The numerical results are compared with the exact solution. The main advantage of this approach is its capability to efficiently address both linear and nonlinear problems in FDDEs and provides accurate and reliable results. Additionally, the paper examines the convergence results and provides a detailed error analysis.

1. Introduction

The concept of fractional order derivatives was introduced in the seventeenth century and has since garnered significant attention due to its many applications in nonlinear complex systems across a variety of important phenomena. These include fields such as mathematical biology [16, 18], electrical networks [48], signal processing [17], fluid mechanics [41], diffusion-reaction processes [40], and relaxation processes, among other scientific and engineering domains [46].

Fractional derivatives (FDs) offer more accurate approximations for real-world problems compared to integer-order derivatives due to their memory effect and non-local behaviors. In many application areas, fractional calculus is essential for explaining complex dynamic behavior. It also enhances our understanding of the structure of matter and facilitates the design of control systems without compromising essential characteristics.

Over the past four decades, significant advancements have been made in the study and development of differential equations and fractional derivative. Fractional-order differential equations generalize traditional differential equations and hold nonlocal and intrinsic significance in understanding material properties. The solutions to fractional differential equations often describe the nonlinear problems encountered in nature. Due to the complexity of solving these equations accurately, various analytical and numerical methods have been employed [1, 4, 5, 40, 42, 43].

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Delay differential equations (DDEs) are widely applicable in various fields, including electrical, chemical, biological, and transportation systems. A delay differential equation is defined as an equation in which the temporal derivatives at the present time depend on both the solution and its potentially earlier derivatives. Recent studies have employed several types of delays, such as proportional, constant, and time-dependent delays. Additionally, fractional delay differential equations (FDDEs) can be used to model a variety of phenomena, including population growth, economic development, and neural networks; further details can be found in [22, 26, 29, 30, 45, 49]. In recent years, FDDEs have gained significant attention, as they are considered suitable for describing various systems and processes in engineering and other scientific disciplines [9]. Time delays find specific applications in industrial operations [32], transportation systems [10], mechanical systems [27], and chemical processes [11]. The advantage of studying FDDEs is that FDs are non-local in nature and can efficiently model memory effects while delay terms present history of the earlier state. Therefore, FDDEs capture the behavior of complex systems more accurately than integer order DDEs. This enhanced flexibility leads to a more correct representation of real world problem. Also, multiplicative differential equations (MDEs) find applications in bio-medical image analysis, time scale-theory, finance etc. Iterative methods such as Adomian decomposition method and variational iteration method (VIM) can be use to obtain solution of MDEs [15].

In recent decades, researchers have employed various methods to tackle DDEs, and FDDEs such as differential transform method [13], Adomian decomposition method [2], homotopy perturbation method [47], generalized Legendre polynomial configuration method [12], residual power series method [8, 25], fuzzy Elzaki transform method and the fuzzy Elzaki decomposition method [44], variational iteration method [25, 31, 46], Elzaki iteration method [20], Krawtchouk wavelet based projection approach [36], Finite element method [24] and spectral Galerkin technique [6]. However, most of these approaches encounter challenges related to convergence, computational complexity, initial approximations, or the selection of basis functions. Additionally, they often struggle to effectively handle solitary, stiff, or highly nonlinear situations. The optimal method depends on the specific issues at hand and the desired balance between computational efficiency and accuracy.

In this paper, we focus on the following FDDEs [33]

$$\begin{aligned} {}^{lc}D^{\alpha}w(t) &= f(t, w(t), w(\tau t)), \quad 0 < t \leq T,\, T \in \mathbb{R}^{+}, \quad 0 < \alpha \leq 1, \\ w(t) &= \phi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{1.1}$$

where $f \in \mathrm{C}[0, \tau]$, $\phi \in \mathrm{C}[-\tau, 0]$ and ${}^{lc}D^{\alpha}$ denotes the Liouville-Caputo fractional derivative and $\phi(t)$ is initial condition.

The Elzaki transform (ET) is a modification of the Sumudu and Laplace transforms. It was first introduced by Elzaki and Elzaki in 2011, as referenced in [14]. This transform is commonly used in applied sciences and engineering to solve mathematical problems, such as integral and differential equations, see [38] for application to the Fokker-Planck equation. The Variational Iteration Method, developed by Odibat [37], is employed for solving a wide range of nonlinear problems. Typically, VIM relies on general Lagrange multipliers, yielding solutions in the form of a convergent series. In this paper, we combine the ET with the VIM to create the Elzaki Variational Iteration Method (EVIM). Notably, this proposed method does not require the calculation of general Lagrange multipliers. Moreover, there are several approaches to integrating the ET with other techniques, such as the decomposition method [19], the residual power series method [8], and the differential transform method [13]. Many researchers utilize these techniques to address various differential equations, including DDEs of both integer and fractional orders, ordinary differential equations (ODEs), and partial differential equations (PDEs). The EVIM has been applied to ODEs and PDEs of both integer and fractional orders, as well as to integer-order DDEs, with further details available in [7, 20]. Consequently, we aim to extend this method to fractional-order delay differential equations with proportional delay.

