



INTRODUCTION TO STATISTICAL CONVERGENCES IN QUANTUM METRIC SPACES

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ABSTRACT. Metric spaces are assuming a progressively significant role in both mathematics and the applied sciences. In the recent times the concept of metric spaces are worked on generalizations of metric spaces. Quantum calculus, frequently known as calculus without limits, is analogous to standard infinitesimal calculus, lacking the idea of limits. In this paper, we introduce a generalized metric space in associate with quantum. We called this metric space by q -metric spaces. Several properties of q -metric spaces are discussed. Further, using the concept of natural density, we have introduced statistical convergence in q -metric space. Lastly, we find several results via the idea of statistical dense on q -metric spaces.

1. INTRODUCTION AND PRELIMINARIES

Metric spaces are playing an increasing role in mathematics and the applied sciences. Over the past two decades, the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. The notion of a metric space was introduced by M. Fréchet [12], serves as a common idealization for numerous mathematical, physical, and other scientific frameworks in which the concept of distance is significant. The subjects of interest can vary greatly. They may consist of points, functions, sets, and even the personal experiences of sensations. Since Banach's fundamental result in 1922 [3], metric fixed point theory has been an area of vigorous scientific research, with applications on various fields of mathematics and other sciences. Over time, several generalizations of metric spaces have been introduced by altering or relaxing the traditional metric axioms. Significant breakthrough in this direction, was the development of pseudo metric [23], semi metric [34] and quasi-metric space [35]. Another one natural generalization, called cone metric [16] which is a vector-valued generalization of a metric space. One significant advancement in this direction was the introduction of b -metric spaces by Czerwik [6]. In b -metric space, a weaker version of the triangle inequality was considered. In a b -metric space, the distance function satisfies a relaxed form of the triangle inequality with a multiplicative constant $K \geq 1$, allowing greater flexibility in mathematical modeling. Many researchers have been interested in the concept of metric spaces and worked on generalizations of metric spaces in the years that followed, one can see in [8, 15, 21, 26] and their references.

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The notion of *infinite series* arose with the advent of algebraic operations in the field of number systems. Gregory focused on infinite series throughout the seventeenth century and wrote several articles about Maclaurin series. The former class of infinite series was referred to by him as *convergent series* (see [4]). Euler advanced the theory of q -series and hyper-geometric series throughout the eighteenth century, contributing to our understanding of infinite and non-convergent series. The notion of infinite was first introduced to mathematical analysis by Gauss. But Cauchy deserves praise for his precise definition of the sum of an infinite series. In 1821, Cauchy formalized his ideas on convergence and divergence of infinite series, explicitly defining the sum of an infinite series, known as the Cauchy sum, based on the concept of limit developed in his book *Analyse Algébrique*. Another significant figure in the early nineteenth century who contributed concepts on convergence and divergence was Abel [1]. Mathematicians developed a number of different techniques for allocating sums to infinite series by the end of the nineteenth century. *Summability Methods* was the moniker given to these summarizing techniques. The summability techniques that bear the names of eminent mathematicians such as Abel, Borel, Cesàro, Euler, Hausdorff, Hölder, Lambert, Nörlund, Reisz, Riemann, and Lebesgue are among the most well-known. Since summability theory was first based on classical analysis, it was regarded as a subfield of classical analysis. Hardy's book [17] is considered the pinnacle of that particular period. Functional analysis methods were first applied in foundational research by Zeller and others (see [37]), and were further advanced by Willansky and others (see [36]). It is encouraging to see that functional analysis basic textbooks have included the topic (see [25]). The theory and applications of summability theory have advanced significantly in the last century, both in the fields of functional analysis and classical analysis. In response, summability theory has given rise to several intriguing spaces of summable sequences and double sequences as well as new classes of matrices. The framework of the summability techniques was often designed to assign convergent series with the same sum that Cauchy assigned since his idea of the sum of a convergent series survived all mathematical scrutiny. A function from the set of sequences of partial sums of a series to a value is known as a summability technique. Therefore, summability, in its widest sense, is the theory of assigning limits, which is essential to topology, functional analysis, analysis, and function theory. If a summability technique adds up all convergent series to its Cauchy's sum, it is considered regular. If two summability techniques provide the same sum to the same series, they are considered consistent (see [17]). The fundamental idea behind all summability techniques is to convert an infinite series or sequence of partial sums into another series or sequence that may be used with Cauchy's approach. When applied to convergent series, the transformation selected is typically linear and ensures that Cauchy's sum is preserved. All summability methods are considered to be derived from two general basic processes, methods based on sequence-to-sequence transformation and methods based on sequence-to-function transformation. The two broad processes described above provide the basis for a variety of summability approaches that are crucial to summability theory and practice. Because of these approaches' advantages and applicability in other relevant sectors, they have become more important. Below is a quick description of matrix summability, one of these specialized

techniques. On the other hand, the principle of convergence for sequences of real numbers has been extended to the realm of statistical convergence by Fast [10] and Steinhaus [33]. This extension is mainly based on the concept of natural density of subsets within the set of all natural numbers, and various studies have explored this concept (see [13, 14, 27, 32]). The issue of convergence through various approaches of a real (or complex) valued divergent sequence dates back to the early nineteenth century. Many different methods of convergence have been defined (such as Cesaro, Nörlund, Weighted Mean, Abel, etc.) and have been applied in various branches of mathematics. Almost every convergence method is contingent upon the algebraic structure of the space. One can see [28–31] and references therein for several latest approach of summability. It is apparent that a metric space typically does not possess an algebraic structure. However, the concept of statistical convergence is easily extendable to arbitrary metric spaces.