In our work, we address the nonlinear term using a clearly defined formula discussed in [34] and in one of the example we apply the method of steps to transform a constant neutral delay differential equation into a proportional delay differential equation, as referenced in [35]. To the best of our knowledge, this technique has not been previously employed by any researcher for FDDEs.

It is important to note that while the ET is effective on its own for linear FDDEs, it struggles with nonlinearity. On the other hand, the VIM can adapt to nonlinear scenarios, although it may converge slowly. The novelty of our approach lies in the fact that by combining the VIM with the ET, we enhance both the accuracy and convergence of the solution, making the method more reliable and effective.

This paper is organized as follows: Section 2 introduces the fractional integral and derivative operators, the Elzaki transform (ET), and several properties and results related to the Elzaki transform. In Section 3, we discuss fractional delay differential equations along with the existence and uniqueness theorem. The methodology for the proposed approach is presented in section 4, while Section 5 outlines the specific steps involved in this method. In Section 6, we provide numerical examples to illustrate the proposed technique. Finally, a conclusion is presented in Section 7.

2. Preliminaries

In this section, we present the Gamma function, the Riemann–Liouville integral operator, the Liouville-Caputo fractional derivative, and present several fundamental concepts of ET.

Definition 2.1 ([21])**.** The Gamma function, $\Gamma(n)$ is given as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \;\; n > 0. \tag{2.1}$$

Definition 2.2 ([20, 28])**.** The Riemann–Liouville integral operator with fractional order α is defined as

$${}^{RL}I^{\alpha} w(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} w(\xi) d\xi, \quad 0 < \alpha \leq 1, \quad \xi > 0. \tag{2.2}$$

Definition 2.3 ([20, 23, 28])**.** The Liouville-Caputo fractional derivative of order α is defined as

$$
{}^{lc}D_t^{\alpha}w(x) = \begin{cases} I^{n-\alpha}\left(\frac{d^n w(x)}{dx^n}\right) & \text{for} \quad n-1<\alpha<n, \\ \frac{d^n w(x)}{dx^n} & \text{for} \quad \alpha = n,\, n\in\mathbb{N}. \end{cases} \tag{2.3}
$$

Definition 2.4 ([20, 50])**.** Let

$$
H = \left\{ w(x)\colon \exists M, k_1, k_2 > 0, |w(x)| < Me^{\frac{|x|}{k_i}} \quad \text{if} \quad x\in(-1)^i \times [0,\infty) \right\},
$$

and w is of exponential order. Then, the Elzaki transform (ET) of $w(x)$ is defined as

$$
E[w(x)] = T(\nu) = \nu\int_0^{\infty} w(x)e^{-\frac{x}{\nu}}dx, \quad x\geq 0, \quad k_1\leq \nu\leq k_2. \tag{2.4}
$$

Theorem 2.5 ([50])**.** (i) *The ET of the power series function*

$$
w(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2.5}
$$

is given by

$$
E[w(x)] = T(\nu) = \sum_{n=0}^{\infty} n! a_n \nu^{n+2}. \tag{2.6}
$$

(ii) *Let $w(x)\in H$ and suppose that $T_n(\nu)$ is the ET of the nth derivative of $w(x)$, for $n\geq 1$. Then*

$$
T_n(\nu) = \frac{T(\nu)}{\nu^n} - \sum_{i=0}^{n-1}\nu^{2-n+i}w^i(0). \tag{2.7}
$$

(iii) *If $w(x) = e^{nx}$, then $E[w(x)] = \frac{-\nu^2}{n\nu-1}$.*

(iv) *If $w(x) = x^n$, $n\in\mathbb{N}$, then $E[w(x)] = \Gamma(n+1)\nu^{n+2}$.*

3. Fractional delay differential equation with proportional delay

Consider the following intial value fractional delay problem

$$
\begin{aligned} {}^{lc}D^{\alpha}w(x) &= f(x, w(x), w(qx)), \quad 0<\alpha\leq 1, \quad q\in(0,1), \\ w(0) &= w_0, \end{aligned} \tag{3.1}
$$

where ${}^{lc}D^{\alpha}$ represent for the Liouville-Caputo fractional derivative, $w(qx)$ is delay term and f is continuous function defined on a rectangle

$$
X = \{|x|\leq a, \quad |w(x)-w_0|\leq b, \quad |w(qx)-w_0|\leq b, \quad a,b>0\}. \tag{3.2}
$$

Theorem 3.1. *[39] Let $f\colon X\to\mathbb{R}$ be a continuous function and $|f|$ be bounded by a positive integer R, where X is defined in* (3.2)*. Suppose the sequence of iterations described by*

$$
\aleph_0(x) = w_0,
$$

$$\aleph_{k+1}(t) = w_0 + \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \aleph_k(s), \aleph_k(qs))ds, \quad k = 0, 1, 2, \dots, \tag{3.3}$$

exists and remain continuous on the interval $I = [-\gamma, \gamma]$, *where* $\gamma = \min\left\{a, \left(\frac{\Gamma(\alpha+1)b}{R}\right)^{\frac{1}{\alpha}}\right\}$. *If* $x \in I$, *then* $(x, w(x), w(qx))$ *is found inside* X *and bounds*

$$|\aleph_k(x) - w_0| \le R\frac{|x|^\alpha}{\Gamma(\alpha+1)} \quad \text{and} \quad |\aleph_k(qx) - w_0| \le R\frac{|x|^\alpha}{\Gamma(\alpha+1)}$$

are accurate.