In q -calculus we are looking for q -analogues of mathematical objects that have the original object as limits when q tends to 1. There are two types of q -addition, the Nalli-Ward-Al-Salam q -addition (NWA) and the Jackson-Hahn-Cigler q -addition (JHC). The first one is commutative and associative, while the second one is neither. This is one of the reasons why sometimes more than one q -analogue exists. The two operators above form the basis of the method which unities hypergeometric series and q -hypergeometric series and which gives many formulas of q -calculus a natural form reminding directly of their classical origin. The method is reminiscent of Eduard Heine (1821–1881), who mentioned the case where one parameter in a q -hypergeometric series is $+\infty$. The q -addition is the natural way to extend addition to the q -case, as can be seen when restating addition formulas for q -trigonometric functions. The history of q -calculus (and q -hypergeometric functions) dates back to the eighteenth century. It can in fact be taken as far back as Leonhard Euler (1707–1783), who first introduced the q in his *Introductio* [9] in the tracks of Newton's infinite series. The formal power series were introduced by Christoph Gudermann (1798–1852) and Karl Weierstrass (1815–1897). In England, Oliver Heaviside (1850–1925) made yet another contribution to this subject. q -calculus is a mathematical approach that serves as an alternative to traditional definitions of differential and integral calculus. Its origins can be traced back to the work of Euler and Jacobi in the eighteenth century. This methodology was further enhanced by Jackson's comprehensive studies conducted in the early 1900's [18]. The q -analog of a mathematical expression refers to the extension of that expression through the introduction of the parameter q . As q approaches 1, the generalized expression reverts to the original form. q -calculus, particularly through the application of q -derivatives and q -integrals, has emerged as a significant tool in the fields of differential equations, special functions, and combinatorial analysis. During the latter half of the twentieth century, there was a notable surge in research interest in q -calculus, driven by its relevance to various domains of mathematics and physics. More recently, q -calculus has garnered increasing focus within mathematical analysis and applied sciences. q -extensional operators such as q -Bernstein polynomials, q -Szász-Mirakyan operators and q -Weierstrass operators are used as powerful tools in approximate analysis. Moreover, the relation of q -integrals to quantum mechanics and quantum groups offers new possibilities in modeling physical systems. Also, a

tremendous interest was driven by high demands for mathematical models in quantum computing. It is a wide and an interesting area of research in recent times. Several researchers are engaged in the field of q -calculus due to its vast applications in mathematics, physics and engineering sciences. In the field of mathematics, it is widely used by researchers in approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, etc. One can see [2, 7, 11, 18–20] and references therein for detailed of q -calculus related work and applications.

The importance of generalized metric spaces of [8, 15, 21, 26] and references therein motivated us to introduce the idea of q -metric spaces. Further, the work of [5] motivated us to introduce statistical convergence on q -metric spaces.

The structure of the manuscript is as follows: In Section 1, we recall several definitions and results that are important for our next Section. In Section 2, we introduce q -metric spaces. Several properties of q -metric spaces are investigated. In particular q -convergent and q -Cauchy sequence in q -metric space. We have shown that every q -convergent sequence are q -Cauchy sequence. Moreover, we have shown separability, compactness of q -metric spaces. Lastly, we have developed some results related to q -continuity, and uniformly q -continuity on q -metric spaces. In Section 3, we introduce statistical convergence on q -metric space. Several important properties of statistical convergence on q -metric spaces. Finally, we present statistically dense set and its relation with statistical convergence on q -metric space.

In 1906, Fréchet [12] gave the formal definition of the distance by introducing a function d that assigns a nonnegative real number $d(x, y)$ (the distance between x and y) to every x, y of elements of a nonempty set X in the following way.

Definition 1.1 ([21]). Let X be a nonempty set. A function $d : X \times X \mapsto \mathbb{R}$ is called a metric on X if:

- (d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
- (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d is a metric on X , then (X, d) is called a metric spaces.

Example 1.2. A function $d : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ defined with $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$, is a metric on $\mathbb{R}$, often called the usual metric on $\mathbb{R}$.

Example 1.3. If $C[0, 1] = \{x : [0, 1] \mapsto \mathbb{R} \mid x \text{ is a continuous function}\}$ and

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|, \quad x, y \in C[0, 1],$$

then $(C[0, 1], d)$ is a metric space and such defined metric is well-known as supremum metric.

Recall the concept of Cauchy and convergent sequence in a metric space as follows.

Definition 1.4 ([21]). If $\{x_n\}$ is a sequence in X such that, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, $m, n \geq n_0$, then $\{x_n\}$ is a Cauchy sequence in X .

Definition 1.5 ([21]). If every Cauchy sequence in a metric space (X, d) is convergent in X , then (X, d) is a complete metric space.

For any x and $\epsilon > 0$, the set $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ is called the open ball centered at x having radius ϵ . A subset of a metric space is called open if it is a union of open balls or if it is empty. A subset of a metric space is called closed if its complement is open. The following properties are easily shown for a metric space.

- (1) An arbitrary union of open sets is open.
- (2) A finite intersection of open sets is open.
- (3) An arbitrary intersection of closed set is closed.

For any set A the intersection of all closed set containing A is a closed set called the closure of A . We denote $\bar{A}$ be the closure of A . We denote the boundary of A by $b(A)$. The complement of the closure is an open set called the interior of A . We denote $In(A)$ be the interior of A .

Next, we recall several definitions, and theorems of q -calculus that will be use in our next Sections.

Definition 1.6 ([18]). Let $0 < q < 1$. The quantum number or q -number of $n \in \mathbb{N}$ is defined by

$$[n]_q = [n] = \begin{cases} \frac{1 - q^n}{1 - q} & , n > 0 \\ 1 & , n = 0. \end{cases}$$

One may notice that when $q \rightarrow 1$ then $[n]_q = n$ for $n > 0$.

The q -analog of binomial coefficient or q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ r \end{bmatrix} \begin{cases} \frac{[n]_q!}{[n-r]_q! [r]_q!} & , n \geq r \\ 0 & , n < r \end{cases}$$

where q -factorial $[n]_q!$ of n is given by

$$[n]_q! = [n]! = \begin{cases} 1 & , n = 0 \\ [n][n-1]\dots[2][1] & , n > 0. \end{cases}$$

The q -differential of an arbitrary $\flat$ function is defined by $d_q f(x) = f(qx) - f(x)$. In particular let $d_q x = (q - 1)x$. Then the q -derivative of f defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x}$$

where $x \neq 0$ and $0 < q < 1$. Note that if $\flat$ is differentiable function, then

$$\lim_{q \rightarrow 1} D_q f(x) = \lim_{q \rightarrow 1} \frac{f(qx) - f(x)}{(q - 1)x} = \frac{xf'(x)}{x} = f'(x) = \frac{df(x)}{dx}.$$

One can see [2, 7, 11, 18–20] and references therein for details of q -differential of an arbitrary f functions and their related work. The q -analogue of $(a - b)^n$ is defined by

$$(a - b)_q^n = \begin{cases} 1 & , n = 0 \\ (a - b)(a - qb) \cdots (a - q^{n-1}b) & , n \geq 1 \end{cases}$$

for every $a, b \in \mathbb{R}$. In other words

$$(a - b)_q^n = \prod_{i=1}^{n-1} (a - q^i b) \quad \text{and} \quad (a - b)_q^0 = 1, \quad n \in \mathbb{N}.$$

We recall several definitions and theorems of the concept of statistically convergent.

Definition 1.7 ([13, 14]). Suppose $\mathcal{U} \subseteq \mathbb{N}$ and let $\mathcal{U}_n = \{k : k \leq n, k \in \mathcal{U}\}$. Then the natural density $d(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$d(\mathcal{U}) = \lim_{n \rightarrow \infty} \frac{|\mathcal{U}_n|}{n} = u$$

where the u is a real number and finite, $|\mathcal{U}_n|$ is the cardinality of $\mathcal{U}_n$.

A given sequence (x_n) is called statistically convergent to L , if for each $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \in \mathbb{N} : |x_k - L| \geq \epsilon \right\} \right| = 0.$$

Here we write

$$\text{stat} \lim_{n \rightarrow \infty} (x_n) = L.$$

A given sequence (x_n) is called statistically Cauchy, if for each $\epsilon > 0$, there exists an integer number $m > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \in \mathbb{N} : |x_k - x_m| \geq \epsilon \right\} \right| = 0.$$

Definition 1.8 ([24]). Let (X, d) be a metric space and $(x_n) \subset X$ be a sequence of elements of X .

The given sequence (x_n) is called statistically convergent to $x \in X$, if for every $\epsilon > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \in \mathbb{N}, d(x_k, x) \geq \epsilon\}| = 0$.

The given sequence (x_n) is called statistically Cauchy, if for every $\epsilon > 0$, there exist a integer number $m > 0$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \in \mathbb{N}, d(x_k, x_m) \geq \epsilon\}| = 0$.

The given sequence (x_n) is called statistically bounded, if for every $\epsilon > 0$, there exist a real number $M > 0$ and $x \in X$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k : k \in \mathbb{N}, d(x_k, x) \geq M\}| = 0$.

Theorem 1.9 ([24]). Let (X, d) be a metric space. Then the followings are satisfied.

- (1) Every convergent sequence is statistically convergent.
- (2) Every statistically convergent sequence is statistically Cauchy sequence.
- (3) Every statistically convergent sequence is statistically bounded.

Definition 1.10 ([10, 13]). Let $B \in \mathbb{N}$. If $\lim_{n \rightarrow \infty} \frac{1}{n} |B(n)| = 1$ whenever $B(n) = \{k \in B : k \leq n\}$, then B is called statistically dense subset of $\mathbb{N}$.

2. CONSTRUCTION OF THE QUANTUM METRIC SPACES

In this section we have introduced the concept of quantum metric space in short q -metric space. We have investigated the q -convergent and q -Cauchy sequences in q -metric space. Further, we have shown that every q -convergent sequence are q -Cauchy sequence, but converse may not be true. We have presented some properties of q -metric space. We start the section with following q -metric definition.

Definition 2.1. Let (Z, d) be a metric space. d_q real valued function is called q -metric on Z if for $0 < q < 1$ and $n \in \mathbb{N}$, $d_q : Z \times Z \rightarrow \mathbb{R}$ satisfy for all $x, y, z \in Z$,

- i. $[d_q(x, y)]_q \geq 0$
- ii. $[d_q(x, y)]_q = 0$ if and only if $x = y$
- iii. $[d_q(x, y)]_q = [d_q(y, x)]_q$
- iv. $[d_q(x, z)]_q \leq [d_q(x, y)]_q + [d_q(y, z)]_q$ where $[n]_q = \frac{1-q^n}{1-q}$.

Followings are some examples of q -metric spaces.

Example 2.2. Let $Z = \mathbb{N}$ or $Z = \mathbb{N}_0$. The function $[d_q(x, y)]_q = |[x]_q - [y]_q|$ is a q -metric on Z for $0 < q < 1$. Then (Z, d_q) is a q -metric space.

Example 2.3. If we take $Z = \mathbb{Z}$, then $[d_q(x, y)]_q = \frac{1-q^{|x-y|}}{1-q}$ is a q -metric on $\mathbb{Z}$ and (Z, d_q) is a q -metric space.

Example 2.4. Let $Z = \mathbb{R}^+$ and $\alpha > 0$ be a scalar, then for $x, y \in Z$, d_q is a q -metric on Z defined as:

$$[d_q(x, y)]_q = \frac{|x - y|}{1 + \alpha(1 - q)|x - y|}.$$

The q -metric d_q approach to classic metric on $\mathbb{R}^+$ when q approach to 1.

Example 2.5. Consider the set $Z = \mathbb{R}_q = \{q^n : n \in \mathbb{Z}\} \subset \mathbb{R}^+$ whose elements are sequences of real numbers with q -scale for $0 < q < 1$. The function $[d_q(q^m, q^n)]_q = |q^m - q^n|$ is a q -metric on Z .

Example 2.6. Let Z be a metric space and $x, y \in Z$,

$$[d_q(x, y)]_q = \begin{cases} 1, & x = y \\ 0, & x \neq y. \end{cases}$$

Then d_q is a q -trivial metric on Z .

Next, we state q -bounded metric space, q -convergent and q -Cauchy sequences on q -metric space as follows.

Definition 2.7. A q -metric space (Z, d_q) is called q -bounded if there exists a $M > 0$ real number such that $[d_q(x, y)]_q \leq M$ for all $x, y \in Z$.

Definition 2.8. A sequence $(x_n)_{n=1}^\infty$ in q -metric space (Z, d_q) is called q -converges to x subordinate to q -metric if and only if there exists $x \in Z$ such that $[d_q((x_n), x)]_q \rightarrow 0$ when $n \rightarrow \infty$.

We write $\lim_{n \rightarrow \infty} (x_n) = x$ or $(x_n) \rightarrow x$ and x is called the q -limit of sequence (x_n) .

Remark 2.9. It is not hard to see that every q -convergent sequences are q -bounded but the converse may not be true.

The following example shows that q -boundedness in a q -metric space does not imply q -convergent in the same space.

Example 2.10. Let $([x]_q)_n = \{[1]_q, [1.4]_q, [1.41]_q, [1.414]_q, \dots\}$ be a sequence of q -decimal defined on $Z = \mathbb{Q}$. Clearly this sequence is converges to $\sqrt{[2]_q}$. This sequence is q -bounded in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$.