Theorem 3.2. (Existence theorem) *[3, 39] Let* $f\colon X \to \mathbb{R}$ *be a continuous function and* $|f|$ *be bounded by a positive integer* R, *where* X *is defined in* (3.2). *Assume that* f *satisfies the Lipschitz condition*

$$|f(x, w_1(x), w_1(qx)) - f(x, w_2(x), w_2(qx))| \le L_1|w_1(x) - w_2(x)| + L_2|w_1(qx) - w_2(qx)|,$$

with the Lipschitz constants $L_1, L_2 > 0$. *Then, under these conditions, the iterative solution by* (3.3) *converges to a solution of* $\aleph$ *of the initial value problem* (3.1) *across the interval* $I = [-\gamma, \gamma]$, *where* $\gamma = \min\left\{a, \left(\frac{\Gamma(\alpha+1)b}{R}\right)^{\frac{1}{\alpha}}\right\}$.

4. Technique of Elzaki transform iteration method

In this section, we discuss implementation of Elzaki transform on FDDEs. Consider the fractional differential equation

$$^{lc}D_t^\alpha w(x) + Rw(x) + Nw(x) = G(x), \quad 0 < \alpha \le 1, \tag{4.1}$$

with initial condition

$$^{lc}D_t^{\alpha-1} w(x)|_{x=0} = f_{\alpha-1}(x),$$

where $^{lc}D_t^\alpha w(x)$ is the derivative of order α, $Nw(x)$ is nonlinear term, R is linear term, and $G(x)$ is alternate source term.
Applying the Elzaki transform to (4.1), we get

$$E[^{lc}D_t^\alpha w(x)] + E[Rw(x)] + E[Nw(x)] = E[G(x)].$$

Now, rearranging the above equation, we have

$$E[^{lc}D_t^\alpha w(x)] = E[G(x)] - E[Rw(x)] - E[Nw(x)].$$

By using Theorem 2.5, we obtain

$$E[w(x)] = \sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0) + \nu^\alpha E[G(x)] - \nu^\alpha \left[E[Rw(x) + Nw(x)]\right]. \tag{4.2}$$

Applying the inverse Elzaki operator E^{-1} on both sides of (4.2), we get

$$\begin{aligned} w(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] + E^{-1}\left[\nu^\alpha E[G(x)]\right] \\ - E^{-1}\left[\nu^\alpha E[Rw(x) + Nw(x)]\right], \end{aligned}$$

which we can rewrite as

$$w(x) = K(x) - E^{-1}\left[\nu^{\alpha} E[Rw(x) + Nw(x)]\right], \tag{4.3}$$

where $K(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] + E^{-1}\left[\nu^{\alpha} E[G(x)]\right]$ represents the term from source term and the initial conditions. Finally, according to the variational iteration method [37], we can write the correctional functional as follows:

$$w_{n+1}(x) = K(x) - E^{-1}\left[\nu^{\alpha} E[Rw_n(x) + Nw_n(x)]\right], \tag{4.4}$$

and the approximate analytical solution is given by

$$w(x) = \sum_{n=0}^{\infty} w_n(x). \tag{4.5}$$

Theorem 4.1. *[37] Let $X\colon H \to H$. Then the series solution $w(x)$, given in* (4.5)*, converges if there exists $0 < \beta < 1$ such that*

$$\|X[w_0 + w_1 + \cdots + w_{n+1}]\| \leq \beta \|X[w_0 + w_1 + \cdots + w_n]\|,$$

that is, $\|w_{n+1}(x)\| \leq \|w_n(x)\|$, $n \in \mathbb{N} \cup \{0\}$.

Theorem 4.2. *Suppose $Nw(x)$ is the nonlinear term in* (4.1) *and satisfies a Lipschitz condition*

$$\|N(w) - N(v)\| \leq L\|w - v\|, \quad w, v \in X, \quad 0 \leq L < 1.$$

Then the sequence $\{w_n(x)\}$ converges uniquely to the exact solution $w^*(x)$.

Proof. Consider the iteration function

$$T(w) = w + E^{-1}R(w). \tag{4.6}$$

The EVIM iteration may be expressed as

$$w_{n+1} = T(w_n), \tag{4.7}$$

and $R(w_n) = G(x) - {}^{lc}D_t^{\alpha} w_n - Rw_n - N(w_n)$ is the residual function. In order to demonstrate convergence, we must demonstrate that $T(w)$ is a contraction mapping. Take a Cauchy sequence $\{w_n\}$ converges to a distinct fix point w^*.

Let w^* be the exact solution of (4.1). Then

$$w^* = T(w^*) = w^* + E^{-1}R(w^*). \tag{4.8}$$

Define the error at n^{th} step as follows:

$$e_n = w_n - w^*. \tag{4.9}$$

From (4.7) and (4.8), we have

$$\begin{aligned} e_{n+1} &= w_{n+1} - w^* = T(w_n) - T(w^*), \\ &= w_n + E^{-1}R(w_n) - w^* - E^{-1}R(w^*), \end{aligned}$$

that is, $e_{n+1} = (w_n - w^*) - E^{-1}(R(w_n) - R(w^*))$. Taking norm on both sides, we obtain

$$\|e_{n+1}\| = \|(w_n - w^*) - E^{-1}(R(w_n) - R(w^*))\|,$$

that is,

$$\|e_{n+1}\| \leq \|e_n\| - \|E^{-1}(R(w_n) - R(w^*))\|. \tag{4.10}$$

Including the nonlinear function $N(w)$ in $R(w)$, from (4.7) we get

$$\begin{aligned} R(w_n) - R(w^*) = &-\big({}^{lc}D^{\alpha} w_n - {}^{c}D^{\alpha} w^*\big) - \big(Rw_n - Rw^*\big) \\ &- \big(N(w_n) - N(w^*)\big). \end{aligned} \tag{4.11}$$

For the fractional derivative operator ${}^{lc}D_t^{\alpha}$, there exists a constant $C_\alpha > 0$ such that

$$\|{}^{lc}D^{\alpha} w_n - {}^{lc}D^{\alpha} w^*\| \leq C_\alpha \|w_n - w^*\|. \tag{4.12}$$

For the linear operator R, there exists a bound $C_R > 0$ such that

$$\|Rw_n - Rw^*\| \leq C_R \|w_n - w^*\|. \tag{4.13}$$

For the nonlinear operator $N(w)$, using the Lipschitz condition there exists $L \geq 0$ such that

$$\|N(w_n) - N(w^*)\| \leq L\|w_n - w^*\|. \tag{4.14}$$

Thus, taking norm of (4.11) and using (4.12)–(4.14), we get

$$\|R(w_n) - R(w^*)\| \leq -(C_\alpha + C_R + L)\|w_n - w^*\|.$$

Applying E^{-1} and assuming that it is a bounded operator with M, we obtain

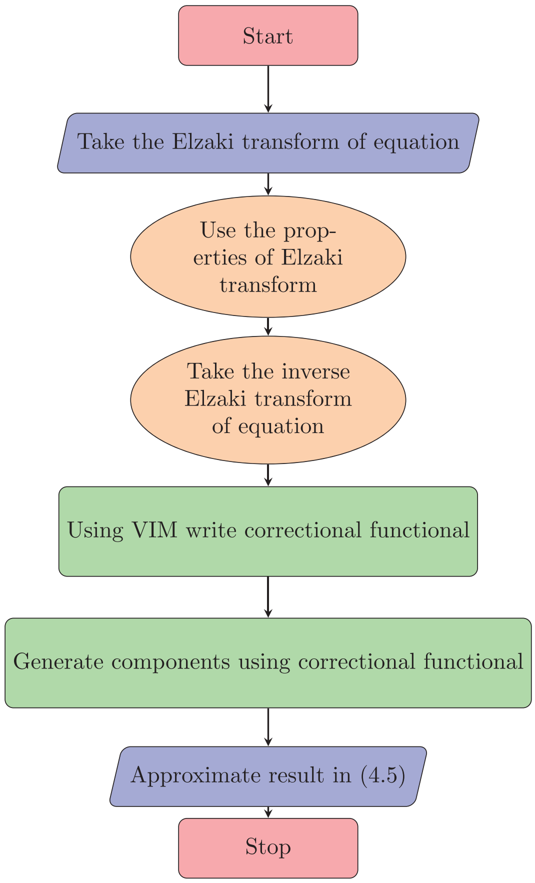
$$\|E^{-1}R(w_n) - E^{-1}R(w^*)\| \leq -M(C_\alpha + C_R + L)\|w_n - w^*\|.$$

Now, using (4.9) and (4.10), we get $\|e_{n+1}\| \leq \big(1 + M(C_\alpha + C_R + L)\big)\|e_n\|, \quad n \in \mathbb{N}$. Continuing in this way, we obtain $\|e_{n+1}\| \leq \big(1 + M(C_\alpha + C_R + L)\big)^{n+1}\|e_0\|$. Now, in order for the convergence to occur, we need $1 + M(C_\alpha + C_R + L)$ to be a contraction constant that satisfies $0 < M(C_\alpha + C_R + L) < 1$. Since C_α, C_R, and M rely on the Lipschitz constant $L < 1$, R, and the operator ${}^{lc}D^{\alpha}$, e_n falls exponentially if the nonlinear term fulfills the Lipschitz condition with constant $L < 1$. Since $\big(1 + M(C_\alpha + C_R + L)\big)^{n+1} \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \|e_n\| = 0$. Thus, $\{w_n\}$ is a Cauchy sequence and converges to w^*. This completes the proof. □

5. Steps of proposed method

In this section, we present step-by-step implementation and flow chart of the proposed method to solve FDDEs.