Definition 2.11. Let $(x_n)_{n=1}^{\infty}$ be a sequence of q -metric space (Z, d_q) . If for every $\epsilon > 0$, there exists a natural number $n_0 > 0$ such that for all $m, n > n_0$, $[d_q((x_n), (x_m))]_q < \epsilon$ then the sequence (x_n) is called a q -Cauchy sequence in q -metric space.

Next, the following theorem gives relation between q -convergent sequence and q -Cauchy sequence on q -metric space.

Theorem 2.12.

- i. *A q -convergent sequence has a unique q -limit point.*
- ii. *Every q -convergent sequence is also a q -Cauchy sequence on q -metric space. Converse may not be true.*
- iii. *If a q -Cauchy sequence has a convergent subsequence then the all sequence is q -convergent sequence on q -metric space.*

Proof.

- i. If $(x_n) \rightarrow x$ and $(x_n) \rightarrow y$ in the q -metric d_q , then by the triangle inequality $0 \leq [d_q(x, y)]_q \leq [d_q(x, (x_n))]_q + [d_q((x_n), y)]_q \rightarrow 0$ where $n \rightarrow \infty$. Thus $[d_q(x, y)]_q = 0$, so by the q -metric definition, we have $x = y$ in other words the q -limit is unique.
- ii. Let (x_n) be q -converges to x . Then by the q -metric definition $0 \leq [d_q((x_n), (x_m))]_q \leq [d_q((x_n), x)]_q + [d_q(x, (x_m))]_q \rightarrow 0$ where $n, m \rightarrow \infty$. Therefore (x_n) is a q -Cauchy sequence, but converse is not true.

Example 2.13. Let $Z = (0, 1)$ a subset of $\mathbb{R}$ and the q -metric on Z defined by $[d_q(x, y)]_q = |x - y|$. Consider $(x_n) = \frac{1}{[n]_q} \subset \mathbb{R}$ for $n \in \mathbb{R}$,

$$\begin{aligned} \left| \frac{1}{[n]_q} - \frac{1}{[m]_q} \right| &= \left| \frac{1}{\frac{1-q^n}{1-q}} - \frac{1}{\frac{1-q^m}{1-q}} \right| \\ &= \left| \frac{1-q}{1-q^n} - \frac{1-q}{1-q^m} \right| \\ &\leq \frac{1}{1-q^n} + \frac{1}{1-q^m} \rightarrow 2 \end{aligned}$$

where $n, m \rightarrow \infty$. Hence 2 is the q -limit point but $2 \notin Z$. So the given sequence is a q -Cauchy sequence but not q -convergent on Z .

- iii. Let (x_n) be a q -Cauchy sequence in (Z, d_q) . Suppose that (x_{n_k}) is a q -convergent subsequence where $n_1 < n_2 < n_3 \dots$ are positive integer and also that $(x_{n_k}) \rightarrow x$ as $m \rightarrow \infty$ on q -metric space. Thus we have

$$\begin{aligned} 0 &\leq [d_q((x_n), x)]_q \\ &\leq [d_q((x_n), (x_{n_k}))]_q + [d_q((x_{n_k}), x)]_q \\ &\rightarrow 0 \end{aligned}$$

for n and k sufficiently large. So the q -Cauchy sequence (x_n) converges to q -limit of the q -convergent subsequence (x_{n_k}) . $\square$

2.1. Several properties of q -metric spaces. In this Section of the paper, we discuss some properties of q -metric spaces. We start with the following definitions.

- Definition 2.14.** (1) A q -neighbourhood of a point x is the set of all points within a distance of $\epsilon > 0$ from x , denoted as $B_q(x, \epsilon) = \left\{ y : [d_q(x, y)]_q < \epsilon \right\}$.
- (2) In an q -metric space, a set is considered to be q -open if for every point within the set there exists a q -neighbourhood that is entirely contained within the set.

We need the following Lemma for our next theorem.

Lemma 2.15. *Let Z be a q -metric space. A subset O of Z is q -open if and only if O is a union of q -open ball.*

Proof. Let Z be a q -metric space and O is a subset of Z . Let O be q -open set and O is a union of q -open balls. Since O is q -open ball, there exists $\epsilon > 0$ so that $B_q(a, \epsilon) \subseteq O$. We shall show that $O = \bigcup_{a \in O} B_q(a, \epsilon)$. We choose ϵ , $B_q(a, \epsilon) \subseteq O$ for every $a \in O$. So, $\bigcup_{a \in O} B_q(a, \epsilon) \subseteq O$.

Conversely, let $x \in O$. Then $x \in B_q(x, \epsilon)$ and so $x \in \bigcup_{a \in O} B_q(a, \epsilon)$. Thus $O \subseteq \bigcup_{a \in O} B_q(a, \epsilon)$. So, $O = \bigcup_{a \in O} B_q(a, \epsilon)$. $\square$

Theorem 2.16. (1) *An arbitrary union of q -open set is q -open in q -metric space.*

Proof. Let Z be a q -metric space. Let $\{O_\alpha\}$ is a collection of q -open sets in Z for α in some indexing set I and let $O = \bigcup_{\alpha \in I} O_\alpha$. By Lemma 2.15, we know that O_α is a union of q -open balls for each $\alpha \in I$. Combining all these q -open balls together shows that O is a union of q -open balls and therefore by Lemma 2.15, O is q -open set. $\square$

(2) *A finite intersection of q -open set is q -open in q -metric space.*

Proof. Let $O_1, O_2, \dots, O_n$ are q -open sets in Z for some $n \in \mathbb{N}$. In order to prove $O = \bigcap_{k=1}^n O_k$ is a q -open set, we shall show that O is q -neighborhood of each of its points. Let $x \in O$. Then $x \in O_k$ for each $1 \leq k \leq n$. Let k be between 1 and n . Since O_k is q -open, we know that O_k is a q -neighborhood of each of its points. So there exists $\epsilon_k > 0$ such that $B_q(x, \epsilon_k) \subseteq O_k$. Since there are only finitely many values of k , let $\epsilon = \min\{\epsilon_k : 1 \leq k \leq n\}$. Then $B_q(x, \epsilon) \subseteq B_q(x, \epsilon_k)$ for each k and so $B_q(x, \epsilon) \subseteq B_q(x, \epsilon_k)$ for each k and so $B_q(x, \epsilon) \subseteq \bigcap_{k=1}^n O_k = O$. Therefore O is a q -neighborhood of each of points and O is a q -open set. $\square$

(3) *A finite union of q -closed set is q -closed in q -metric space.*

Proof. Let $F_1, F_2, \dots, F_n$ are q -closed set of Z . If p is a q -limit of $\bigcup_{i=1}^n F_i$ then in every neighborhood there is a point $q \neq p$, such that $q \in \bigcup_{i=1}^n F_i$. Since $q \in \bigcup_{i=1}^n F_i$ then q belongs to at least one F_i , then p is a limit point of F_i . Since F_i are q -closed, then $p \in F_i$; then $p \in \bigcup_{i=1}^n F_i$. Therefore if p is a limit point of $\bigcup_{i=1}^n F_i$, then $p \in \bigcup_{i=1}^n F_i$. $\square$

Theorem 2.17. *An entire q -metric space is both q -open and q -closed.*

Proof. The proof is an analogous to the proof of entire metric space is both open and closed of classical metric space. Let (Z, d_q) be any q -metric space. Let x be any point in Z , the open ball $B_q(x, r) = \{y \in Z : [d_q(x, y)]_q < r\}$ centered at x with any radius $r > 0$ is clearly a subset of Z . So, for any point x in Z , we can find an q -open ball around is that is contained within Z . Therefore, Z is a q -open set.

Again, since complement of Z is $\emptyset$. The $\emptyset$ is q -open. Therefore Z is q -closed. $\square$

Definition 2.18. We called a q -metric space to be q -compact if every q -open cover has a finite sub cover.

We shall prove “Every q -compact metric space is separable”. In order to prove this, we need the followings.

Definition 2.19. A set E in a q -metric space (Z, d_q) is called totally bounded if for all $\epsilon > 0$ there exist finitely many balls $B_q(x_1, \epsilon), B_q(x_2, \epsilon), B_q(x_3, \epsilon), \dots, B_q(x_n, \epsilon)$ such that $E \subset \bigcup_{i=1}^n B_q(x_i, \epsilon)$.

Definition 2.20. A given q -metric space is called q -sequentially compact if every sequence with in the q -metric space has a q -convergent subsequence that converges to a point with in the space.

Lemma 2.21. Every sequentially q -compact set is totally bounded.

Proof. Let E be sequentially q -compact set. For $\epsilon > 0$, we pick any $(x_1 \in E$ and consider $E \setminus B_q(x_1, \epsilon)$. If $E \setminus B_q(x_1, \epsilon) = \emptyset$. The proof is complete, if $E \setminus B_q(x_1, \epsilon) \neq \emptyset$ and we can pick a point $x_{q_2} \in E \setminus B_q(x_1, \epsilon)$ and consider $E \setminus \left(B_q(x_1, \epsilon) \cup B_q(x_2, \epsilon) \right)$.

If $E \setminus \left(B_q(x_1, \epsilon) \cup B_q(x_2, \epsilon) \right) = \emptyset$ then our proof is complete If not, we can pick $x_3 \in E \setminus \left(B_q(x_1, \epsilon) \cup B_q(x_2, \epsilon) \right)$ and consider $E \setminus \left(B_q(x_1, \epsilon) \cup B_q(x_2, \epsilon) \cup B_q(x_3, \epsilon) \right)$ and so on. If we stop at a finite step, then there exist $x_1, x_2, \dots, x_n$ such that $E \setminus \left(\bigcup_{i=1}^n B_q(x_i, \epsilon) \right) = \emptyset$. The proof is completed. If not, we obtained a sequence $\{x_{q_1}, x_2, \dots\}$ such that

$$\begin{aligned} [d_q(x_2, x_1)]_q &\geq \epsilon \\ [d_q(x_3, x_i)]_q &\geq \epsilon \text{ for } i = 1, 2 \\ &\dots \\ [d_q(x_n, x_i)]_q &\geq \epsilon \text{ for } i = 1, 2, \dots, n-1. \end{aligned}$$

By sequential q -compactness of E , there exists a sequence (x_{n_i}) and $x \in E$ such that $(x_{n_i}) \rightarrow x$ as $j \rightarrow \infty$. Then there exists $j_0 > 0$ such that

$$\begin{aligned} [d_q((x_{n_j}), (x_{n_k}))]_q &< [d_q((x_{n_j}), x)]_q + [d_q((x_{n_k}), x)]_q \\ &< \epsilon \text{ for } n_j, n_k \geq n_{j_0}. \end{aligned}$$

We may assume $n_j \geq n_k$, then $\epsilon \leq d_q((x_{n_j}), (x_{n_k})) < \epsilon$ a contradiction. Hence we must stop at a finite step and hence we must stop at a finite step and hence we find many points $x_1, x_2, \dots, x_n$ such that $E \subset \bigcup_{i=1}^n B_q(x_i, \epsilon)$. $\square$

Theorem 2.22. *Every q -compact metric space is separable.*

Proof. By the Lemma 2.21, every q -compact set is totally bounded. This implies for all n , there exist finitely many points $(x_1)^n, (x_2)^n, \dots, (x_{N_n})^n$ such that $\left\{ B_q((x_i)^n, \frac{1}{n}) \right\}_{i=1,2,\dots,N_n}$ covers the space. Then $E = \left\{ (x_i)^n : n = 1, 2, \dots; i = 1, 2, \dots, N_n \right\}$ is a countable set. Since for all $x \in Z$, and for all $n \geq 1$ there exist $(x_i)^n \in E$ such that $[d_q(x, (x_i)^n)]_q < \frac{1}{n}$ so, E is dense. Hence proved. $\square$

Next, we show a relationship of q -compactness and sequentially q -compactness on q -metric space.

Theorem 2.23. *A subset of (Z, d_q) is q -compact if and only if it is sequentially q -compact.*

Proof. Let (Z, d_q) be a q -compact metric space. Let (Z, d_q) is not sequentially compact. Then there exists a sequence x_n in Z that has no convergent subsequence. Since there is no convergent subsequence, x_n must contain an infinite number of distinct points. Let $x \in Z$. If for every $\epsilon > 0$, the ball $B_q(x, \epsilon)$ contains a point in the sequence (x_n) that is distinct from x , the x will be the limit of a subsequence. Since we would be able to choose points from (x_n) from shrinking balls around x so, there is a $\epsilon_n > 0$ such that $B_q(x, \epsilon_n)$ contains no points from (x_n) except possibly x itself. The collection of open balls $\left\{ B_q(x, \epsilon_n) : x \in Z \right\}$ is an open cover of Z . The union of every finite number of these balls contains at most n terms in the sequence as there are an infinite number of distinct terms in the sequence. That is no finite subcollection of these balls will cover Z , since no finite subcollection will ever cover the terms of the sequence x_n in Z . As a result, we can find an open cover of Z that has no finite subcover. This is a contradiction of q -compactness. So, Z must be sequentially q -compact metric space.