Step-1: Take Elzaki transform on both sides of the given equation.
Step-2: Use properties of the Elzaki transform and transform the function.
Step-3: Take inverse Elzaki transform.
Step-4: Apply variational iteration method by writing correctional functional.
Step-5: Obtain the components and write the solution.



6. Numerical examples

This section covers both linear and nonlinear FDDEs with proportional and neutral delays to demonstrate the method's robustness and dependability. Numerical calculations are performed using software Mathematica 11.1.1 and Python.

Example 6.1. Consider a linear fractional differential equation with proportional delay [34].

$$ {}^{lc}D^{\alpha} w(x) = \frac{1}{2} e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + \frac{1}{2} w(x), \quad 0 < \alpha \leq 1, \tag{6.1} $$

with initial condition $w(0) = 1$. The exact solution of (6.1) is given by $w(x) = e^{x}$. Applying ET to (6.1), we get

$$ E[{}^{lc}D^{\alpha} w(x)] = \frac{1}{2} E\left[e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + w(x)\right]. \tag{6.2} $$

Using (2.7), we have $\frac{1}{\nu^{\alpha}} E[w(x)] = \sum_{i=0}^{\alpha-1} \nu^{2-\alpha+i} w^{i}(0) + \frac{1}{2} E\left[e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + w(x)\right]$,

$$ E[w(x)] = \sum_{i=0}^{\alpha-1} \nu^{2+i} w^{i}(0) + \frac{1}{2} \nu^{\alpha} E\left[e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + w(x)\right]. \tag{6.3} $$

Now, taking the inverse ET of (6.3), we have

$$w(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] + \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + w(x)\right]\right],$$

which may be written as $w(x) = K(x) + \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} w\left(\frac{x}{2}\right) + w(x)\right]\right]$.
For the correctional functional, we use variational iteration method which is given by

$$w_{n+1}(x) = K(x) + \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} w_n\left(\frac{x}{2}\right) + w_n(x)\right]\right],$$

where $K(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] = w(0) = 1$, and the correctional functional is given by

$$w_{n+1}(x) = \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} w_n\left(\frac{x}{2}\right) + w_n(x)\right]\right].$$

Now, for $n = 0$, we have

$$\begin{aligned} w_1(x) &= \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} w_0\left(\frac{x}{2}\right) + w_0(x)\right]\right], \\ &= \frac{1}{2}E^{-1}\left[\nu^\alpha E\left[e^{\frac{x}{2}} + 1\right]\right], \end{aligned}$$

which, using Theorem 2.5, can be written as

$$w_1(x) = \frac{1}{2}E^{-1}\left[\nu^\alpha\left(\frac{\nu^2}{1-0.5\nu}\right) + \nu^\alpha \nu^2\right],$$

which yields

$$w_1(x) = \frac{x^\alpha}{\alpha!} + \frac{1}{4}\frac{x^{\alpha+1}}{(\alpha+1)!} + \frac{1}{8}\frac{x^{\alpha+2}}{(\alpha+2)!} + \frac{1}{16}\frac{x^{\alpha+3}}{(\alpha+3)!} + \dots,$$

Similarly, we get further components for $n = 1, 2, \dots$ as

$$w_2(x) = \frac{3}{4}\frac{x^{\alpha+1}}{(\alpha+1)!} + \frac{13}{32}\frac{x^{\alpha+2}}{(\alpha+2)!} + \frac{39}{128}\frac{x^{\alpha+3}}{(\alpha+3)!} + \dots,$$

and

$$w_3(x) = \frac{15}{32}\frac{x^{\alpha+2}}{(\alpha+2)!} + \frac{189}{512}\frac{x^{\alpha+3}}{(\alpha+3)!} + \dots,$$

and so on. Hence, the solution of (6.1) is given by $w(x) = w_0(x) + w_1(x) + w_2(x) + w_3(x) + \dots$.

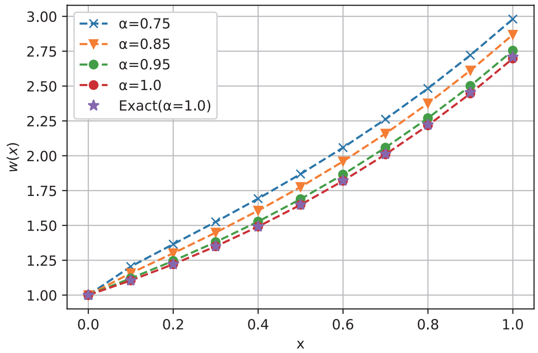


FIGURE 1. Graphical representation of $w(x)$ for Example 6.1.

TABLE 1. Numerical solution of $w(x)$ for Example 6.1.

x	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1.0$	Exact($\alpha = 1.0$)
0.1	1.204954832	1.157741497	1.120580472	1.105169735	1.105170833
0.2	1.365407728	1.300483889	1.245494051	1.221382422	1.221400000
0.3	1.525402396	1.448539013	1.380551889	1.349748511	1.349837500
0.4	1.691981578	1.606122573	1.527784273	1.491452083	1.491733333
0.5	1.868723622	1.775739207	1.688809881	1.647750854	1.648437500
0.6	2.058114443	1.959356323	1.865148491	1.819976172	1.821400000
0.7	2.262188049	2.158732101	2.058311378	2.009533016	2.012170833
0.8	2.482768462	2.375542286	2.269837055	2.217900000	2.222400000
0.9	2.721579720	2.611438845	2.501308093	2.446629370	2.453837500
1.0	2.980302310	2.868080472	2.754359976	2.697347005	2.708333333

TABLE 2. Absolute and relative errors when $\alpha = 1.0$ for Example 6.1.

x	EVIM	Exact	Absolute error	Relative error
0.1	1.105169735	1.105170833	1.09863×10^{-6}	9.94084×10^{-7}
0.2	1.221382422	1.221400000	1.75781×10^{-5}	1.43918×10^{-5}
0.3	1.349748511	1.349837500	8.89893×10^{-5}	6.59259×10^{-5}
0.4	1.491452083	1.491733333	2.81250×10^{-4}	1.88539×10^{-4}
0.5	1.647750854	1.648437500	6.86646×10^{-4}	4.16543×10^{-4}
0.6	1.819976172	1.821400000	1.42383×10^{-3}	7.98193×10^{-4}
0.7	2.009533016	2.012170833	2.63782×10^{-3}	1.31093×10^{-3}
0.8	2.217900000	2.222400000	4.48000×10^{-3}	2.02484×10^{-3}
0.9	2.446629370	2.453837500	7.20813×10^{-3}	2.93749×10^{-3}
1.0	2.697347005	2.708333333	1.09863×10^{-2}	4.05649×10^{-3}

The graphical representation of $w(x)$ for different values of α is shown in Figure 1. The efficiency and accuracy of the proposed technique are demonstrated through numerical results, which are detailed in Table 1. It is evident from Table 1 and Figure 1 that the generated numerical solution converges to the correct value as α increases from 0.75 to 1.0. The absolute and relative errors for $\alpha = 1.0$ are summarized in Table 2, indicating an accuracy range between 10^{-7} and 10^{-3}.

Example 6.2. Consider a nonlinear fractional differential equation with proportional delay [34].

$$
{}^{lc}D^{\alpha} w(x) = 1 - 2w^2\left(\frac{x}{2}\right), \quad x \in [0,1], \quad 0 < \alpha \le 1, \tag{6.4}
$$

with initial condition $w(0) = 0$. The exact solution of (6.4) is $w(x) = \sin x$.
Apply ET to (6.4), we get $E[{}^{lc}D^{\alpha} w(x)] = E\left[1 - 2w^2\left(\frac{x}{2}\right)\right]$. Using (2.7), we have

$$
E[w(x)] = \sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0) + \nu^{\alpha} E\left[1 - 2w^2\left(\frac{x}{2}\right)\right]. \tag{6.5}
$$

Now, taking inverse ET of (6.5), we get

$$
w(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] + E^{-1}\left[\nu^{\alpha} E\left[1 - 2w^2\left(\frac{x}{2}\right)\right]\right].
$$

The correctional functional using VIM is given as $w_{n+1}(x) = E^{-1}\left[\nu^{\alpha} E\left[1 - 2w_n^2\left(\frac{x}{2}\right)\right]\right]$. Thus, for $n = 0, 1, 2, \ldots$, we obtain following components:

$$
w_1(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}, \quad w_2(x) = \frac{-2(2\alpha)!}{2^{2\alpha}(\alpha!)^2}\frac{x^{3\alpha}}{(3\alpha)!}, \quad w_3(x) = \frac{8(2\alpha)!(4\alpha)!}{2^{6\alpha}(\alpha!)^3(3\alpha)!}\frac{x^{5\alpha}}{(5\alpha)!},
$$

and so on. Hence, the solution of (6.4) is given by $w(x) = w_0(x) + w_1(x) + w_2(x) + w_3(x) + \ldots$.