Conversely, suppose (Z, d_q) be sequentially q -compact metric space. Let $\{G_\alpha\}$ be an arbitrary open cover of Z . Let B be the collection of open balls with rational radius and center in A where A be a countable subset of Z . Since A is countable and the rationals are countable, B is countable. Let C be the subcollection of balls in B that are contained in at least one of the q -open sets in the cover $\{G_\alpha\}$. Since C is a subset of B and B is countable, C is countable. That is for every $x \in Z$ there is a G_α such that $x \in G_\alpha$. Since G_α is q -open, there exists an $\epsilon > 0$ such that $B_q(x, \epsilon) \subseteq G_\alpha$. Since A is dense in Z , there exists a point $y \in A$ that is within $\frac{\epsilon}{3}$ of x . Then $x \in B_q(y, \frac{\epsilon}{3})$ and $B_q(y, \frac{2\epsilon}{3}) \subseteq G_\alpha$. Let $p \in \mathbb{Q}$ such that $\frac{\epsilon}{3} < p < \frac{2\epsilon}{3}$. Then $x \in B_q(y, p) \subseteq B(y, \frac{2\epsilon}{3}) \subseteq G_\alpha$. Since $B_q(y, p)$ has rational radius and center in A it is a ball in B . Furthermore, since it is a ball in B that is contained in a G_α , it is in the collection C . Thus, every $x \in Z$ belongs to a ball in C . So, C is a countable q -open cover of Z . Clearly every ball $B \in C$ is in at least one set G_α in $\{G_\alpha\}$. Let α_B such that $B \subseteq G_{\alpha_B}$. Since C is countable and covers Z and since $\left\{ G_{\alpha_B} : B \in C \right\}$ countable subcover of Z . Lastly, we will show that a countable open cover of a sequentially q -compact space has a finite subcover. Since $\{G_n\}$ has

no finite subcover, $\bigcup_{k=1}^n G_k$ does not contain Z for any n . Choose $(x_1) \in Z$ such that $(x_2) \notin \bigcup_{n=1}^{n_1} G_n$. We can do this because we have assumed that Z can not be covered by a finite subset of $\{G_n\}$. Since $\{G_n\}$ covers Z , there exists an n_2 such that $(x_2) \in G_{n_2}$. Choose $(x_3) \in Z$ such that $(x_3) \notin \bigcup_{n=1}^{n_2} G_n$. Choose n_3 so that $(x_3) \in G_{n_3}$. Proceeding this way, $(x_k) \in G_{n_k}$ and $(x_k) \notin \bigcup_{n=1}^{n_{k-1}} G_n$. So, G_{n_k} is not equal to G_n for any $n = 1, 2, \dots, n_{k-1}$ and the sequence (n_k) is strictly increasing. Since Z is sequentially compact, (x_n) must have a subsequence that converges to a point $x \in Z$. Since $\{G_n\}$ covers Z , $x \in G_n$ for some n . By our construction, of sequence there exists an integer K_n such that $(x_k) \notin G_n$ for all $k \geq K_n$. As $x \in G_n$, yet the sequence (x_n) and hence any subsequence of (x_n) can not be in G_n after some point. This contradicts the statement that (x_n) must have a subsequence converging to x and the sequential compactness of Z . Therefore the q -open cover $\{G_n\}$ must have a finite subcover and Z is q -compact metric space. $\square$

Proposition 2.24. (1) *A q -closed subset of a q -compact metric space is q -compact.*

(2) *A q -compact subset of any q -metric space is q -closed.*

(3) *A finite union of q -compact sets is q -compact.*

We shall now find some results of q -continuous functions on q -compact metric spaces.

Theorem 2.25. *The direct image of a q -compact metric space by a q -continuous function is q -compact.*

Proof. Consider a sequence (x_n) of elements of Z . Let f be a q -continuous function on the given q -compact metric space. Since f is surjective, we can choose a sequence (y_n) of points in Z such that $f((x_n)) = (y_n)$. Since Z is q -compact, there exists a subsequence (x_{n_k}) that converges to some point $x \in Z$. But since f is q -continuous at x , the sequence (y_{n_k}) converges to $f(x)$. This proves that Z is sequentially q -compact. Hence the direct image of a q -compact metric space by a q -continuous function is q -compact. $\square$

Theorem 2.26. *Let Z be a q -compact metric space, and let $f : Z \rightarrow \mathbb{R}$ be q -continuous. Then $f(Z)$ is q -bounded and there exist points $a, b \in Z$ such that $f(a) = \inf_{x \in Z} f(x)$ and $f(b) = \sup_{x \in Z} f(x)$.*

Proof. By Theorem 2.25, $f(Z)$ is a q -compact subset of $\mathbb{R}$, hence q -closed and bounded. Now, any bounded set $A \subset \mathbb{R}$ has a least upper bound $\sup A$ and a greatest lower bound $\inf A$, and these two points belong to the closure $\bar{A}$. Applying this, we can see $A = f(Z)$ which is q -closed. We conclude that $\sup f(Z)$ and $\inf f(Z)$ belongs to $f(Z)$ itself. This is what is our claim. $\square$

Finally, we find q -continuity and q -uniformly continuity on q -metric space.

Definition 2.27. (1) Let (Z, d_q) and (Y, d_q) are two q -metric spaces. A mapping $f : Z \rightarrow Y$ is continuous at a point $x \in Z$ if for each $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ and x) such that for all $x' \in Z$, $[(d_q)_Z(x, x')]_q < \delta$ implies $[(d_q)_Y(f(x), f(x'))]_q < \epsilon$.

- (2) Let (Z, d_q) and (Y, d_q) are two q -metric spaces. A mapping $f : Z \rightarrow Y$ is uniformly continuous at a point $x \in Z$ if for each $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ) such that for all $x' \in Z$, $[(d_q)_Z(x, x')]_q < \delta$ implies $[(d_q)_Y(f(x), f(x'))]_q < \epsilon$.