TABLE 3. Numerical solution of $w(x)$ for Example 6.2.

x	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1.0$	Exact($\alpha = 1.0$)
0.1	0.191047636	0.148527049	0.114216912	0.0998334166	0.0998334166
0.2	0.313919220	0.264293912	0.219133329	0.1986693333	0.1986693307
0.3	0.412845968	0.366198738	0.318583536	0.2955202500	0.2955202066
0.4	0.494263547	0.456746053	0.412500617	0.3894186666	0.3894183423
0.5	0.561066156	0.536745310	0.500383336	0.4794270833	0.4794255386
0.6	0.615083912	0.606530505	0.581631828	0.5646480000	0.5646424733
0.7	0.657752654	0.666301806	0.655650714	0.6442339166	0.6442176872
0.8	0.690364856	0.716257888	0.721894725	0.7173973333	0.7173560909
0.9	0.714181715	0.756657620	0.779894361	0.7834207500	0.7833269096
1.0	0.730489045	0.787852710	0.829273113	0.8416666666	0.8414709848

TABLE 4. Absolute and relative errors when $\alpha = 1.0$ for Example 6.2.

x	EVIM	Exact	Absolute error	Relative error
0.1	0.0998334166667	0.0998334166468	1.91000×10^{-11}	1.99332×10^{-10}
0.2	0.1986693333330	0.1986693307950	2.53800×10^{-9}	1.27750×10^{-8}
0.3	0.2955202500000	0.2955202066610	4.33390×10^{-8}	1.46653×10^{-7}
0.4	0.3894186666670	0.3894183423090	3.24358×10^{-7}	8.32929×10^{-7}
0.5	0.4794270833330	0.4794255386040	1.54473×10^{-6}	3.22204×10^{-6}
0.6	0.5646480000000	0.5646424733950	5.52661×10^{-6}	9.78780×10^{-6}
0.7	0.6442339166670	0.6442176872380	1.62294×10^{-5}	2.51925×10^{-5}
0.8	0.7173973333330	0.7173560909000	4.12424×10^{-5}	5.74923×10^{-5}
0.9	0.7834207500000	0.7833269096270	9.38404×10^{-5}	1.19797×10^{-4}
1.0	0.8416666666670	0.8414709848080	1.95682×10^{-4}	2.32547×10^{-4}

The values of $w(x)$ for various values of α are illustrated in Figure 2. This figure also shows that as α approaches 1.0, the numerical solution increasingly aligns with the precise value, thereby confirming the accuracy and reliability of the numerical method for these parameter values. Furthermore, Table 3 indicates that both the precise and numerical solutions converge and become more similar as α gets closer to 1.0. Additionally, Table 4 presents the absolute and relative errors for $\alpha = 1.0$, demonstrating an accuracy range from 10^{-11} to 10^{-4}.

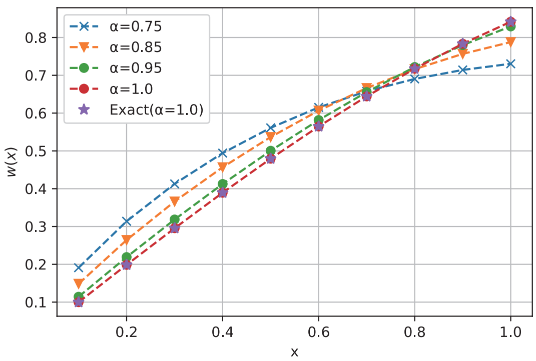


FIGURE 2. Graphical representation of $w(x)$ for Example 6.2.

Example 6.3. Consider a nonlinear fractional neutral differential equation with delay

$$ {}^{lc}D^{\alpha}w(x) = w'(x-2) + w^2\left(\frac{x}{2}\right) - e^{x-2}, \quad x \in [0,1], \quad 0 < \alpha \leq 1, \tag{6.6}$$

with initial function $f(x) = e^x$ and initial condition $w(0) = 1$. The exact solution of (6.6) is $w(x) = e^x$.

Firstly, applying method of steps in $x \in [0,1]$ to (6.6) and using initial function, we convert fractional neutral differential equation with delay (6.6) into fractional differential equation with proportional delay

$$^{lc}D^{\alpha}w(x) = w^2\left(\frac{x}{2}\right). \tag{6.7}$$

Applying ET to (6.7), we get $E\left[{}^{lc}D^{\alpha}w(x)\right] = E\left[w^2\left(\frac{x}{2}\right)\right]$, Then, using (2.7), we have

$$E[w(x)] = \sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0) + \nu^{\alpha} E\left[w^2\left(\frac{x}{2}\right)\right].$$

Now, taking inverse ET of (6.3), we get

$$w(x) = E^{-1}\left[\sum_{i=0}^{\alpha-1} \nu^{2+i} w^i(0)\right] + E^{-1}\left[\nu^{\alpha} E\left[w^2\left(\frac{x}{2}\right)\right]\right].$$

The correctional functional using VIM given as: $w_{n+1}(x) = E^{-1}\left[\nu^{\alpha}E\left[w_n^2\left(\frac{x}{2}\right)\right]\right]$. Thus, for $n = 0, 1, 2, \ldots$, we obtain following components:

$$w_1(x) = \frac{x^{\alpha}}{(\alpha)!}, \quad w_2(x) = \frac{2}{2^{\alpha}}\frac{x^{2\alpha}}{(2\alpha)!}, \quad w_3(x) = \left[\frac{4}{2^{3\alpha}} + \frac{(2\alpha)!}{2^{2\alpha}(\alpha!)^2}\right]\frac{x^{3\alpha}}{(3\alpha)!},$$

and so on. Hence, the solution of (6.6) is given by $w(x) = w_0(x) + w_1(x) + w_2(x) + w_3(x) + \ldots$.

Table 5. Numerical solution of $w(x)$ for Example 6.3 for different α.

x	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1.0$	Exact($\alpha = 1.0$)
0.1	1.224859984	1.164684974	1.121939560	1.105166666	1.105170918
0.2	1.420083627	1.321526863	1.250003827	1.221333333	1.221402758
0.3	1.624562674	1.488870672	1.389575470	1.349500000	1.349858807
0.4	1.843327163	1.670044473	1.542375507	1.490666666	1.491824697
0.5	2.078480122	1.866820453	1.709679822	1.645833333	1.648721270
0.6	2.331202125	2.080443092	1.892604095	1.816000000	1.822118800
0.7	2.602274136	2.311910810	2.092185140	2.002166666	2.013752707
0.8	2.892270431	2.562084371	2.309413187	2.205333333	2.225540928
0.9	3.201645028	2.831737521	2.545247364	2.426500000	2.459603111
1.0	3.530775720	3.121583937	2.800624149	2.666666666	2.718281828

Table 6. Absolute and relative errors when $\alpha = 1.0$ for Example 6.3.

x	EVIM	Exact	Absolute error	Relative error
0.1	1.105166666667	1.105170918076	4.25141×10^{-6}	3.84683×10^{-6}
0.2	1.221333333333	1.221402758160	6.94248×10^{-5}	5.68402×10^{-5}
0.3	1.349500000000	1.349858807576	3.58808×10^{-4}	2.65811×10^{-4}
0.4	1.490666666667	1.491824697641	1.15803×10^{-3}	7.76251×10^{-4}
0.5	1.645833333333	1.648721270700	2.88794×10^{-3}	1.75162×10^{-3}
0.6	1.816000000000	1.822118800391	6.1188×10^{-3}	3.35807×10^{-3}
0.7	2.002166666667	2.013752707470	1.1586×10^{-2}	5.75346×10^{-3}
0.8	2.205333333333	2.225540928492	2.02076×10^{-2}	9.07986×10^{-3}
0.9	2.426500000000	2.459603111157	3.31031×10^{-2}	1.34587×10^{-2}
1.0	2.666666666667	2.718281828459	5.16152×10^{-2}	1.89882×10^{-2}

Figure 3 illustrates the graphical representation of $w(x)$ for different values of α. The numerical results from Example 6.3, as summarized in Table 5, showcase the effectiveness and accuracy of the proposed method. It is clear from Table 5 and Figure 3 that as α approaches 1.0, the computed numerical solutions closely match the exact solution. Furthermore, Table 6 presents the absolute and relative errors for $\alpha = 1.0$, indicating an accuracy range from 10^{-6} to 10^{-2}.

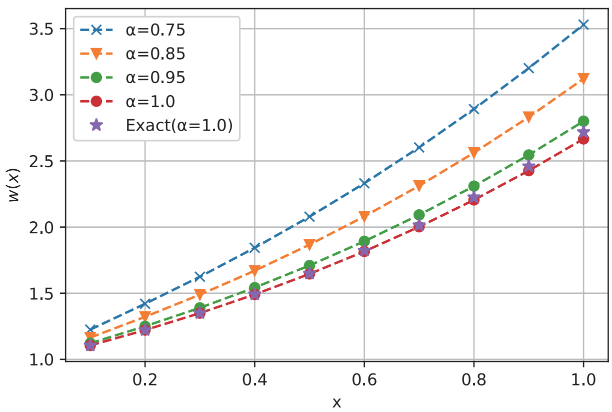


Figure 3. Graphical representation of $w(x)$ for Example 6.3.

7. Conclusion

This paper presents an approximate numerical solution for fractional delay differential equations (FDDEs) that involve proportional delay term. The solution is obtained using the Elzaki transform and the variational iteration method. The results obtained are in close agreement with the exact solution, in one of the problem discussed in the paper we used method of steps to convert the neutral constant delay differential equation into proportional delay differential equation, which is then solved by proposed method. The proposed method is easy to implement and provides accurate result. Additionally, we confirm through an existence and uniqueness theorem that solutions to FDDEs are well-defined and unique. Error analysis and convergence results further validate the accuracy of the numerical solutions. Moreover, graphical representations are provided for various values of α. In the future, the proposed approach may be extended to include nonlinear fractional constant and time-dependent delay differential equations, as well as systems of FDDEs with delays and other mathematical models.

Conflict of Interest

The authors declare that they have no conflict of interest.

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