Theorem 2.28. *Let f be a continuous mapping of a q -compact metric space Z into a q -metric space Y . Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$ and $x, x' \in Z$. Since f is continuous mapping, by definition there exists a $\delta_x > 0$ such that $[(d_q)_Z(x, x')]_q < \delta_x$ implies $[(d_q)_Y(f(x), f(x'))]_q < \epsilon$. Let $U_x = B_q(x, \frac{\delta_x}{2})$, the q -open ball of centre x and radius $\frac{\delta_x}{2}$. The collection $\{U_x\}$ is an q -open covering of Z , so it has a finite subcovering $\{U_{(x_1)}, \dots, U_{(x_n)}\}$. Let $\delta = \frac{1}{2} \min\{\delta_{(x_1)}, \dots, \delta_{(x_n)}\}$. Clearly $\delta > 0$. Next, given two points $y, z \in Z$ such that $[(d_q)_Z(y, z)]_q < \delta$, then y must belong to some $U_{(x_i)}$ and $[(d_q)_Y(y, f(x_i))]_q < \frac{\delta_{(x_i)}}{2}$. Then

$$\begin{aligned} [(d_q)_Z(z, (x_i))]_q &\leq [(d_q)_Z(z, y)]_q + [(d_q)_Z(y, (x_i))]_q \\ &< \delta + \frac{1}{2}\delta_{(x_i)} \\ &< \delta_{(x_i)}. \end{aligned}$$

So, both y and z lie at a distance less than $\delta_{(x_i)}$ from the point (x_i) . This gives that $[(d_q)_Y(f(y), f((x_i)))]_q < \frac{\epsilon}{2}$ and $[(d_q)_Y(f(z), f((x_i)))]_q < \frac{\epsilon}{2}$. Hence $[(d_q)_Y(f(y), f(z))]_q < \epsilon$, which shows that f is uniformly continuous. $\square$

3. STATISTICAL CONVERGENCE IN q -METRIC SPACE

In this section we introduce the concept of statistical convergent on q -metric space. We introduce some properties of statistical convergent sequence on q -metric space. Additionally we present statistically dense set and its relation with statistical convergent on q -metric space. We start the section with following theorem.

Definition 3.1. Let (Z, d_q) be a q -metric space. For $0 < q < 1$ and $n \in \mathbb{N}$, $(x_n) \in Z$ is called statistical converges to $x \in Z$ on q -metric, if for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : [d_q((x_k), x)]_q \geq \epsilon \right\} \right| = 0.$$

In other words, for every $\epsilon > 0$ the set $G_\epsilon = \{k \leq n : [d_q((x_k), x)]_q \geq \epsilon\}$ has zero density. If $(x_n) \subset Z$ sequence is statistical converges to x on q -metric space, then we write $st\text{-}\lim_{n \rightarrow \infty} (x_n) = x$.

Theorem 3.2. *Every q -convergent sequence is statistically convergent on q -metric space.*

Proof. The proof is straight from the definition of q -convergent sequence. $\square$

The subsequent example suffices to assert that the converse of the theorem is not necessarily true.

Example 3.3. Let us consider q -metric space Z be $Z = \mathbb{R}$ and $d_q : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ be defined by $[d_q(x, y, z)]_q = \max \left\{ |x - y|, |x - z|, |y - z| \right\}$.

$$\text{Let } (x_k) = \begin{cases} k, & k \text{ is square} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly (x_k) is statistical convergent but not convergent in general.

Theorem 3.4. If a sequence (x_n) is statistically converges to x in q -metric space, then x is unique.

Proof. Suppose (Z, d_q) is a q -metric space and the sequence $(z_n) \subset Z$ has two limit points. Therefore we can write, $st\text{-}\lim_{n \rightarrow \infty} (z_n) = x$ and $st\text{-}\lim_{n \rightarrow \infty} (z_n) = y$. Thus for given $\epsilon > 0$, $\partial(G_{\epsilon_1}) = 0$ and $\partial(G_{\epsilon_2}) = 0$ where $G_{\epsilon_1} = \{k \leq n : [d_q((z_k), x)]_q \geq \epsilon\}$ and $G_{\epsilon_2} = \{k \leq n : [d_q((z_k), y)]_q \geq \epsilon\}$. Let $G_\epsilon = G_{\epsilon_1} \cup G_{\epsilon_2}$, then $\partial(G_\epsilon) = 0$ gives $\mathbb{N}/\partial(G_\epsilon) = 1$. Next, we write

$$\begin{aligned} G_\epsilon &= \left\{ k \leq n : [d_q((z_k), x)]_q + [d_q((z_k), y)]_q \geq \epsilon \right\} \\ &\geq \left\{ k \leq n : [d_q(x, y)]_q \geq \epsilon \right\}. \end{aligned}$$

Since $\partial(G_\epsilon) = 0$, then $[d_q(x, y)]_q = 0$. So, $x = y$. □

Linearity of statistical convergent sequences on q -metric space are follows from the following results. The proofs are directly based on the definition.

Theorem 3.5. If $st\text{-}\lim_{n \rightarrow \infty} (x_n) = x$, $st\text{-}\lim_{n \rightarrow \infty} (y_n) = y$ and λ is any real number, then

- (1) $st\text{-}\lim_{n \rightarrow \infty} ((x_n) + (y_n)) = x + y$.
- (2) $st\text{-}\lim_{n \rightarrow \infty} (\lambda(x_n)) = \lambda x$.

At this point, we present the statistically Cauchy sequence on q -metric space, which is intimately connected to statistically convergent sequences in q -metric space.

Definition 3.6. A sequence (x_n) is called statistically Cauchy sequence in q -metric space if for any $\epsilon > 0$ there exists a natural number n such that for every $k, m \leq n$ the density $\partial \left(\{k \leq n : [d_q((x_k), (x_m))]_q \geq \epsilon\} \right) = 0$.

Theorem 3.7. Every sequence of statistically convergent in q -metric space is statistically Cauchy sequence in q -metric space.

Proof. Let (x_n) be statistical convergent sequence which is converges to x in q -metric space. For every $\epsilon > 0$, $\partial(G_\epsilon) = 0$ where $G_\epsilon = \{k \leq n : [d_q((x_k), x)]_q \geq \epsilon\}$. Next, we choose $m < k$ such that for the set $H_\epsilon = \{m : [d_q((x_m), x)]_q \geq \epsilon\}$, the density $\partial(H_\epsilon) = 0$. Thus we have

$$\begin{aligned} \partial \left(\left\{ k, m : [d_q((x_k), (x_m))]_q \geq \epsilon \right\} \right) &\leq \partial \left(\left\{ k : [d_q((x_k), x)]_q \geq \epsilon \right\} \right) + \partial \left(\left\{ m : [d_q((x_m), x)]_q \geq \epsilon \right\} \right) \\ &= \partial(G_\epsilon) + \partial(H_\epsilon) \end{aligned}$$

$$= 0.$$

Therefore (x_n) is statistical Cauchy sequence on q -metric space. $\square$

We shall introduce statistically bounded sequence on q -metric space.

Definition 3.8. A sequence (x_n) is said to be statistically bounded in q -metric space if there exist a real number $N > 0$ such that the set $\{k \leq n : [d_q((x_k), x)]_q \geq N\}$ has zero density.

The theorem presented below establishes the connection between statistically bounded and statistically convergent sequences, and the proof relying directly on their definitions.

Theorem 3.9. Every statistically convergent sequence in q -metric space is statistically bounded in q -metric space.

Next, we present the necessary and sufficient condition for a statistical convergent sequence in q -metric space.

Theorem 3.10. A sequence (x_n) is statistical convergent in q -metric space if and only if for every $\epsilon > 0$ the following condition satisfied:

If for the set

$$(3.1) \quad G_\epsilon = \left\{ k, m \leq n : [d_q((x_k), (x_m))]_q \geq \epsilon \right\}$$

has zero density that is $\partial(G_\epsilon) = 0$, whenever (x_m) is convergence subsequence of (x_n) .

Proof. Suppose that (x_n) sequence is statistically converges to x in a q -metric space. For any $\epsilon > 0$ $\lim_{n \rightarrow \infty} \frac{1}{n} |H_\epsilon| = 0$ where $H_\epsilon = \{k \leq n : [d_q((x_k), x)]_q \geq \epsilon\}$. Let (x_m) be convergent subsequence of (x_n) and using the q -metric space properties we get;

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k, m : [d_q((x_k), (x_m))]_q \geq \epsilon \right\} \right| &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k, m : \left(d_q[d_q((x_k), x)]_q + [d_q(x, (x_m))]_q \geq \epsilon \right) \right\} \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k : [d_q((x_k), x)]_q \geq \epsilon \right\} \right| \\ &\quad + \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ m : [d_q((x_m), x)]_q \geq \epsilon \right\} \right|. \end{aligned}$$

Since (x_m) is convergent subsequence, then $\partial(G_\epsilon) = 0$ where G_ϵ is Eqn.3.1.

Conversely, for G_ϵ in Eqn.3.1 assume that $\lim_{n \rightarrow \infty} \frac{1}{n} |G_\epsilon| = 0$ for (x_m) convergent subsequence of (x_n) , such that $st\text{-}\lim_{m \rightarrow \infty} (x_m) = x$. Next, we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : [d_q((x_k), x)]_q \geq \epsilon \right\} \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k, m \leq n : [d_q((x_k), (x_m))]_q \geq \epsilon \right\} \right| + \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ m \leq n : [d_q((x_m), x)]_q \geq \epsilon \right\} \right|. \end{aligned}$$

Since (x_m) converges to x and $\partial(G_\epsilon) = 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : [d_q((x_k), x)]_q \geq \epsilon \right\} \right| = 0$. Consequently, (x_n) is statistically convergent in q -metric space. $\square$

Definition 3.11. A subsequence (x_{n_k}) of a sequence (x_n) in a q -metric space (Z, d_q) is statistically dense, if the index set $\{n_k : k \in \mathbb{N}\}$ is statistically dense subset of $\mathbb{N}$, i.e. $\partial(\{n_k : k \in \mathbb{N}\}) = 1$.

Theorem 3.12. Let (x_n) be a sequence in a q -metric space (Z, d_q) . Then the followings are equivalent:

- (1) (x_n) is statistically convergent in (Z, d_q) .
- (2) There is a convergent sequence (y_n) in Z such that $(x_n) = (y_n)$ for almost $n \in \mathbb{N}$.
- (3) There is a statistically dense subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is convergent.
- (4) There is a statistically dense subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is statistically convergent.

Proof. For (1) $\Rightarrow$ (2): Assume (x_n) be a statistically converges to l in q -metric space. Let (y_n) be defined as

$$(y_n) = \begin{cases} (x_n), & \text{if } [d_q((x_k), l)]_q \geq \epsilon \\ l, & \text{otherwise.} \end{cases}$$

Let $A = \{n : [d_q((x_k), l)]_q \geq \epsilon\}$ has density zero. Then (y_n) only differ from (x_n) on a set of density zero. Hence (y_n) also converges to l . So (2) holds.

For (2) $\Rightarrow$ (3): Let (y_n) be a convergent sequence in Z such that $(x_n) = (y_n)$ for almost all n . That is (y_n) is convergent and differ from (x_n) only at a set of density zero, then other-side this set the index n_k , we have $(x_{n_k}) = (y_{n_k})$ are statistically dense subsequence and (x_{n_k}) is convergent.

For (3) $\Rightarrow$ (4): Since every convergent sequence are statistically convergent. So, (x_{n_k}) is convergent and statistically dense implies it is statistically convergent. So (4) hold true.

For (4) $\Rightarrow$ (1): Let (x_{n_k}) be a statistically dense subsequence which is statistically converges to l . Then for any $\epsilon > 0$, the set $A = \{n : [d_q((x_k), l)]_q \geq \epsilon\}$ has density zero with respect to $\mathbb{N}$. It is easy to see the sequence of index (n_k) form a statistically dense set in $\mathbb{N}$, it follows that most of original sequence also statistically converges to l . Hence (x_n) is also statistically converges to l . Hence (4) $\Rightarrow$ (1) is true. $\square$

CONCLUSION

This paper introduces a generalized metric space related to quantum mechanics. We designate this metric space as q -metric spaces. Several characteristics of q -metric spaces are analyzed. Moreover, utilizing the concept of natural density, we have established statistical convergence in the framework of q -metric space. In conclusion, we obtain various results through the principle of statistical density in q -metric spaces. The concept of the q -metric can be regarded as an extension of traditional metrics. This framework can facilitate the exploration of concepts like functions, sequences, convergence, and continuity within the realm of q -calculus in a novel context. It can serves as a tool, especially in the applications of differential equations based on q -calculus, functional analysis, and numerical analysis.

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